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ON THE FINITE TIME BLOW-UP OF BIHARMONIC MAP FLOW IN DIMENSION FOUR

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Abstract. In this paper, we show that for certain initial values, the (extrinsic) biharmonic map flow in dimension four must blow up in finite time.

1. Introduction

Let (M,g) be a compact Riemannian manifold without boundary of dimension four and (N,h) be another compact Riemannian manifold without boundary, which is isometrically embedded in $\mathbb{R}^K$. The critical points of the following functional

$$E(u)=\int_M |\triangle u|^2\,dv$$

are called (extrinsic) biharmonic maps. We also define

$$\mathcal{E}(u)=\int_M \left|\nabla^2 u\right|^2+|\nabla u|^4\,dv$$

and notice that since the target manifold is compact, we can bound $\mathcal{E}(u)$ by $E(u)$.

The associated heat flow of $E(u)$ was first studied by Lamm [8]. In [8], the author proved that in dimension four, the following evolution equation

$$\partial_t u=-\triangle^2 u+\triangle(B(u)(\nabla u,\nabla u))+2\nabla\langle\triangle u\nabla P(u)\rangle-\langle\triangle P(u),\triangle u\rangle \tag{1.1}$$

has a unique short time smooth solution for all smooth initial value. Here B is the second fundamental form of $N\subset\mathbb{R}^K$ and $P(u)$ is the projection from $\mathbb{R}^K$ to the tangent space T_uN. Moreover, the solution is global if the $W^{2,2}$ norm of the initial value is small. Following the famous work of Struwe on harmonic map flow [13], Gastel [7] and Wang [17] showed the existence of a global weak solution with at most finitely many singular times.

It is a natural question whether the flow develops finite time singularity. The problem is particularly interesting given that all weak biharmonic maps with bounded $W^{2,2}$ norm in dimension four are known to be smooth (see [15]). The corresponding problem for harmonic map flow was answered by Chang, Ding and Ye [3]. After that, more finite-time singularity examples were found by Topping [14], Li and Wang [9] and very recently by Chen and Li [4]. The last construction shows that the blow-up could be forced by topological reason and its proof relies on the no neck theorem for approximate harmonic maps of Qing and Tian

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[11]. In fact, it was pointed out by Qing and Tian that the no neck theorem could be used in showing finite time blow-up.

Recently, the authors proved the no neck theorem for the blow-up of a sequence of (extrinsic) biharmonic maps with bounded energy. The proof follows from an idea of [11], which was motivated by the three circle lemma of Simon [12]. In light of [4], it is very natural to extend the argument to the case of biharmonic map flow and this is the purpose of this paper. More precisely, we show

Theorem 1. *Suppose that* N' *is any closed manifold of dimension* $m > 4$ *with nontrivial* $\pi_4(N')$ *and let* $N = N'\#T^m$ *be the connected sum of* N' *with the torus of the same dimension. For any Riemannian metric* g *on* N*, we can find (infinitely many) initial map* $u_0 : S^4 \to N$ *such that the biharmonic map flow (1.1) starting from* u_0 *develops finite time singularity.*

As remarked earlier, the proof relies on the idea of [4]. However, we give a slightly different presentation. Our assumption on the topology of N enables us to be more specific in the construction. Moreover, we define and use the concept of the width of a biharmonic map u from S^4 to N. Roughly speaking, the idea of the proof is the following. By a compactness argument, we show that the width of biharmonic maps from S^4 to N is bounded by a constant depending on the energy of the map (and the geometry of N of course). However, we can construct initial map u_0 with bounded energy but in a homotopy class in which every smooth representation must have very large width. If no finite-time singularity occurs, we may choose a sequence $t_i \to \infty$ such that the bi-tension field of $u(t_i)$ goes to zero in L^2 norm. Hence, $u(t_i)$ is a sequence of approximate biharmonic maps. $u(t_i)$ either converges to a smooth biharmonic map in the same homotopy class, which is not possible because the energy of the limit is smaller than that of u_0, or blows up. In the latter case, the total number of bubbles and energy of each bubble, as well as the weak limit is bounded and the no neck theorem (Theorem 2) implies a contradiction as well.

The rest of the paper is organized as follows. In Section 2, we generalize the no neck result in [10] to the case of approximate biharmonic maps. The generalization is in two directions. The first is to involve a non-zero bi-tension field and the second is to show the neck analysis works on round sphere instead of flat domains in $\mathbb{R}^4$.

Remark 1. For many PDE theorems, especially about regularity of geometric PDE, the curvature of the domain is not essential. Hence, it suffices to prove the theorem in the case of domains of Euclidean space. In this paper, we think it may not be very obvious that the neck analysis of biharmonic maps works on curved space. Hence, we present a detailed proof in the case of round metric on S^4, which is needed by the proof of Theorem 1.

In spite of the complexity caused by the round metric, we still believe that the neck analysis works in general. However, that would require greater efforts. We also note that this is not an issue for the neck analysis of harmonic maps, because of the conformal invariance.

In Section 3, the width of a map from S^4 to N is defined and the width of biharmonic maps from both S^4 and $\mathbb{R}^4$ are bounded by the energy. Finally, Theorem 1 is proved in Section 4.

Remark 2. Recently, we notice that Breiner and Lamm [2] proved a no neck theorem for a sequence of biharmonic maps with bi-tension fields in $L\log L$ when the target manifold is a sphere. In this paper, by approximate biharmonic maps, we mean bi-tension field is bounded in L^2.

We also want to mention a paper of Cooper [5]. The author constructed blow-up solution from B^4 to S^4 by using symmetry.

2. No neck for approximate biharmonic maps

In this section, we show that the main result of [10] can be generalized to a sequence of approximate biharmonic maps u_i defined on S^4.

