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NEW GEOMETRIC ESTIMATES FOR EULER ELASTICA

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ABSTRACT. Elastic curves, whose curvature depends linearly on height, were studied by L. Euler. We generalize an elementary comparison theorem relating pairs of these curves.

1. INTRODUCTION

If a planar curve is given locally by the graph of a function $u = u(x)$, and the corresponding curvature satisfies

$$\frac{u''}{(1+u'^2)^{3/2}} = u, \tag{1.1}$$

we say the curve is an *elastic curve.* The following comparison result for elastic graphs is essentially due to R. Finn [Fin10].

Theorem 1. *If u and v are solutions of* (1.1) *on a common interval* $[a, b]$ *with* $u(a) < v(a)$ *and* $u'(a) = v'(a)$*, then*

$$u(x) < v(x) \quad \text{and} \quad u'(x) < v'(x) \quad \text{for } a < x \le b.$$

We wish to generalize Theorem 1 in several ways to allow for certain parametric elastic curves. The curvature condition defining elastic graphs readily generalizes to parametric curves, and it is customary to introduce an arclength parameter s and the *inclination angle* ψ along the curve defined by

$$\begin{cases} \dot{x}(s) = \cos\psi(s) \\ \dot{z}(s) = \sin\psi(s). \end{cases}$$

The curvature condition then becomes

$$\dot{\psi}(s) = z(s).$$

Under this rephrasing, notice that the initial conditions of Theorem 1 may be rewritten as

$$x_1(0) = x_2(0) = a, \quad z_1(0) < z_2(0), \quad \text{and} \quad \psi_1(0) = \psi_2(0)$$

where the common value $\psi_1(0) = \psi_2(0)$ lies between $-\pi/2$ and $\pi/2$. See Figure 1(left). We wish to allow this initial value for ψ to range also between $-3\pi/2$ and $-\pi/2$ as indicated in Figure 1(right). Notice that each of the curves indicated possesses a second point

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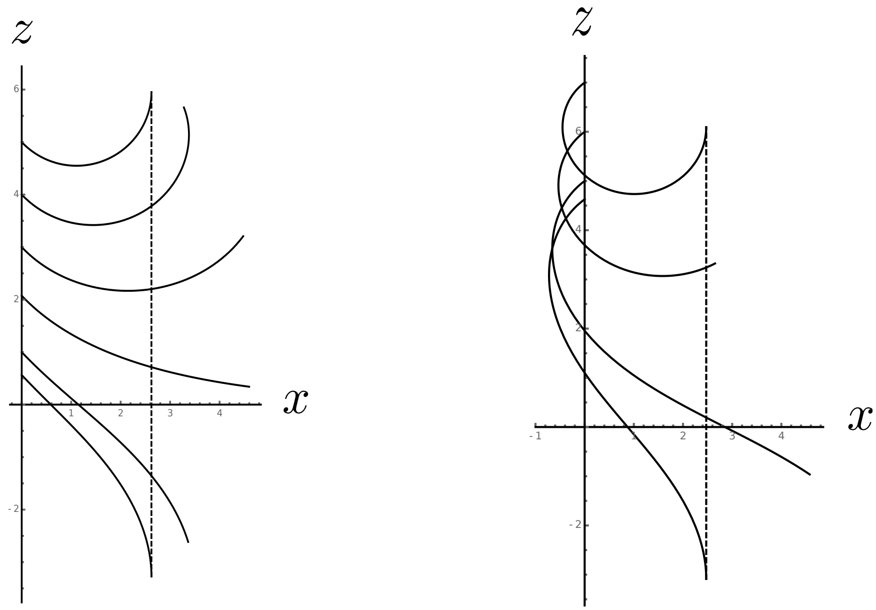

FIGURE 1. Elastic graphs (left) and parametric elastic curves (right)

$(x_j(\sigma_j), z_j(\sigma_j))$ with $x_j(\sigma_j) = a$ and $z_j(\sigma_j) < z_j(0)$. Among the conclusions of Theorem 2 below is that if $z_1(0) < z_2(0)$, then

$$z_1(\sigma_1) < z_2(\sigma_2) \quad \text{and} \quad \psi_1(\sigma_1) < \psi_2(\sigma_2). \tag{1.2}$$

Intuition might suggest that these assertions follow simply from the fact that "the curvature of the higher curve is greater at corresponding points." That the situation is more subtle and that this intuition, though correct in the non-parametric case, is actually incorrect in the parametric case may be illustrated by considering arcs of circles. Let $x_1(0) = x_2(0) = a$ with $\dot{\psi}_1 = z_1(0) < \dot{\psi}_2 = z_2(0)$ and $\psi_1(0) = \psi_2(0) \in (-\pi/2, \pi/2)$ so that (x_j, z_j) parameterizes a circular graph with curvature $z_j(0)$ for $j = 1, 2$ over some interval to the right of $x = a$. See Figure 2(left). We have then the conclusion of Theorem 1: If $s_1, s_2 > 0$ with $x_1(s_1) = x_2(s_2)$, then

$$z_1(s_1) < z_2(s_2) \quad \text{and} \quad \psi_1(s_1) < \psi_2(s_2). \tag{1.3}$$

If, on the other hand, the same conditions hold for arcs of circles with $\psi_1(0) = \psi_2(0) \in (-3\pi/2, -\pi/2)$, then one finds immediately that

$$\psi_1(\sigma_1) = \psi_2(\sigma_2)$$

where σ_j is the first positive arclength for which $x_j(\sigma_j) = a$, $j = 1, 2$. Thus, there is clearly something more involved in the parametric case of elastic curves.

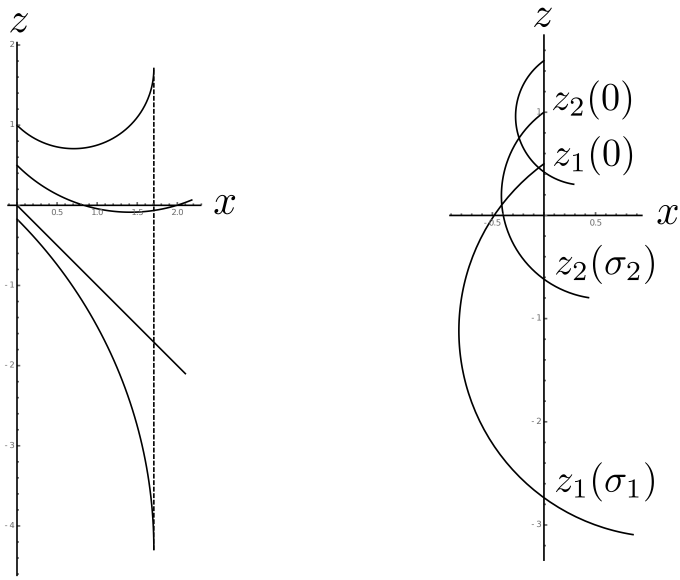

FIGURE 2. Circular graphs (left) and parametric circles (right)

2. CORRESPONDENCE AND COMPARISON

We are primarily interested in solutions of

$$\left\{\begin{array}{ll} \dot{x} = \cos\psi, & x(0) = a \\ \dot{z} = \sin\psi, & z(0) = \zeta \\ \dot{\psi} = z, & \psi(0) = \theta \end{array}\right. \tag{2.1}$$

for which $-3\pi/2 < \theta < -\pi/2$ and there is some first positive arclength $\sigma > 0$ for which $x(\sigma) = a$. We begin with a characterization of solutions for which this is the case. This will put us in a position to state a somewhat technical generalization of Theorem 1 to parametric elastic curves. First observe that the initial value problem

$$\left\{\begin{array}{ll} \dot{z} = \sin\psi, & z(0) = \zeta \\ \dot{\psi} = z, & \psi(0) = \theta \end{array}\right.$$

decouples from the conditions on x, namely $\dot{x} = \cos\psi$ and $x(0) = a$, and admits a conserved quantity

$$h(\psi, z) = \frac{z^2}{2} + \cos\psi = \frac{\zeta^2}{2} + \cos\theta.$$

See Figure 3.