We use a subscript g to denote operators defined on S^4 with round metric, such as $\triangle_g$ and ∇_g. $\triangle$ and ∇ are reserved for the Laplace and gradient with respect to the flat metric given by normal coordinates around some point in S^4. We always take the normal coordinates x so that the scaling $u(\lambda x)$ is well defined for small λ. Moreover, due to the Gauss Lemma, the geodesic ball B_r is the same as the ball of radius r with respect to the flat metric given by the normal coordinates. Finally, there is no need to distinguish the L^p norms defined with round metric g and the flat metric in normal coordinates for our purpose.

We will prove

Theorem 2. *Let u_i be a sequence of approximate biharmonic maps from B^4 to N satisfying*

$$\triangle_g^2 u = \triangle_g(B(u)(\nabla_g u, \nabla_g u)) + 2\nabla_g \cdot \langle \triangle_g u, \nabla(P(u))\rangle - \langle \triangle_g(P(u)), \triangle_g u\rangle + \tau(u). \tag{2.1}$$

with

$$\int_{B_1} \left|\nabla_g^2 u_i\right|^2 + \left|\nabla_g u_i\right|^4 dv_g < \Lambda \quad and \quad \|\tau(u_i)\|_{L^p(B_1)} < \Lambda \tag{2.2}$$

for some $\Lambda > 0$ and $p \geq \frac{4}{3}$. Assume that there is a positive sequence $\lambda_i \to 0$ such that

$$u_i(\lambda_i x) \to \omega$$

in $W^{4,p} \subset W^{2,2} \cap C^0$ on any compact set $K \subset \mathbb{R}^4$, that u_i converges weakly in $W^{2,2}$ to u_∞ and that ω is the only bubble. Then,

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{i\to\infty} osc_{B_\delta(0)\setminus B_{\lambda_i R}(0)} u_i = 0. \tag{2.3}$$

Remark 3. In Theorem 2, we assume that there is only one bubble. The same result holds in the case of multiple bubbles. The proof is routine argument by now and hence is omitted.

The proof is similar to the proof of Theorem 1.1 in [10], which we outline below. We first recall some definitions and results, which are modified only slightly.

2.1. Minor modifications.

The following theorem is a modified version of $\varepsilon-$regularity, proved in the Appendix of [10].

Theorem 3 (ε_0-regularity). *Let $u \in W^{4,p}(B_1)(p > 1)$ be an approximate biharmonic map. There exists $\varepsilon_0 > 0$ such that if $\int_{B_1} |\nabla^2 u|^2 + |\nabla u|^4 dx \leq \epsilon_0$ then*

$$\|u - \overline{u}\|_{W^{4,p}(B_{1/2})} \leq C(\|\nabla^2 u\|_{L^2(B_1)} + \|\nabla u\|_{L^4(B_1)} + \|\tau(u)\|_{L^p(B_1)}),$$

where $\overline{u}$ is the mean value of u over B_1.

Remark 4. We may very well use ∇_g in the above lemma. It is the type of result that Riemannian metric does not make any difference.

Next, we modify the definition of $\eta-$approximate biharmonic map as follows.

Definition 1. Let u be a smooth function defined on $B_{r_2}\backslash B_{r_1}$, u is called an $\eta-$approximate biharmonic function if it satisfies

$$\begin{aligned}\triangle_g^2 u(r,\theta) &= a_1\nabla_g\triangle_g u + a_2\nabla_g^2 u + a_3\nabla_g u + a_4 u \\ &+\frac{1}{|\partial B_r|}\int_{\partial B_r} b_1\nabla_g\triangle_g u + b_2\nabla_g^2 u + b_3\nabla_g u + b_4 u d\sigma + h(x).\end{aligned} \tag{2.4}$$

where a_i, b_i and h are smooth functions satisfying, for any $\rho \in [r_1, r_2/2]$,

(a) $\|g_{ij}(\rho x) - \delta_{ij}\|_{C^4(B_2\backslash B_1)} < \eta$. Namely, the metric after scaling to $B_2 \setminus B_1$ is close to the flat metric in C^4 norm.

(b)

$$\||x|^{4(1-1/p)}h\|_{L^p(B_{r_2}\backslash B_{r_1})} \le \eta \tag{2.5}$$

and

(c)

$$\sum_{i=1}^4 \|a_i\|_{\tilde{W}^{4-i,p}(B_{2\rho}\backslash B_\rho)} + \|b_i\|_{\tilde{W}^{4-i,p}(B_{2\rho}\backslash B_\rho)} \le \eta.$$

Here $\tilde{W}^{4-i,p}$ norm is defined as

$$\|w\|_{\tilde{W}^{4-i,p}(B_{2\rho}\backslash B_\rho)} = \left\|\rho^i w(\rho x)\right\|_{W^{4-i,p}(B_2\backslash B_1)}.$$

Remark 5. The condition (c) implies that if we compute the equation satisfied by $\tilde{u}(x) = u(\rho x)$ (defined on $B_2 \setminus B_1$), then the coefficients appearing in the place of a_i (or b_i) become $\rho^i a_i(\rho x)$ (or $\rho^i b_i(\rho x)$), which is assumed to be small in $W^{4-i,p}$ norm.

Moreover, one can check that if u is an $\eta-$approximate biharmonic function on $B_{r_2} \setminus B_{r_1}$, then $w(x) = u(\frac{x}{\lambda})$ is also an $\eta-$approximate biharmonic function on $B_{\lambda r_2} \setminus B_{\lambda r_1}$.

The following is a version of interior L^p estimate for approximate biharmonic function. It is used in the proof of three circle lemma.

Lemma 1. *There is some* $\eta_0 > 0$ *such that for any* $\eta < \eta_0$ *if* $u : B_4 \setminus B_1 \to \mathbb{R}$ *is a* $\eta-$*approximate biharmonic function with*

$$\sum_{i=1}^4 \|a_i\|_{W^{4-i,p}(B_4\backslash B_1)} + \|b_i\|_{W^{4-i,p}(B_4\backslash B_1)} \le \eta \quad \text{and} \quad \|h\|_{L^p(B_4\backslash B_1)} \le C.$$

Then, for any $4 > p > 1$, *we have*

$$\|u\|_{W^{4,p}(B_3\backslash B_2)} \le C(\|u\|_{L^p(B_4\backslash B_1)} + \|h\|_{L^p(B_4\backslash B_1)}).$$

Proof. Without loss of generality, we assume the metric g is the standard Euclidean metric. The main idea is similar to the lemma 3.3 in [10], but the assumptions on a_i and b_i are different from [10]. Next, we sketch the proof here.

For $0 < \sigma < 1$, set $A_\sigma = B_{3+\sigma} \setminus B_{2-\sigma}$ and $A'_\sigma = B_{3+\frac{1+\sigma}{2}} \setminus B_{2-\frac{1+\sigma}{2}}$. Let φ be a cut-off function supported in A'_σ satisfying: (1) $\varphi \equiv 1$ in A_σ; (2) $\left|\nabla^j\varphi\right| \le \frac{c}{(1-\sigma)^j}$ for $j = 1, 2, 3, 4$ and some universal constant c; (3) φ is a function of $|x|$.

Computing directly, we have

$$\begin{aligned}\triangle^2(\varphi u) &= \triangle(\varphi\triangle u + 2\nabla\varphi\nabla u + u\triangle\varphi)\\ &= \varphi\triangle^2 u + 4\nabla\triangle u\nabla\varphi + 4\nabla^2 u\nabla^2\varphi + 2\triangle u\triangle\varphi + 4\nabla\triangle\varphi\nabla u + \triangle^2\varphi u\\ &= \varphi a_1\nabla\triangle u + \varphi a_2\nabla^2 u + \varphi a_3\nabla u + \varphi a_4 u + \varphi h\\ &\quad +\varphi\frac{1}{|\partial B_r|}\int_{\partial B_r} b_1\nabla\triangle u + b_2\nabla^2 u + b_3\nabla u + b_4 u d\sigma\\ &\quad +4\nabla\triangle u\nabla\varphi + 4\nabla^2 u\nabla^2\varphi + 2\triangle u\triangle\varphi + 4\nabla\triangle\varphi\nabla u + \triangle^2\varphi u.\end{aligned}$$

Next, we estimate the $L^p (p > 1)$ norm of the right hand side of the above equation. By our choice of φ and the assumption of a_1, we have

$$\|\nabla\triangle u\nabla\varphi\|_{L^p(A'_\sigma)} \leq \frac{C}{1-\sigma}\left\|\nabla^3 u\right\|_{L^p(A'_\sigma)}$$

and

$$\begin{aligned}&\|\varphi a_1\nabla\triangle u\|_{L^p(A'_\sigma)}\\ \leq\ &\|a_1\nabla\triangle(\varphi u)\|_{L^p(A'_\sigma)} + \left\|a_1\nabla^2 u\nabla\varphi\right\|_{L^p(A'_\sigma)} + \left\|a_1\nabla u\nabla^2\varphi\right\|_{L^p(A'_\sigma)} + \left\|a_1 u\nabla^3\varphi\right\|_{L^p(A'_\sigma)}\\ \leq\ &C\|a_1\|_{W^{3,p}}\left\|\nabla^3(\varphi u)\right\|_{W^{1,p}(A'_\sigma)} + \frac{C}{1-\sigma}\left\|a_1\nabla^2 u\right\|_{L^p(A'_\sigma)} + \frac{C}{(1-\sigma)^2}\|a_1\nabla u\|_{L^p(A'_\sigma)}\\ &+\frac{C}{(1-\sigma)^3}\|a_1 u\|_{L^p(A'_\sigma)}\\ \leq\ &C\eta\|\varphi u\|_{W^{4,p}(A'_\sigma)} + C\left(\frac{\left\|\nabla^2 u\right\|_{W^{1,p}(A'_\sigma)}}{1-\sigma} + \frac{\|\nabla u\|_{W^{1,p}(A'_\sigma)}}{(1-\sigma)^2} + \frac{\|u\|_{W^{1,p}(A'_\sigma)}}{(1-\sigma)^3}\right)\\ \leq\ &C\eta\|\varphi u\|_{W^{4,p}(A'_\sigma)} + C\left(\frac{\left\|\nabla^3 u\right\|_{L^p(A'_\sigma)}}{1-\sigma} + \frac{\left\|\nabla^2 u\right\|_{L^p(A'_\sigma)}}{(1-\sigma)^2} + \frac{\|\nabla u\|_{L^p(A'_\sigma)}}{(1-\sigma)^3} + \frac{\|u\|_{L^p(A'_\sigma)}}{(1-\sigma)^4}\right).\end{aligned}$$

Here in the third and fourth line above, we use the Sobolev multiplication theorem which says that $\|f_1 f_2\|_{L^p} \leq C\|f_1\|_{W^{4-i,p}}\|f_2\|_{W^{i,p}}$.

Moreover, Jensen's inequality implies that

$$\begin{aligned}&\int_{A'_\sigma}\frac{\varphi^p}{|\partial B_r|^p}\left(\int_{\partial B_r} b_1\nabla\triangle u\right)^p dx\\ \leq\ &\int_{A'_\sigma}\varphi^p\frac{1}{|\partial B_r|}\left(\int_{\partial B_r}|b_1\nabla\triangle u|^p\right)dx\\ \leq\ &C\int_{A'_\sigma}\varphi^p\left|b_1\nabla^3 u\right|^p dx.\end{aligned}$$

Now, the same estimate used for $\|\varphi a_1\nabla\triangle u\|_{L^p(A'_\sigma)}$ can be used again to get the same upper bound.