Lemma 1 (preliminary characterization of solutions). *Given θ with $-3\pi/2 < \theta < -\pi/2$, there is a unique value $\zeta_- = \zeta_-(\theta) > 0$ for which solutions of (2.1) with $\zeta > \zeta_-$ are precisely those having*

$$\sup_{s>0} x(s) > a.$$

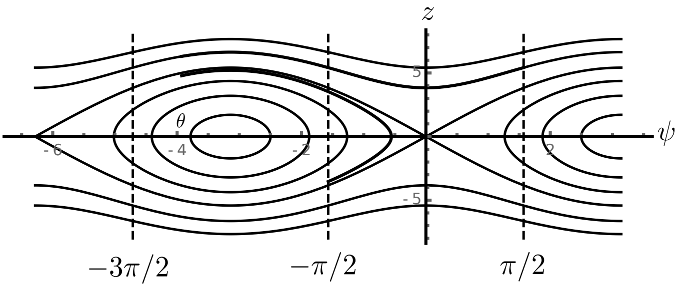

FIGURE 3. Phase plane diagram for ψ and z.

For each $\zeta \geq \zeta_-$ there is a unique first positive arclength $\ell = \ell(\theta, \zeta)$ for which $\psi(\ell) = -\pi/2$. Moreover, there is a unique value $\zeta_* = \zeta_*(\theta) > \zeta_-$ determined by

$$\frac{\zeta_*^2}{2} + \cos\theta = 1, \tag{2.2}$$

and we have

(1) *If $\zeta < \zeta_-$, then $x(s) < a$ for $s > 0$.*

(2) *If $\zeta_- \leq \zeta < \zeta_*$, then there is a unique second positive arclength $r = r(\theta, \zeta)$ for which the solution is vertical; $\psi(r) = -\pi/2$, and*

 (a) *$\dot{x}(s) = \cos\psi(s) < 0$ for $0 \leq s < \ell$, and $\dot{x}(s) = \cos\psi(s) > 0$ for $\ell \leq s < r$.*

 (b)
$$\lim_{\zeta \searrow \zeta_-} x(r(\theta, \zeta)) = a.$$

 (c)
$$\frac{d}{d\zeta} x(r(\theta, \zeta)) > 0, \quad \zeta_- < \zeta < \zeta_*.$$

 (d)
$$\lim_{\zeta \nearrow \zeta_*} x(r(\theta, \zeta)) = +\infty.$$

(3) *If $\zeta = \zeta_*$, then $\dot{x}(s) = \cos\psi(s) < 0$ for $0 \leq s < \ell$, and $\dot{x}(s) = \cos\psi(s) > 0$ for $\ell \leq s$ with*

$$\lim_{s \nearrow \infty} x(s) = +\infty. \tag{2.3}$$

(4) *If $\zeta > \zeta_*$, then there is a unique second positive $r = r(\theta, \zeta)$ for which the solution is vertical; $\psi(r) = \pi/2$, and*

 (a) *$\dot{x}(s) = \cos\psi(s) < 0$ for $0 \leq s < \ell$, and $\dot{x}(s) = \cos\psi(s) > 0$ for $\ell \leq s < r$.*

 (b)
$$\lim_{\zeta \searrow \zeta_*} x(r(\theta, \zeta)) = +\infty.$$

 (c)
$$\frac{d}{d\zeta} x(r(\theta, \zeta)) < 0, \quad \zeta > \zeta_*.$$

 (d)
$$\lim_{\zeta \nearrow \infty} x(r(\theta, \zeta)) = a.$$

Proof. The existence of ζ_* determined by (2.2) and the conserved quantity $z^2/2+\cos\psi=1$ is immediate; we start the proof from this point. The existence and uniqueness of the values ℓ and r also follow from the phase diagram in Figure 3. Notice that ℓ and r will be well-defined as long as $\zeta \geq \sqrt{-2\cos\theta}$. We will see below that $\zeta_- > \sqrt{-2\cos\theta}$.

To see the limit (2.3) of the third assertion, note that $\dot{\psi} = z > 0$, so $x = x(s)$ may be expressed as a function $\xi = \xi(\psi)$ for $\theta \leq \psi < 0$ with

$$\frac{d\xi}{d\psi} = \frac{\dot{x}}{\dot{\psi}} = \frac{\cos\psi}{z} = \frac{\cos\psi}{\sqrt{2}\sqrt{1-\cos\psi}}$$

since $z^2/2+\cos\psi = 1$. Thus, observing that $\psi(s) \nearrow 0$ as $s \nearrow +\infty$,

$$\begin{aligned} x(s) = \xi(\psi) &= a + \frac{1}{\sqrt{2}}\int_\theta^\psi \frac{\cos t}{\sqrt{1-\cos t}}\,dt \\ &\geq a + \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/4} \frac{\cos t}{\sqrt{1-\cos t}}\,dt + \int_{-\pi/4}^{\psi} \frac{1}{|t|}\,dt\right) \\ &\to \infty \quad \text{as } \psi \to 0, \end{aligned}$$

and the third assertion is established. We next turn to the second assertion and consider $0 < \zeta < \zeta_*$. In this case, a maximum inclination angle $\psi_m \in (-\pi, 0)$ is determined by

$$\cos\psi_m = c := \frac{\zeta^2}{2} + \cos\theta \quad \text{so that} \quad \frac{dc}{d\zeta} = \zeta \quad \text{and} \quad \frac{d\psi_m}{d\zeta} = -\zeta\csc\psi_m > 0. \tag{2.4}$$

Note also in this case, the existence of a unique first positive arclength $m = m(\theta,\zeta) > \ell$ for which $\psi(m) = \psi_m$, and this is also the first positive arclength for which $z(m) = 0$.