Similar argument applies to the remaining terms and gives an estimate of L^p norm of $\triangle^2(\varphi u)$, if we choose η sufficiently small, by which the L^p estimate of bi-Laplace operator implies

$$\|\varphi u\|_{W^{4,p}(A'_\sigma)} \leq C\left(\frac{\left\|\nabla^3 u\right\|_{L^p(A'_\sigma)}}{1-\sigma} + \frac{\left\|\nabla^2 u\right\|_{L^p(A'_\sigma)}}{(1-\sigma)^2} + \frac{\|\nabla u\|_{L^p(A'_\sigma)}}{(1-\sigma)^3} + \frac{\|u\|_{L^p(A'_\sigma)}}{(1-\sigma)^4} + \|h\|_{L^p}\right).$$

In particular, we have

$$\begin{aligned}(1-\sigma)^4\left\|\nabla^4 u\right\|_{L^p(A_\sigma)} \quad\le\quad & C\Big((1-\sigma)^3\big\|\nabla^3 u\big\|_{L^p(A'_\sigma)}+(1-\sigma)^2\big\|\nabla^2 u\big\|_{L^p(A'_\sigma)}\\ & +(1-\sigma)\|\nabla u\|_{L^p(A'_\sigma)}+\|u\|_{L^p(A'_\sigma)}+\|h\|_{L^p}\Big).\end{aligned}$$

By setting

$$\Psi_j=\sup_{0\le\sigma\le 1}(1-\sigma)^j\left\|\nabla^j u\right\|_{L^p(A_\sigma)}$$

and noting that

$$A'_\sigma=A_{\frac{1+\sigma}{2}}\quad\text{and}\quad 1-\sigma=2(1-\frac{1+\sigma}{2}),$$

we obtain

$$\Psi_4\le C(\Psi_3+\Psi_2+\Psi_1+\Psi_0+\|h\|_{L^p}). \tag{2.6}$$

We claim that for $j=1,2,3$, the following interpolation inequality holds for any $\epsilon>0$,

$$\Psi_j\le\epsilon^{4-j}\Psi_4+\frac{C}{\epsilon^j}\Psi_0.$$

In fact, by the definition of Ψ_j, for any $\gamma>0$, there is $\sigma_\gamma\in[0,1]$ such that

$$\begin{aligned}\Psi_j &\le (1-\sigma_\gamma)^j\left\|\nabla^j u\right\|_{L^p(A_{\sigma_\gamma})}+\gamma\\ &\le \epsilon^{4-j}(1-\sigma_\gamma)^4\left\|\nabla^4 u\right\|_{L^p(A_{\sigma_\gamma})}+\frac{C}{\epsilon^j}\|u\|_{L^p(A_{\sigma_\gamma})}+\gamma\\ &\le \epsilon^{4-j}\Psi_4+\frac{C}{\epsilon^j}\Psi_0+\gamma.\end{aligned}$$

Here we used the interpolation inequality

$$\left\|\nabla^j u\right\|_{L^p(A_{\sigma_\gamma})}\le\eta^{4-j}\left\|\nabla^4 u\right\|_{L^p(A_{\sigma_\gamma})}+\frac{C_3}{\eta^j}\|u\|_{L^p(A_{\sigma_\gamma})} \tag{2.7}$$

with $\eta=\epsilon(1-\sigma_\gamma)$. We remark that the constant in the above interpolation inequality are independent of $\sigma\in[0,1]$ (see the proof of Lemma 5.6 in [1]).

By sending γ to 0 and choosing small ϵ, we obtain from (2.6)

$$\Psi_4\le C\left(\Psi_0+\|h\|_{L^p}\right),$$

from which our lemma follows. □

For the universal constant $L>0$ given in Section 3 of [10], set

$$A_i=B_{e^{-(i-1)L}}\setminus B_{e^{-iL}}$$

and

$$F_i(u)=\int_{A_i}\frac{1}{|x|^4}u^2dx.$$

Remark 6. Here is a technical issue. We use dx instead of dv_g in the definition of $F_i(u)$. The advantage is that $F_i(u)$ is invariant under the scaling $x\to\lambda x$. Since g is close to Euclidean metric, this difference does not matter when we use $F_i(u)$ as a control of L^2 norm.

Theorem 4. *There is some constant $\eta_0 > 0$ such that the following is true. Assume that $u : A_{i-1} \cup A_i \cup A_{i+1} \to \mathbb{R}^K$ is an η_0-approximate biharmonic function in the sense of (2.4). Suppose*

$$\max_{i-1,i,i+1} \||x|^{4(1-1/p)} h\|^2_{L^p(A_i)} \le \eta_0 F_i(u) \tag{2.8}$$

and

$$\int_{\partial B_r} u d\theta = 0 \tag{2.9}$$

for $r \in [e^{-l_1 L}, e^{-(l_2-1)L}]$. Then

(a) if $F_{i+1}(u) \le e^{-L} F_i(u)$, then $F_i(u) \le e^{-L} F_{i-1}(u)$;
(b) if $F_{i-1}(u) \le e^{-L} F_i(u)$, then $F_i(u) \le e^{-L} F_{i+1}(u)$;
(c) either $F_i(u) \le e^{-L} F_{i-1}(u)$, or $F_i(u) \le e^{-L} F_{i+1}(u)$.

Proof. (The proof is almost the same as Theorem 3.4 in [10]. For reader's convenience, we repeat it below.)

The exact value of i does not matter, because F_i is invariant under scaling. Hence, we consider only the case of $i = 2$. Assume the theorem is not true. We have a sequence of $\eta_k \to 0$ and a sequence of u_k defined on $A_1 \cup A_2 \cup A_3$ (and a sequence of g_k defined on $A_1 \cup A_2 \cup A_3$ as required in (a) of Definition 1) satisfying

$$\begin{aligned} \triangle^2_{g_k} u_k(r,\theta) &= a_{k1} \nabla_{g_k} \triangle_{g_k} u_k + a_{k2} \nabla^2_{g_k} u_k + a_{k3} \nabla_{g_k} u_k + a_{k4} u_k \\ &+ \frac{1}{|\partial B_r|} \int_{\partial B_r} b_{k1} \nabla_{g_k} \triangle_{g_k} u_k + b_{k2} \nabla^2_{g_k} u_k + b_{k3} \nabla_{g_k} u_k + b_{k4} u_k + h_k(x) \end{aligned} \tag{2.10}$$

with

$$\max_{1,2,3} \||x|^{4(1-1/p)} h_k\|^2_{L^p(A_i)} \le \eta_k F_2(u_k) \tag{2.11}$$

and

$$\sum_{i=1}^4 \|a_{ki}\|_{\tilde{W}^{4-i,p}(B_{2\rho} \setminus B_\rho)} + \|b_{ki}\|_{\tilde{W}^{4-i,p}(B_{2\rho} \setminus B_\rho)} \le \eta_k, \tag{2.12}$$

for any $B_{2\rho} \setminus B_\rho \subset A_1 \cup A_2 \cup A_3$.

By taking subsequence, we assume that one of (a), (b) and (c) is not true for u_k. If (a) is not true, then we have

$$F_2(u_k) \ge e^L F_3(u_k) \quad \text{and} \quad F_2(u_k) > e^{-L} F_1(u_k).$$

If (b) is not true, then

$$F_2(u_k) \ge e^L F_1(u_k) \quad \text{and} \quad F_2(u_k) > e^{-L} F_3(u_k).$$

If (c) is not true, then

$$F_2(u_k) > e^{-L} \max\{F_1(u_k), F_3(u_k)\}.$$

In any case, we control $F_1(u_k)$ and $F_3(u_k)$ by $F_2(u_k)$. Multiplying by a constant to u_k if necessary, we assume that $F_2(u_k) = 1$ for all k. The above discussion shows that

$$\|u_k\|_{L^2(A_1 \cup A_2 \cup A_3)} \le C.$$

Lemma 1 shows that (by passing to a subsequence) we have

$$\begin{aligned} u_k &\rightharpoonup u \qquad \textit{weakly} \quad \textit{in} \quad L^2(A_1\cup A_2\cup A_3),\\ u_k &\to u \qquad \textit{strongly} \quad \textit{in} \quad L^2(A_2). \end{aligned}$$

By (2.10), (2.11) and (2.12), we know that u is a nonzero biharmonic function with respect to the flat metric defined on $A_1\cup A_2\cup A_3$ satisfying (2.9), because g_k converges strongly in C^3 norm to the flat metric. The three circle lemma for biharmonic function (Theorem 3.1 in [10]) implies that

$$2F_2(u) < e^{-L}(F_1(u)+F_3(u)). \tag{2.13}$$

If (c) does not hold for u_k, we have

$$2F_2(u_k) \geq e^{-L}(F_1(u_k)+F_3(u_k)).$$

By the strong convergence of u_k in $L^2(A_2)$ and weak convergence in $L^2(A_1\cup A_2\cup A_3)$, we have

$$2F_2(u) \geq e^{-L}(F_1(u)+F_3(u)),$$

which is a contradiction to (2.13). Similar argument works for other cases. □

2.2. **Estimate of the tangential energy.** Let u_i be the sequence in Theorem 2. Assume that

$$\Sigma = B_\delta\setminus B_{\lambda_i R} = \bigcup_{l=l_0}^{l_i} A_l$$

and for any $\varepsilon>0$, by choosing δ small and R large, we may also assume (by an induction argument of Ding and Tian [6])

$$\int_{A_l} \left|\nabla^2 u_i\right|^2 + \left|\nabla u_i\right|^4 dx < \varepsilon^4 < \varepsilon_0 \tag{2.14}$$

for $l=l_0,\cdots,l_i$. Set $\widetilde{u_i}(x)=u_i(e^{-lL}x)$, by ε_0-regularity Theorem 3, we have

$$\|\widetilde{u_i}-\overline{\widetilde{u_i}}\|_{W^{4,p}(A_0)} \leq C(\|\nabla^2\widetilde{u_i}\|_{L^2(A_{-1}\cup A_0\cup A_1)}+\|\nabla\widetilde{u_i}\|_{L^4(A_{-1}\cup A_0\cup A_1)}+\|\tau(\widetilde{u_i})\|_{L^p(A_{-1}\cup A_0\cup A_1)}),$$

where $\overline{\widetilde{u_i}}$ is the mean value of $\widetilde{u_i}$ over A_0.

Scaling back, if δ is sufficiently small, we will get

$$\begin{aligned} &\sum_{k=0}^{4}\||x|^{k-4/p}\nabla^k(u_i-\overline{\widetilde{u_i}})\|_{L^p(A_l)}\\ \leq\ & C(\|\nabla^2 u_i\|_{L^2(A_{l-1}\cup A_l\cup A_{l+1})}+\|\nabla u_i\|_{L^4(A_{l-1}\cup A_l\cup A_{l+1})}+e^{-lL4(1-1/p)}\|\tau(u_i)\|_{L^p(A_{l-1}\cup A_l\cup A_{l+1})})\\ \leq\ & C\varepsilon. \end{aligned}$$

Let $r=e^t$, then as a function of (t,θ), we have

$$\|u_i-\overline{\widetilde{u_i}}\|_{W^{4,p}(-lL,-(l-1)L)\times S^3} \leq C\varepsilon, \tag{2.15}$$

for any $l_0\leq l\leq l_i$.

The theorem is equivalent to the statement that for any $\varepsilon>0$, we can find δ small and R large such that

$$\operatorname{osc}_{B_\delta\setminus B_{\lambda_i R}} u_i < C\varepsilon$$

for i sufficiently large.

Set

$$u_i^*(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u_i(r,\theta) d\sigma.$$

The Poincaré inequality and (2.14) imply

$$\int_{A_l} \frac{1}{|x|^4} |u_i - u_i^*|^2 \, dx \le C\varepsilon^2.$$

Lemma 2. *There exists some $\varepsilon_1 > 0$ such that if $\varepsilon < \varepsilon_1$ in (2.14) and $\delta < \varepsilon_1$, $w_i = u_i - u_i^*$ is an η_0-approximate biharmonic function defined on $B_\delta \setminus B_{\lambda_i R}$ in the sense of (2.4), where η_0 is the constant in Theorem 4.*

Remark 7. Although the proof is parallel to Lemma 4.1 in [10]. We reproduce it because (1) we now use the sphere metric instead of the flat one; (2) the definition of $\eta-$approximate biharmonic function is different.