If $0 < \zeta \leq \sqrt{-2\cos\theta}$, then $\psi_m \leq -\pi/2$ so $\dot{x} = \cos\psi \leq 0$ and $x(s) < a$ for $s > 0$. The same conclusion holds for $\zeta \leq 0$, so these cases comprise part of the first assertion. For $\sqrt{-2\cos\theta} < \zeta < \zeta_*$, we find

$$\begin{aligned} x(r) - a &= \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c-\cos t}}\,dt + \int_{-\pi/2}^{\psi_m} \frac{\cos t}{\sqrt{c-\cos t}}\,dt - \int_{\psi_m}^{-\pi/2} \frac{\cos t}{\sqrt{c-\cos t}}\,dt\right) \\ &= \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c-\cos t}}\,dt + 2\int_{-\pi/2}^{\psi_m} \frac{\cos t}{\sqrt{c-\cos t}}\,dt\right) \\ &= \frac{1}{\sqrt{2}}\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c-\cos t}}\,dt + 2\sqrt{2}\int_{-\pi/2}^{\psi_m} \csc^2 t\sqrt{c-\cos t}\,dt. \end{aligned}$$

In particular, it follows from the last expression for $x(r)$ that

$$\frac{d}{d\zeta}x(r) = \zeta\left(-\frac{1}{\sqrt{2}}\int_\theta^{-\pi/2} \frac{\cos t}{2(c-\cos t)^{3/2}}\,dt + \sqrt{2}\int_{-\pi/2}^{\psi_m} \frac{\csc^2 t}{\sqrt{c-\cos t}}\,dt\right) > 0 \tag{2.5}$$

since $\cos t < 0$ for $\theta \leq t < -\pi/2$. This establishes part (c) of the second assertion. Next, using (2.2) and (2.4) we can write

$$c = \frac{\zeta^2}{2} + 1 - \frac{\zeta_*^2}{2} = \cos\psi_m.$$

Thus, using a Taylor approximation of $c - \cos t$ at $t = \psi_m$, we obtain some small positive ϵ for which $0 < c-\cos t = \sin\psi_m(t-\psi_m) + \circ(|t-\psi_m|) < 2\sin\psi_m(t-\psi_m)$ for $\psi_m - \epsilon < t < \psi_m$.

Using the second expression for $x(r)$ and assuming $\psi_m - \epsilon > \pi/4$, we find

$$\begin{aligned}
x(r) - a &\geq \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt + 2\int_{\psi_m - \epsilon}^{\psi_m} \frac{\cos \pi/4}{\sqrt{2 \sin \psi_m (t - \psi_m)}}\, dt\right) \\
&= \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt + \frac{1}{\sqrt{|\sin \psi_m|}}\int_{\psi_m - \epsilon}^{\psi_m} \frac{1}{\sqrt{\psi_m - t}}\, dt\right) \\
&= \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt + 2\sqrt{\frac{\epsilon}{|\sin \psi_m|}}\right) \\
&\to x(\ell(\theta, \zeta_*); \zeta_*) - a + \infty \quad \text{as } \zeta \nearrow \zeta_*
\end{aligned}$$

since $\psi_m \nearrow 0$ as $\zeta \nearrow \zeta_*$ and $\epsilon > 0$ is fixed. This establishes part (d) of the second assertion. Part (b) of the second assertion and the rest of the first assertion now follow from the monotonicity (2.5) if we let ζ_- be the unique value of ζ for which $x(r(\theta, \zeta); \zeta) = a$.

Finally, we turn to the fourth and last assertion of the lemma. In this case ψ increases monotonically and

$$\begin{aligned}
x(r) - a &= \frac{1}{\sqrt{2}}\int_\theta^{\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt \\
&= \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt + \int_{-\pi/2}^{\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt\right) \\
&= \frac{1}{\sqrt{2}}\left(-\int_{\pi+\theta}^{\pi/2} \frac{\cos \tau}{\sqrt{c + \cos \tau}}\, d\tau + \int_{-\pi/2}^{\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt\right) \\
&\geq \frac{1}{\sqrt{2}}\left(-\int_{\pi+\theta}^{\pi/2} \frac{\cos \tau}{\sqrt{c + \cos \tau}}\, d\tau + \int_{-\pi/4}^{0} \frac{\cos \pi/4}{\sqrt{(\zeta^2 - \zeta_*^2)/2 + t^2/2}}\, dt\right) \\
&= \frac{1}{\sqrt{2}}\left(-\int_{\pi+\theta}^{\pi/2} \frac{\cos \tau}{\sqrt{c + \cos \tau}}\, d\tau + \sinh^{-1}\left(\frac{\pi/4}{\sqrt{\zeta^2 - \zeta_*^2}}\right)\right) \\
&\to x(\ell(\theta, \zeta_*); \zeta_*) - a + \infty \quad \text{as } \zeta \searrow \zeta_*.
\end{aligned}$$

This is part (b). To see the monotonicity of part (c), we return to the third expression for $x(r)$.

$$\begin{aligned}
\frac{d}{d\zeta} x(r) &= -\frac{\zeta}{2\sqrt{2}}\left(-\int_{\pi+\theta}^{\pi/2} \frac{\cos \tau}{(c + \cos \tau)^{3/2}}\, d\tau + \int_{-\pi/2}^{\pi/2} \frac{\cos t}{(c - \cos t)^{3/2}}\, dt\right) \\
&\leq -\frac{\zeta}{2\sqrt{2}}\int_{-\pi/2}^{-\pi+\theta} \frac{\cos \tau}{(c - \cos \tau)^{3/2}}\, d\tau \\
&< 0.
\end{aligned}$$

Finally, we return once again to the third expression for $x(r)$ and estimate directly to find

$$x(r) > a + \frac{1}{\sqrt{2}}\int_{-\pi/2}^{\pi+\theta} \frac{\cos t}{\sqrt{c - \cos t}}\, dt > a.$$

Furthermore,

$$\lim_{\zeta\nearrow\infty} x(r) \le a + \frac{1}{\sqrt{2}} \lim_{\zeta\nearrow\infty} \int_{-\pi/2}^{\pi/2} \frac{\cos t}{\sqrt{c-\cos t}}\, dt \le a + \lim_{\zeta\nearrow\infty} \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{c}}\, dt = a + \pi \lim_{\zeta\nearrow\infty} \frac{1}{\sqrt{c}} = a.$$

This establishes the last part of the fourth assertion. □

In view of Lemma 1, our first comparison result may be stated as follows:

Theorem 2. *If (x_1, z_1, ψ_1) and (x_2, z_2, ψ_2) are solutions of (2.1) with $-3\pi/2 < \theta < -\pi/2$ and $\zeta = \zeta_1 = z_1(0)$ and $\zeta = \zeta_2 = z_2(0)$ respectively with $\zeta_- = \zeta_-(\theta) \le \zeta_1 < \zeta_2$, then for $j = 1, 2$, there is a unique first positive arclength σ_j for which $x_j(\sigma_j) = a$ and*

$$\{(x_j(s), z_j(s)) : 0 \le s \le \sigma_j\}$$

consists of an upper graph

$$\mathcal{U}_j = \{(x_j(s), z_j(s)) : 0 \le s \le \ell_j\}$$

where $\ell_j = \ell(\theta, \zeta_j)$ and a lower graph

$$\mathcal{L}_j = \{(x_j(s), z_j(s)) : \ell_j \le s \le \sigma_j\}.$$

The inclination angle ψ_j increases monotonically from θ to $-\pi/2$ with s for $0 \le s \le \ell_j$, and this provides a natural one-to-one correspondence between the upper graphs $\mathcal{U}_1$ and $\mathcal{U}_2$.
1. *Each upper graph may be parameterized by inclination angle $t \mapsto (\alpha_j(t), \beta_j(t)) = (x_j(s), z_j(s))$ for $t \in [\theta, -\pi/2]$ (and $s \in [0, \ell_j]$). Comparison of the points $p = (\alpha_1(t), \beta_1(t))$ and $q = (\alpha_2(t), \beta_2(t))$ in the upper graphs, as indicated in Figure 4(left), is given by*

$$\alpha_1(t) < \alpha_2(t) \quad \text{and} \quad \beta_1(t) < \beta_2(t) \quad \text{for } \theta < t \le -\pi/2. \tag{2.6}$$

A one-to-one correspondence between the lower graphs $\mathcal{L}_1$ and $\mathcal{L}_2$ is determined as follows:
2. *Let $B_j = B_j(\xi)$ for $x_j(\ell_j) \le \xi \le a$ express $\mathcal{L}_j$ as a graph over the horizontal x-axis. Two points $P = (\xi_1, B_1(\xi_1))$ and $Q = (\xi_2, B_2(\xi_2))$, in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, correspond to one another if for some $t \in [\theta, -\pi/2]$ (an inclination angle along the upper graphs) we have*

$$\alpha_1(t) = \xi_1 \quad \text{and} \quad \alpha_2(t) = \xi_2,$$

i.e., the x coordinates of the points on the lower graphs are shared with points on the upper graphs having the same inclination. This correspondence is illustrated in Figure 4(right). Under this correspondence, if $s_1 > \ell_1$ and $s_2 > \ell_2$ with

$$P = (x_1(s_1), z_1(s_1)) = (\xi_1, B_1(\xi_1)) \quad \text{and} \quad Q = (x_2(s_2), z_2(s_2)) = (\xi_2, B_2(\xi_2)),$$

then

$$z_1(s_1) < z_2(s_2) \quad \text{and} \quad \psi_1(s_1) < \psi_2(s_2). \tag{2.7}$$

The last assertion may be compared with (1.3). Among its corollaries is assertion (1.2) mentioned in the introduction which is obtained by taking $\xi_1 = \xi_2 = a$ and $t = \theta$.

In the next section we will prove the following generalization of Finn's result which allows us to concatenate the nonparametric comparison onto the parametric comparison.

Theorem 3. *If u and v are solutions of (1.1) on a common interval $[a, b]$ with $u(a) \le v(a)$ and $u'(a) \le v'(a)$, then either $u \equiv v$ or*

$$u(x) < v(x) \quad \text{and} \quad u'(x) < v'(x) \quad \text{for } a < x \le b.$$

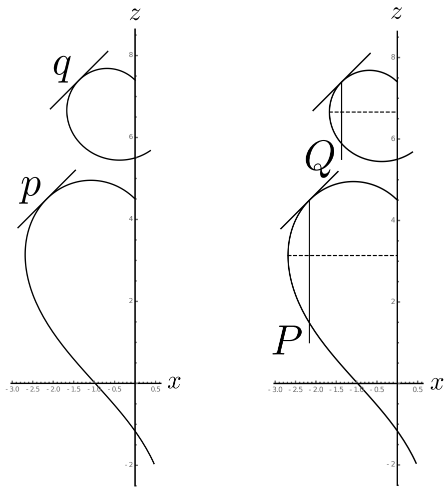

FIGURE 4. One-to-one correspondence of the upper graphs by inclination angle (left) and one-to-one correspondence of the lower graphs (right); the points P and Q in the lower graphs correspond because they share abscissas with p and q respectively which correspond in the upper graphs (by inclination angle).

Corollary 2. *If* (x_1, z_1, ψ_1) *and* (x_2, z_2, ψ_2) *are solutions of* (2.1) *with* $-3\pi/2 < \theta < -\pi/2$ *and* $\zeta = \zeta_1 = z_1(0)$ *and* $\zeta = \zeta_2 = z_2(0)$ *respectively with* $\zeta_- = \zeta_-(\theta) \leq \zeta_1 < \zeta_2$ *as described in Theorem* 2*, and both solutions extend as graphs over some common interval* $[a, b]$ *on the horizontal axis so that* $\sigma_j \leq s_j$ *for* $j = 1, 2$ *and* $a \leq x_1(s_1) = x_2(s_2) \leq b$*, then*

$$z_1(s_1) < z_2(s_2) \quad \text{and} \quad \psi_1(s_1) < \psi_2(s_2). \tag{2.8}$$

Remarks

(1) A minor modification of the proofs below allows the generalization of the curvature condition to $\dot{\psi} = \kappa z$ where κ is a positive constant.
(2) The results can also be generalized (with some minor technical changes) to allow the initial inclination angle $\theta = -3\pi/2$. It is probably possible to obtain some kind of generalization for $\theta < -3\pi/2$ as well, but that would require some fundamentally new ideas.
(3) The results for graphs, Theorem 1 and Theorem 3, can be generalized to graphs whose curvature is given by any strictly increasing function of height. Our proof below uses continuity of u and v on the closed interval $[a, b]$, but the values of the derivatives may be allowed to take the values $\pm\infty$ at the endpoints.

3. PROOFS

We begin with a proof of Theorem 3 (which also provides a proof of Theorem 1)
proof of Theorem 3: If $u(a) = v(a)$ and $u'(a) = v'(a)$, then $u \equiv v$.

In the complementary case, there is always some $\epsilon > 0$ for which $u(x) < v(x)$ and $u'(x) < v'(x)$ for $a < x < a + \epsilon$. Thus, taking

$$x_* = \min\{x > a : u(x) \geq v(x) \text{ or } u'(x) \geq v'(x)\},$$

assuming such a point exists, we can set $c = \max\{v(x) - u(x) : a \leq x \leq x_*\}$ and $w = u + c$. We see that w is a smooth function with $w \geq v$, and equality holds for at least one point $x_b \in (a, x_*]$. Also $w'(x_b) = v'(x_b)$. In a neighborhood of $x_b \in (a, x_*]$ we have $w \geq v$ but the graph of w has curvature strictly less than that of the graph of v. This leads to a contradiction. □

Proof of the upper graph comparison of Theorem 2:

We begin with a general comparison which applies to any continuous correspondence by inclination angle starting at $t = \theta$ with $\beta_1(\theta) < \beta_2(\theta)$.

Lemma 3. $\beta_1(t) < \beta_2(t)$.