Proof. For simplicity, we omit the subscript i. Recall that u satisfies

$$\begin{aligned}\triangle_g^2 u &= \alpha_1(u)\nabla_g \triangle_g u \# \nabla_g u + \alpha_2(u)\nabla_g^2 u \# \nabla_g^2 u \\ &+\alpha_3(u)\nabla_g^2 u \# \nabla_g u \# \nabla_g u + \alpha_4(u)\nabla_g u \# \nabla_g u \# \nabla_g u \# \nabla_g u + \tau(u).\end{aligned} \tag{2.16}$$

Here $\alpha_i(u)$ is a smooth function of u and $\#$ is the contraction of tensors with respect to g, for which we have for example,

$$|\nabla_g \triangle_g u \# \nabla_g u| \le C |\nabla_g \triangle_g u| |\nabla_g u| .$$

Since $\triangle_g = \frac{\partial^2}{\partial r^2} + \frac{3\cos r}{\sin r}\frac{\partial}{\partial r} + \frac{1}{\sin^2 r}\triangle_{S^3}$ and $\int_{S^3} \triangle_{S^3} f d\theta = 0$ for any f, we have

$$\begin{aligned}\triangle_g^2 u^*(r) &= \frac{1}{|\partial B_r|}\int_{\partial B_r} \triangle_g^2 u d\sigma \\ &= \frac{1}{|\partial B_r|}\int_{\partial B_r} \alpha_1(u)\nabla_g \triangle_g u \# \nabla_g u + \alpha_2(u)\nabla_g^2 u \# \nabla_g^2 u \\ &\quad +\alpha_3(u)\nabla_g^2 u \# \nabla_g u \# \nabla_g u + \alpha_4(u)\nabla_g u \# \nabla_g u \# \nabla_g u \# \nabla_g u d\sigma \\ &\quad +\frac{1}{|\partial B_r|}\int_{\partial B_r} \tau(u) d\sigma \\ &= I + II + III + IV + \frac{1}{|\partial B_r|}\int_{\partial B_r} \tau(u) d\sigma.\end{aligned}$$

Remark 8. Here we make essential use of the symmetry of spherical metric to simplify the computation in the first line above. This is partially the reason that we work on round S^4.

Computing directly, we get

$$\begin{aligned} I &= \frac{1}{|\partial B_r|}\int_{\partial B_r} \alpha_1(u)\nabla_g \triangle_g u \# \nabla_g u - \alpha_1(u^*)\nabla_g \triangle_g u \# \nabla_g u \\ &\quad +\alpha_1(u^*)\nabla_g \triangle_g u \# \nabla_g u - \alpha_1(u^*)\nabla_g \triangle_g u^* \# \nabla_g u \\ &\quad +\alpha_1(u^*)\nabla_g \triangle_g u^* \# \nabla_g u - \alpha_1(u^*)\nabla_g \triangle_g u^* \# \nabla_g u^* d\sigma \\ &\quad +\alpha_1(u^*)\nabla_g \triangle_g u^* \# \nabla_g u^* \\ &= \frac{1}{|\partial B_r|}\int_{\partial B_r} \beta_4[u](u-u^*) + \beta_1[u]\nabla_g \triangle_g (u-u^*) \\ &\quad +\beta_3[u]\nabla_g (u-u^*) d\sigma + \alpha_1(u^*)\nabla_g \triangle_g u^* \# \nabla_g u^*.\end{aligned}$$

Here $\beta_i[u]$ is some expression depending on u, u^* and their derivatives. Precisely, we have

$$\begin{aligned}\beta_4[u] &= \alpha_1'(*)\nabla_g\triangle_g u\#\nabla_g u\\ \beta_1[u] &= \alpha_1(u^*)\nabla_g u\\ \beta_3[u] &= \alpha_1(u^*)\nabla_g\triangle_g u^*.\end{aligned}$$

In what follows, those β_i's may differ from line to line in the following. However, thanks to Theorem 3, we have

$$\|\beta_i\|_{\tilde{W}^{4-i,p}(B_{2\rho}\setminus B_\rho)} \leq \eta_0 \quad \text{for} \quad \rho\in[\lambda_i R, \delta/2],$$

if ε in (2.14) is smaller than some ε_1. We shall require the above holds for all β_i and β_i' below by asking ε_1 to be smaller and smaller.

The same computation gives

$$\begin{aligned}II &= \frac{1}{|\partial B_r|}\int_{B_r}\beta_4[u](u-u^*)+\beta_2[u]\nabla_g^2(u-u^*)d\sigma+\alpha_2(u^*)\nabla_g^2u^*\#\nabla_g^2u^*,\\ III &= \frac{1}{|\partial B_r|}\int_{\partial B_r}\beta_4[u](u-u^*)+\beta_2[u]\nabla_g^2(u-u^*)+\beta_3[u]\nabla_g(u-u^*)d\sigma\\ &\quad +\alpha_3(u^*)\nabla_g^2u^*\#\nabla_g u^*\#\nabla_g u^*\end{aligned}$$

and

$$IV = \frac{1}{|\partial B_r|}\int_{\partial B_r}\beta_4[u](u-u^*)+\beta_3[u]\nabla_g(u-u^*)d\sigma+\alpha_4(u^*)\nabla_g u^*\#\nabla_g u^*\#\nabla_g u^*\#\nabla_g u^*.$$

In summary, u^* satisfies an equation similar to (2.16) except an error term of the form

$$\frac{1}{|\partial B_r|}\int_{\partial B_r}\beta_1[u]\nabla_g\triangle_g w+\beta_2[u]\nabla_g^2 w+\beta_3[u]\nabla_g w+\beta_4[u]w d\sigma.$$