Proof. If for some $t_0 > \theta$, we have $\beta_1(t_0) = \beta_2(t_0)$, then the horizontal translation of the first/lower curve parameterized by

$$t \mapsto (\alpha_1(t) + \alpha_2(t_0) - \alpha_1(t_0), \beta_1(t))$$

is an elastic curve which agrees with the second/upper elastic curve up to first order at $(\alpha_2(t_0), \beta_2(t_0))$. It follows that these two elastica are identical. In particular, $\beta_1(\theta) = \beta_2(\theta)$, which is a contradiction. □

On an upper graph, $\cos t \leq 0$ and

$$\alpha_j'(t) = \frac{\dot{x}_j}{\dot{\psi}_j} = \frac{\cos t}{\beta_j(t)}.$$

Thus, $\alpha_1'(t) \leq \alpha_2'(t) < 0$ with strict inequality except for $t = -\pi/2$. Thus, $\alpha_1(t) < \alpha_2(t)$ for $\theta < t \leq -\pi/2$, and this completes the first part of the proof of Theorem 2.

Proof of the lower graph comparison of Theorem 2:

As might be expected, this is rather more complicated than the first part. However, we have most of the framework set up in the statements of Lemma 1 and the result itself. As a matter of technical convenience, we rename the points P, Q, p, and q as P_1, P_2, P_3, and P_4 respectively so that $P_j = (x_1(s_j), z_1(s_j))$ for j odd and $P_j = (x_2(s_j), z_2(s_j))$ for j even. For the lower graph comparison it is natural to use $\xi = \xi_2 \in [x_2(\ell_2), a]$ as a parameter; it will be noted that each of the arclengths s_j, $j = 1, 2, 3, 4$ is uniquely and continuously determined as a function of ξ, and the dependence is smooth for $x_2(\ell_2) < \xi < a$. Consequently, we can attempt a direct comparison of the derivatives of the quantities in (2.7) with respect to ξ.

Lemma 4 (direct differentiation)**.** *The arclengths s_1 and s_2 are increasing functions of ξ and s_3 and s_4 are decreasing functions of ξ for $x_2(\ell_2) < \xi < a$ with*

(1)
$$\frac{d}{d\xi} z_1(s_1) = \tan\psi_1(s_1) \frac{z_2(s_4)}{z_1(s_3)} \quad \text{and} \quad \frac{d}{d\xi} z_2(s_2) = \tan\psi_2(s_2).$$

(2)
$$\frac{d}{d\xi}\psi_1(s_1) = \frac{z_1(s_1)}{\cos\psi_1(s_1)} \frac{z_2(s_4)}{z_1(s_3)} \quad \text{and} \quad \frac{d}{d\xi}\psi_2(s_2) = \frac{z_2(s_2)}{\cos\psi_2(s_2)}$$

(3) *If* $-\pi/2 < \psi_1(s_1) < 0$ *and* $\psi_1(s_1) < \psi_2(s_2) < \pi/2$, *then*

$$\frac{d}{d\xi} z_1(s_1) < \frac{d}{d\xi} z_2(s_2).$$

(4) *If*
   (a) $z_1(s_1) < 0$,
   (b) $z_1(s_1) < z_2(s_2)$,
   (c) $-\pi/2 < \psi_1(s_1) < 0$, *and*
   (d) $\psi_1(s_1) < \psi_2(s_2)$,

   then

$$\frac{d}{d\xi} \psi_1(s_1) < \frac{d}{d\xi} \psi_2(s_2).$$

Proof. Differentiating directly,

$$\frac{d}{d\xi} z_1(s_1) = \dot{z}_1(s_1) \frac{ds_1}{d\xi} = \sin \psi_1(s_1) \frac{ds_1}{d\xi}.$$

Also, since $x_1(s_1) = x_1(s_3)$ and $\psi_1(s_3) = \psi_2(s_4)$,

$$\cos \psi_1(s_1) \frac{ds_1}{d\xi} = \cos \psi_1(s_3) \frac{ds_3}{d\xi} \quad \text{and} \quad z_1(s_3) \frac{ds_3}{d\xi} = z_2(s_4) \frac{ds_4}{d\xi}.$$

Finally, since $x_2(s_4) = \xi$, we have

$$\frac{ds_4}{d\xi} = \frac{1}{\cos \psi_2(s_4)} \quad \text{and} \quad \frac{d}{d\xi} z_1(s_1) = \tan \psi_1(s_1) \frac{\cos \psi_1(s_3)}{\cos \psi_2(s_4)} \frac{z_2(s_4)}{z_1(s_3)} = \tan \psi_1(s_1) \frac{z_2(s_4)}{z_1(s_3)}.$$

The other differentiation formulas follow similarly.

Since we know from the comparison of the upper graphs that $z_2(s_4) > z_1(s_3)$ and $\tan \psi$ is increasing for $-\pi/2 < \psi < \pi/2$, the inequality of the third assertion is immediate.

For the last assertion, we note that under the hypotheses $\cos \psi_1(s_1) > 0$, and $\cos \psi$ is increasing for $-\pi/2 < \psi < 0$. If $z_2(s_2) < 0$ then $\psi_2(s_2) < 0$, and the inequality follows immediately using the monotonicity of $\cos \psi$. If $z_2(s_2) \geq 0$, then the inequality follows simply because the quantity on the left is negative and the quantity on the right is non-negative. □

Starting at the left-most points, i.e., for $\xi = x_2(\ell_2)$ where $\beta_1(-\pi/2) < \beta_2(-\pi/2)$, we know $z_1(s_1) < z_2(s_2)$ at least initially simply by continuity. Unfortunately, $\psi_2(\ell_2) = -\pi/2 = \psi_1(\ell_1)$, and we do not have the hypotheses of the fourth assertion in order to get started with the desired inequality between $\psi_1(s_1)$ and $\psi_2(s_2)$. Thus, an immediate application of the direct differentiation lemma seems problematic.

We turn instead to a secondary comparison which applies to portions of the lower graphs in one-to-one correspondence by inclination angle.

For each elastic curve, there is some maximal interval $[\ell_j, m_j^*)$ on which $\dot{\psi}_j = z_j > 0$. In fact, if for $\zeta_j^- < \zeta_j < \zeta_*$ we let $m_j = m_j(\theta, \zeta_j)$ denote the first positive arclength m for which $z_j(m) = 0$ and $\psi_j(m) = \psi_m(\theta, \zeta_j)$, then

$$m_j^* = \begin{cases} \min\{m_j, \sigma_j\} & \text{if } \zeta_j^- \leq \zeta_j < \zeta_*(\theta) \\ \sigma_j & \text{if } \zeta_j \geq \zeta_*(\theta). \end{cases}$$

Thus, we see the parameterization $t \mapsto (\alpha_j(t), \beta_j(t))$ by inclination angle is still valid for $\theta \leq t \leq \psi_j(m_j^*)$, and a one-to-one correspondence of the lower graphs is also possible for

$-\pi/2 \le t \le T = \min\{\psi_1(m_1^*), \psi_2(m_2^*)\}$. Lemma 3 still applies on this interval, and we know $\beta_1(t) < \beta_2(t)$.

Now we make the secondary comparison mentioned above.

Lemma 5. *Let $B_j^+ = B_j^+(\xi)$ for $x_j(\ell_j) \le \xi \le a$ express $\mathcal{U}_j$ as a graph over the horizontal x-axis. For each t with $-\pi/2 \le t \le T = \min\{\psi_1(m_1^*), \psi_2(m_2^*)\}$ there are corresponding points $\hat{P}_1 = (\alpha_1(t), \beta_1(t)) \in \mathcal{L}_1$ and $P_2 = (\alpha_2(t), \beta_2(t)) \in \mathcal{L}_2$. The second point $P_2 = (\alpha_2(t), \beta_2(t))$ shares its x-coordinate with a unique point*

$$P_4 = (\xi, B_2^+(\xi)) = (\alpha_2(t), B_2^+(\alpha_2(t))) = (x_2(s_4), z_2(s_4)) \in \mathcal{U}_2.$$

The inclination angle $\psi_2(s_4)$ for $0 \le s_4 \le \ell_2$ determines the point

$$P_3 = (\alpha_1(\psi_2(s_4)), \beta_1(\psi_2(s_4))) \in \mathcal{U}_1$$

by the inclination angle correspondence. Under these conditions

$$\alpha_1(\psi_2(s_4(t))) < \alpha_1(t) \quad \text{for } -\pi/2 < t \le T. \tag{3.1}$$

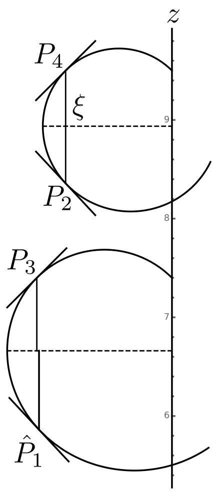

FIGURE 5. Secondary comparison: $\hat{P}_1$ and P_2 correspond by inclination angle; P_3 and P_4 correspond by inclination angle. If P_2 and P_4 have the same x-coordinate, then P_3 is to the left of $\hat{P}_1$.

Proof. An alternative phrasing of the assertion (3.1) is as follows: Each $\xi \in (x_2(\ell_2), \beta_2(T))$, determines points

$$P_2 = (\xi, B_2(\xi)) = (x_2(s_2), z_2(s_2)) \in \mathcal{L}_2 \quad \text{and} \quad P_4 = (\xi, B_2^+(\xi)) = (x_2(s_4), z_2(s_4)) \in \mathcal{U}_2.$$

Each of these points corresponds by inclination angle to a unique point in the first/lower elastic curve:

$$\hat{P}_1 = (x_1(\hat{s}_1), z_1(\hat{s}_1)) \in \mathcal{L}_1 \quad \text{where } \ell_1 < \hat{s}_1 \le \psi_1^{-1}(T) \text{ and } \psi_1(\hat{s}_1) = \psi_2(s_2),$$

and

$$P_3 = (x_1(s_3), z_1(s_3)) \in \mathcal{U}_1 \quad \text{where } 0 \le s_3 < \ell_1 \text{ and } \psi_1(s_3) = \psi_2(s_4). \tag{3.2}$$

Assertion (3.1) then says

$$x_1(s_3) < x_1(\hat{s}_1).$$

The arclength $\hat{s}_1$ is also a continuous function of $\xi \in [x_2(\ell_2), \alpha_2(T)]$ which is smooth on the interior of the interval. (Note that $\hat{s}_1 = s_1$ when $\xi = x_2(\ell_2)$; in this case $(x_1(s_1), z_1(s_1))$ and $(\xi, \alpha_2(-\pi/2))$ are the left-most points of each curve. Generally, $P_1 = (x_1(s_1), z_1(s_1))$ is a different point which lies directly below P_3.)

Differentiating as in the direct differentiation lemma, we find

$$\frac{d}{d\xi} x_1(s_3) = \dot{x}_1(s_3) \frac{ds_3}{d\xi} = \frac{\cos\psi_1(s_3)}{\cos\psi_2(s_4)} \frac{z_2(s_4)}{z_1(s_3)} = \frac{z_2(s_4)}{z_1(s_3)}.$$

Similarly,

$$\frac{d}{d\xi} x_1(\hat{s}_1) = \frac{z_2(s_2)}{z_1(\hat{s}_1)}.$$

We see then that the result will follow if we can show

$$\frac{z_2(s_4)}{z_1(s_3)} < \frac{z_2(s_2)}{z_1(\hat{s}_1)}. \tag{3.3}$$

Given the correspondence of $\hat{P}_1$ to P_2 and P_3 to P_4 by inclination angle t, we compute

$$\frac{d}{dt} \frac{\beta_2(t)}{\beta_1(t)} = \frac{\sin t}{\beta_1^2} \left(\frac{\beta_1}{\beta_2} - \frac{\beta_2}{\beta_1} \right)$$

since $d\beta_j/dt = \dot{z}_j/\dot{\psi}_j = \sin t/\beta_j$. It follows that

$$\frac{d}{dt} \frac{\beta_2(t)}{\beta_1(t)} > 0 \quad \text{for } \max\{-\pi, \psi_2(s_4)\} < t < \min\{0, T\}.$$

We conclude

$$\frac{z_2(s_4)}{z_1(s_3)} < \frac{z_2(\ell_2)}{z_1(\ell_1)} < \frac{z_2(s_2)}{z_1(\hat{s}_1)} \quad \text{for } x_2(\ell_2) < \xi \le \min\{\alpha_2(T), \alpha_2(-\pi)\}. \tag{3.4}$$

Note: It may be that $-\pi < \theta$, but if so, then $\alpha_2(-\pi)$ is still well-defined and $\alpha_2(T) < \alpha_2(-\pi)$. One may also be concerned that $T > 0$; this can only happen if $\zeta_2 > \zeta_*$, and in this case, i.e., in case 4 of Lemma 1, we find

$$\begin{aligned}
\alpha_2(T) &= a + \frac{1}{\sqrt{2}} \left(\int_\theta^{-\pi} \frac{\cos t}{\sqrt{c - \cos t}}\, dt + \int_{-\pi}^{-\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt \right. \\
&\qquad \left. + \int_{-\pi/2}^{0} \frac{\cos t}{\sqrt{c - \cos t}}\, dt + \int_0^T \frac{\cos t}{\sqrt{c - \cos t}}\, dt \right) \\
&= \alpha_2(-\pi) + \frac{1}{\sqrt{2}} \left(\int_{-\pi/2}^{0} \left(\frac{\cos t}{\sqrt{c - \cos t}} - \frac{\cos t}{\sqrt{c + \cos t}} \right) dt + \int_0^T \frac{\cos t}{\sqrt{c - \cos t}}\, dt \right) \\
&> \alpha_2(0) \\
&> \alpha_2(-\pi).
\end{aligned}$$

(See the proof of Lemma 1.)