Subtract the equation of u^* with (2.16) and handle the terms like $\alpha_1(u)\nabla_g\triangle_g u\#\nabla_g u - \alpha_1(u^*)\nabla_g\triangle_g u^*\#\nabla_g u^*$ as before to get

$$\begin{aligned}\triangle_g^2 w &= \beta_1'[u]\nabla_g\triangle_g w+\beta_2'[u]\nabla_g^2 w+\beta_3'[u]\nabla_g w+\beta_4'[u]w\\ &\quad +\frac{1}{|\partial B_r|}\int_{\partial B_r}\beta_1[u]\nabla_g\triangle_g w+\beta_2[u]\nabla_g^2 w+\beta_3[u]\nabla_g w+\beta_4[u]w d\sigma+h,\end{aligned}$$

where

$$h=\tau(u)-\frac{1}{|\partial B_r|}\int_{\partial B_r}\tau(u)d\sigma.$$

To see that h satisfies (b) of Definition 1, we notice that $4(1-\frac{1}{p})>0$ and

$$\left\||x|^{4(1-\frac{1}{p})}h\right\|_{L^p(B_\delta\setminus B_{\lambda_i R})}\leq \delta^{4(1-\frac{1}{p})}\|h\|_{L^p(B_\delta\setminus B_{\lambda_i R})}\leq C\delta^{4(1-\frac{1}{p})}\|\tau(u_i)\|_{L^p(B_1)}.$$

Since $\tau(u_i)$ is uniformly bounded in L^p, the lemma follows by choosing δ small. □

Now we apply Theorem 4 to the function w_i.

Lemma 3. *For any $0<\varepsilon<\varepsilon_1$ and sufficiently small $\delta>0$, we have*

$$F_l(w_i)\leq C\varepsilon^2\left(e^{-\min\{8(1-1/p),1\}(l-l_0)L}+e^{-\min\{8(1-1/p),1\}(l_i-l)L}\right), \tag{2.17}$$

for $l_0<l<l_i$.

Proof. Let the set of $l(l_0 < l < l_i)$, for which the condition (2.8) is not true, be denoted by $\{j_1, \cdots, j_{n_i}\}$ and we assume that

$$l_0 < j_1 < j_2 < \cdots < j_{n_i} < l_i.$$

By definition, for each $l = j_k$,

$$\max_{l-1,l,l+1} \||x|^{4(1-1/p)} h_i\|^2_{L^p(A_l)} \geq \eta_0 F_l(w_i). \tag{2.18}$$

Then we have

$$\begin{aligned} F_l(w_i) &\leq C \max_{l-1,l,l+1} \left\| |x|^{4(1-\frac{1}{p})} h_i \right\|_{L^p(A_l)} \\ &\leq Ce^{-8(1-1/p)lL} \\ &\leq C\delta^{8(1-1/p)} e^{-8(1-1/p)(l-l_0)L} \\ &\leq C\varepsilon^2 e^{-8(1-1/p)(l-l_0)L}, \end{aligned}$$

if we choose δ small.

By the choice of j_k, the condition (2.8) holds for $j_k < l < j_{k+1}$, $k = 1, ..., i-1$. By an application of Theorem 4 (see also Lemma 4.2 in [10]), we have, for $j_k < l < j_{k+1}$

$$\begin{aligned} F_l(w_i) &\leq C \left(e^{-L(l-j_k)} F_{j_k}(w_i) + e^{-L(j_{k+1}-l)} F_{j_{k+1}}(w_i) \right) \\ &\leq C\varepsilon^2 \left(e^{-\min\{8(1-1/p),1\}(l-l_0)L} \right). \end{aligned}$$

So, if $j_1 = l_0 + 1$ and $j_{n_i} = l_i - 1$, the inequality (2.17) follows immediately. If not, assuming $j_1 > l_0 + 1$, by Theorem 4 again, we have, for $l_0 < l < j_1$,

$$\begin{aligned} F_l(w_i) &\leq C \left(e^{-L(l-l_0)} F_{l_0}(w_i) + e^{-L(j_1-l)} F_{j_1}(w_i) \right) \\ &\leq C \left(e^{-L(l-l_0)} F_{l_0}(w_i) + \varepsilon^2 e^{-\min\{8(1-1/p),1\}(l-l_0)L} \right) \\ &\leq C\varepsilon^2 e^{-\min\{8(1-1/p),1\}(l-l_0)L}. \end{aligned}$$

Similarly, if $j_{n_i} < l_i - 1$, we have, for $j_{n_i} < l < l_i - 1$,

$$\begin{aligned} F_l(w_i) &\leq C \left(e^{-L(l-j_{n_i})} F_{j_{n_i}}(w_i) + e^{-L(l_i-l)} F_{l_i}(w_i) \right) \\ &\leq C \left(\varepsilon^2 e^{-\min\{8(1-1/p),1\}(l-l_0)L} + e^{-L(l_i-l)} F_{l_i}(w_i) \right) \\ &\leq C\varepsilon^2 \left(e^{-\min\{8(1-1/p),1\}(l-l_0)L} + e^{-L(l_i-l)} \right) \\ &\leq C\varepsilon^2 \left(e^{-\min\{8(1-1/p),1\}(l-l_0)L} + e^{-\min\{8(1-1/p),1\}(l_i-l)L} \right). \end{aligned}$$

□

Since w_i satisfies (2.4), we may use Lemma 1 to get estimates for the derivatives of w_i and the tangential derivatives of u_i. In the following, (r, θ) is the polar coordinates where $\theta \in S^3$ is a point of the unit sphere. A function $u(r, \theta)$ is also considered a function of $(\tilde{t}, \theta)$, where $r = e^{\tilde{t}}$. We denote the gradient operator on S^3 by ∇_{S^3} and the Laplacian on S^3 by $\triangle_{S^3}$.

Remark 9. Since we have only L^p norm of bi-tension fields bounded, we may not prove pointwise decay bound for tangential derivatives. Hence we need the following lemma as a replacement.