It remains to establish (3.3) when we know $\min\{\alpha_2(T), \alpha_2(-\pi)\} = \alpha_2(-\pi) < \alpha_2(T)$ and $\psi_2(s_4(T)) < -\pi$. For each t with $-3\pi/2 < \theta \le \psi_2(s_4(T)) \le t < -\pi$, and corresponding points

$$P_3 = (x_1(s_3), z_1(s_3)) = (\alpha_1(t), \beta_1(t)) \in \mathcal{U}_1$$

and

$$P_4 = (x_2(s_4), z_2(s_4)) = (\alpha_2(t), \beta_2(t)) \in \mathcal{U}_2,$$

there are symmetric points

$$\tilde{P}_3 = (\alpha_1(-2\pi - t), \beta_1(-2\pi - t)) = (\alpha_1(-2\pi - t), \beta_1(t)) \in \mathcal{U}_1$$

and

$$\tilde{P}_4 = (\alpha_2(-2\pi - t), \beta_2(-2\pi - t)) = (\alpha_2(-2\pi - t), \beta_2(t)) \in \mathcal{U}_2.$$

Since $-\pi < -2\pi - t < -\pi/2$ and the z-coordinates are the same, we have from (3.4) that

$$\frac{z_2(s_4)}{z_1(s_3)} = \frac{\beta_2(t)}{\beta_1(t)} < \frac{z_2(\ell_2)}{z_1(\ell_1)}.$$

The same kind of symmetry applies to the lower graphs with respect to $t = 0$ so that the quotient inequalities of (3.4) hold also for $x_2(\ell_2) < \xi \le \alpha_2(T)$. Thus, (3.3) follows and the lemma is proved.

As a technical note, we mention that the lower bound on s_3 in (3.2) can be given somewhat more precisely as follows. Consider the unique arclength $\mu_4 < \ell_2$ for which $x_2(\mu_4) = \alpha_2(T)$. Consider also the unique arclength $\mu_3 < \ell_1$ for which $\psi_1(\mu_3) = \psi_2(\mu_4)$. Then it is clear that $s_3 \ge \mu_3$. □

We are now in a position to determine how the correspondence of the lower graphs by inclination angle comes to an end.

Corollary 6. $T = \min\{\psi_1(m_1^*), \psi_2(m_2^*)\} = \psi_1(m_1^*) < \psi_2(m_2^*)$.

Proof. Notice that m_j^* is the first positive arclength s for which $(x_j(s), z_j(s))$ is not in the open quadrant $\mathcal{R} = \{(x, z) : x < a, z > 0\}$.

Assume $T = \psi_2(m_2^*) \le \psi_1(m_1^*)$. This means there is some $\hat{s}_1 \le m_1^*$ for which $P_2 = (x_2(s_2), z_2(s_2)) \in \partial\mathcal{R}$ and $\hat{P}_1 = (x_1(\hat{s}_1), z_1(\hat{s}_1)) \in \bar{\mathcal{R}}$ as well.

By Lemma 3, we know that $z_1(\hat{s}_1) < z_2(s_2)$. Since $z_1(\hat{s}_1) \ge 0$, this means $z_2(s_2) > 0$, and $x_2(s_2) = a$. But then $\xi = \xi_2 = \alpha_2(T) = a$. In particular, $x_2(s_4) = a$ and $\psi_2(s_4) = \theta$. This means $x_1(s_3) = a = x_1(\hat{s}_1)$ as well, but this contradicts Lemma 5 which says $x_1(s_3) < x_1(\hat{s}_1)$. □

Let $\mu_2^* < m_2^*$ be defined by

$$(x_2(\mu_2^*), z_2(\mu_2^*)) = (\alpha_2(T), \beta_2(T)).$$

Corollary 7. *If $P = (\xi_1, B_1(\xi_1))$ and $Q = (\xi_2, B_2(\xi_2))$ correspond as in Theorem 2, and $x_2(\ell_2) \le \xi_2 \le x_2(\mu_2^*)$, then*

$$z_1(s_1) < z_2(s_2) \quad \text{and} \quad \psi_1(s_1) < \psi_2(s_2)$$

as asserted in Theorem 2.

Proof. Comparing Figures 4 and 5, we see the identifications $Q = P_2$ and $q = (\xi_2, B_2^+(\xi_2)) = P_4$, and $P_3 = p = (x_1(s_3), z_1(s_3))$ from the proof of Theorem 2 are consistent with the secondary comparison of Lemma 5. Thus, the point $P = P_1$ is directly below $p = P_3$ on the lower graph $\mathcal{L}_1$ while $\hat{P}_1 = (x_1(\hat{s}_1), z_1(\hat{s}_1)) = (\alpha_1(\psi_2(s_2)), \beta_1(\psi_2(s_2)))$ is the point described in the secondary comparison. By Lemma 5,

$$\xi_1 = x_1(s_1) = x_1(s_3) < x_1(\hat{s}_1) = \alpha_1(\psi_2(s_2)).$$

Since x_1 and ψ_1 are both increasing for $\ell_1 \leq s \leq m_1^*$ and $\ell_1 < s_1, \hat{s}_1 \leq m_1^*$, this means $s_1 < \hat{s}_1$ and

$$\psi_1(s_1) < \psi_1(\hat{s}_1) = \psi_2(s_2). \tag{3.5}$$

Thus, we have the angle comparison of (2.7) at least for $x_2(\ell_2) < \xi \leq x_2(\mu_2^*)$.

In view of (3.5), we can apply part 3 of the direct differentiation lemma (Lemma 4) to conclude

$$z_1(s_1) < z_2(s_2)$$

as long as $-\pi/2 < \psi_1(s_1) \leq 0$.

If there are further points for which $0 < \psi_1(s_1) < \psi_2(s_2)$, then since $s_1 < \hat{s}_1$ we have

$$z_1(s_1) < z_1(\hat{s}_1) < z_2(s_2)$$

by Lemma 3. This completes the proof of the corollary. □

We next consider the possibilities when $\xi = \xi_2 > x_2(\mu_2^*)$. We know $(x_1(m_1^*), z_1(m_1^*)) \in \partial\mathcal{R}$, and we consider various cases.

case 1. $x_1(m_1^*) = a$ and $z_1(m_1^*) \geq 0$.

In this case, $m_1^* = \sigma_1$. Let us assume that the closed set

$$F = \{s_2 \in (\mu_2^*, m_2^*] : z_1(s_1) \geq z_2(s_2) \text{ or } \psi_1(s_1) \geq \psi_2(s_2)\}$$

is nonempty. Then there is a first arclength $s = s_2^F \in F$ and a corresponding s_1^F where either $z_1(s_1^F) = z_2(s_2^F)$, $\psi_1(s_1^F) = \psi_2(s_2^F)$, or both.

case 1a. $x_1(m_1^*) = x_1(\sigma_1) = a$, $z_1(m_1^*) = z_1(\sigma_1) \geq 0$, and $\psi_1(\sigma_1) \leq 0$.

In this case $\psi_1(s_1) < 0$ for $\ell_1 \leq s_1 < \sigma_1$, and part 3 of the direct differentiation lemma (Lemma 4) applies for $\ell_1 < s_1 < s_1^F$ so that

$$z_1(s_1) < z_2(s_2) \quad \text{for } x_2(\ell_2) \leq \xi \leq x_2(s_2^F). \tag{3.6}$$

This means, in particular, that $z_2(s_2) > 0$ for $\mu_2^* \leq s_2 \leq s_2^F$. Since $\dot{\psi}_2 = z_2$, it follows that

$$\psi_2(s_2) \geq \psi_2(\mu_2^*) = \psi_1(\sigma_1) > \psi_1(s_1) \quad \text{for } x_2(\ell_2) \leq \xi \leq x_2(s_2^F). \tag{3.7}$$

Together (3.6) and (3.7) contradict the existence of $s_2^F = \min F$.

case 1b. $x_1(m_1^*) = x_1(\sigma_1) = a$, $z_1(m_1^*) = z_1(\sigma_1) \geq 0$, and $\psi_1(\sigma_1) > 0$.

In this case we know $\zeta_1 > \zeta_*$ and $z_1 > 0$ globally. Since $\zeta_2 > \zeta_1$, the same assertion holds for the second/upper curve. It follows that

$$\psi_2(s_2) > \psi_1(\sigma_1) > \psi_1(s_1) \quad \text{for } x_2(\mu_2^*) \leq \xi \leq x_2(s_2^F) \leq a, \tag{3.8}$$

and

$$z_2(s_2) > z_1(\sigma_1) \quad \text{for } x_2(\mu_2^*) \leq \xi_2 \leq a.$$

Since the desired angle comparison of (2.7) is not violated, we must have

$$z_1(s_1^F) = z_2(s_2^F) > z_1(\sigma_1).$$

The only way this can happen is if $\psi_1(s_1^F) < 0$, i.e., $Q = P_2^F = (x_2(s_2^F), z_2(s_2^F))$ is to the right of the global minimum of the second/upper curve and $P_1^F = (x_1(s_1^F), z_1(s_1^F))$ is to the left of the global minimum on the first/lower curve. In summary, letting s_j^0, $j = 1, 2$ denote the arclength for which the respective minima are achieved, we have

$$s_2^0 < s_2^F \le m_2^* = \sigma_2, \quad \ell_1 < s_1^F < s_1^0, \quad \text{and} \quad z_1(s_1^F) = z_2(s_2^F).$$

It follows that for $\epsilon > 0$ small enough, the point $\tilde{Q} = \tilde{P}_2 = (x_2(s_2^F - \epsilon), z_2(s_2^F - \epsilon))$ has

$$z_2(s_2^F - \epsilon) < z_2(s_2^F),$$

while the corresponding point $\tilde{P} = (x_1(s_1^\epsilon), z_1(s_1^\epsilon))$ for some $s_1^\epsilon < s_1^F$ satisfies

$$z_1(s_1^\epsilon) > z_1(s_1^F) = z_2(s_2^F).$$

Since $\xi^\epsilon = x_2(s_2^F - \epsilon) < x_2(s_2^F)$, this contradicts the fact that $s_2^F = \min F$.

These contradictions show that $F = 0$ and the conclusion of Theorem 2 must hold in case 1.

case 2. $x_1(m_1^*) < a$ and $z_1(m_1^*) = 0$.

We begin with $z_2(\mu_2^*) > 0 = z_1(m_1^*) = z_1(m_1)$ and $\psi_1(m_1^*) = \psi_1(m_1) < \psi_2(\mu_2^*)$. It will be noted that the hypotheses of parts 3 and 4 of the direct differentiation lemma hold initially for each $\xi > x_2(\mu_1^*)$. These hypotheses continue to hold and imply (2.7) for $x_2(\mu_1^*) \le \xi \le a$. □

In view of the foregoing proof, Corollary 2 may be sharpened/extended. In order to apply the comparison of graphs in Theorem 3 on an interval extending to the left of $x = a$ it is only necessary to have an initial $\xi \in [x_2(\ell_2), a]$ for which

$$B_1(\xi) \le B_2(\xi) \quad \text{and} \quad B_1'(\xi) \le B_2'(\xi).$$

These conditions will hold for $\xi = x_2(\mu_2^*)$ the termination point on the upper curve for the inclination angle correspondence. Furthermore, $B_1(x_2(\mu_2^*)) < B_2(x_2(\mu_2^*))$ as we now explain.

In the situation of **case 1** we know $x_1(s_1) = x_3(s_3) < x_2(s_4) = x_2(s_2) < x_1(\sigma_1) = a$ and $z_1(s_1) < z_2(s_2)$ for $x_2(\ell_2) \le \xi \le x_2(\mu_2^*)$. But also, $z_1(\hat{s}_1) < z_2(s_2)$ for $x_2(\ell_2) \le \xi \le x_2(\mu_2^*)$ by Lemma 3. Letting s_1^* denote the value of s_1 when $\xi = x_2(\mu_2^*)$, this means

$$x_1(s_1^*) < x_2(\mu_2^*) < x_1(\sigma_1) = a \quad \text{and} \quad B_2(x_2(\mu_2^*)) > \max\{B_1(x_1(s_1^*)), B_1(x_1(\sigma_1))\}.$$

By convexity, $B_2(x_2(\mu_2^*)) > B_1(x_2(\mu_2^*))$. Also, since $\psi_2(\mu_2^*) = \psi_1(\sigma_1)$ in this case, we have $\psi_2(\mu_2^*) > \psi_1(x_1^{-1}(B_1(x_2(\mu_2^*))))$ where x_1^{-1} is the inverse of the restriction of x_1 to $[\ell_1, \sigma_1]$. This implies $B_2'(x_2(\mu_2^*)) > B_1'(x_2(\mu_2^*))$.

In the situation of **case 2**, we again let s_1^* be the value of s_1 when $\xi = x_2(\mu_2^*)$, and we find

$$B_2(x_2(\mu_2^*)) > 0 = z_1(m_1) \quad \text{and} \quad B_2(x_2(\mu_2^*)) > z_1(s_1^*)$$

while

$$\psi_2(\mu_2^*) = \psi_2(m_1).$$

If $x_1(s_1^*) < x_2(\mu_2^*) \le x_1(m_1)$, then the convexity argument we used in the situation of **case1** applies.

If $x_1(m_1) \le x_2(\mu_2^*) < a$, then it is immediate that $B_2(x_2(\mu_2^*)) > 0 \ge B_1(x_2(\mu_2^*))$ and $B_2'(x_2(\mu_2^*)) = B_1'(x_1(m_1)) \ge B_1'(x_2(\mu_2^*))$.

Final remark/summary: We have considered three different kinds of comparisons between pairs of elastic curves. The first (Theorem 2 part 1 and Lemma 5) is a simple comparison by inclination angle which applies for some continuous intervals $0 \le s \le m_1^*$ and $0 \le s \le \mu_2^*$ when $\psi_j(0) = \theta \in (-3\pi/2, -\pi/2)$. The second (Theorem 2) is a somewhat complicated combination of inclination angle correspondence (on the upper graphs) and vertical correspondence by shared x-coordinates. The third (Theorems 1 and 3) is simple comparison of graphs over the x-axis. The three corresponding ranges of application overlap with the first comparison extending into the lower graphs (Lemma 5), the second applying precisely to the lower graphs, and the third applying at least where the first ceases to apply.

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