Lemma 4.

$$\begin{aligned}\int_{(-lL,-(l-1)L)\times S^3}&\left(|\triangle_{S^3}u_i|^2+|\partial_{\tilde{t}}\nabla_{S^3}u_i|^2\right)d\tilde{t}d\theta\\ \leq\ & C\varepsilon^2\left(e^{-\min\{8(1-1/p),1\}(l-l_0)L}+e^{-\min\{8(1-1/p),1\}(l_i-l)L}\right).\end{aligned}\tag{2.19}$$

Or equivalently,

$$\begin{aligned}\int_{[\tilde{t},\tilde{t}+1]\times S^3}&\left(|\triangle_{S^3}u_i|^2+|\partial_{\tilde{t}}\nabla_{S^3}u_i|^2\right)d\tilde{t}d\theta\\ \leq\ & C\varepsilon^2\left(e^{-\min\{8(1-1/p),1\}(\log\delta-\tilde{t})}+e^{-\min\{8(1-1/p),1\}(\tilde{t}-\log\lambda_i R)}\right).\end{aligned}$$

Proof. Setting

$$\tilde{w}(x)=w_i(e^{-(l-1)L}x),$$

we have

$$\begin{aligned}\|\tilde{w}\|^2_{L^2(A_0\cup A_1\cup A_2)}\ &\leq\ C(F_{l-1}(w_i)+F_l(w_i)+F_{l+1}(w_i))\\ &\leq\ C\varepsilon^2\left(e^{-\min\{8(1-1/p),1\}(l-l_0)L}+e^{-\min\{8(1-1/p),1\}(l_i-l)L}\right).\end{aligned}$$

By scaling, $\tilde{w}$ satisfies

$$\begin{aligned}\triangle_g^2\tilde{w}(r,\theta)&=\tilde{a}_1\nabla_g\triangle_g\tilde{w}+\tilde{a}_2\nabla_g^2\tilde{w}+\tilde{a}_3\nabla_g\tilde{w}+\tilde{a}_4\tilde{w}\\ &+\frac{1}{|\partial B_r|}\int_{\partial B_r}\tilde{b}_1\nabla_g\triangle_g\tilde{w}+\tilde{b}_2\nabla_g^2\tilde{w}+\tilde{b}_3\nabla_g\tilde{w}+\tilde{b}_4\tilde{w}d\theta+\tilde{h}(x).\end{aligned}\tag{2.20}$$

Here

$$\tilde{h}(x)=e^{-4(l-1)L}h(e^{-(l-1)L}x)$$

and

$$h(x)=\tau(u_i)-\frac{1}{|(\partial B_r)|}\int_{\partial B_r}\tau(u_i)d\sigma.$$

Letting $\lambda=e^{-(l-1)L}$, we have

$$\begin{aligned}&\left\|\tilde{h}\right\|_{L^p(A_0\cup A_1\cup A_2)}\\ =\ &\left(\int_{A_0\cup A_1\cup A_2}\left|\lambda^4h(\lambda x)\right|^p dx\right)^{\frac{1}{p}}\\ =\ &\lambda^{4(1-1/p)}(\int_{A_{l-1}\cup A_l\cup A_{l+1}}|h(x)|^p dx)^{1/p}\\ \leq\ &Ce^{-4(1-1/p)(l-1)L}\\ \leq\ &C\delta^{4(1-1/p)}e^{-4(1-1/p)(l-l_0)L}\\ \leq\ &C\varepsilon e^{-4(1-1/p)(l-l_0)L},\end{aligned}\tag{2.21}$$

if δ is small.

Lemma 1 and the Sobolev embedding theorem imply that

$$\begin{aligned} &\int_{(-lL,-(l-1)L)\times S^3} (|\triangle_{S^3} u_i|^2 + |\partial_{\tilde{t}} \nabla_{S^3} u_i|^2) d\tilde{t} d\theta \\ \leq\ & C \int_{A_1} (|\nabla^2 \tilde{w}|^2 + |\nabla \tilde{w}|^2) dx \\ \leq\ & C\varepsilon^2 \left(e^{-\min\{8(1-1/p),1\}(l-l_0)L} + e^{-\min\{8(1-1/p),1\}(l_i-l)L} \right). \end{aligned}$$

□

2.3. **Proof of Theorem 2.** With the preparations of previous subsections, we may now prove Theorem 2. For the rest of the proof, we require $p \geq \frac{4}{3}$ and hence $\min\left\{8(1-\frac{1}{p}), 1\right\} = 1$.

The rest of the proof is some type of Pohozaev argument. It follows the same line of Section 5 of [10]. However, the proof there made use of the explicit expression of bi-Laplace operator in polar coordinates of $\mathbb{R}^4$. Since we are now using the round metric on S^4, we think it is necessary to justify the reason why the proof still works. As it can be seen from below, this is not obvious and the proof depends on some detailed computation.

To begin with, we define a function (for $r < 1$)

$$t(r) = \int_1^r \frac{1}{\sin s} ds.$$

Obviously, $t'(r) = \frac{1}{\sin r}$. One may want to compare it with $\tilde{t}(r) = \log r$. In fact, we have

$$0 < \tilde{t}(r) - t(r) < C \quad \text{for} \quad r < 1$$

and $\tilde{t}'(r)$ is comparable with $t'(r)$. As a consequence, the result of Lemma 4 can be further rewritten as (noting that $p \geq 4/3$ here)

$$\begin{aligned} &\int_{[t,t+1]\times S^3} \left(|\triangle_{S^3} u_i|^2 + |\partial_t \nabla_{S^3} u_i|^2 \right) dt d\theta \\ \leq\ & C\varepsilon^2 \left(e^{-(t(\delta)-t)} + e^{-(t-t(\lambda_i R))} \right). \end{aligned} \tag{2.22}$$

Recall that the metric is given by $g = dr^2 + \sin^2 r d\theta^2$. (Here $d\theta^2$ is the standard metric on the unit sphere.) To simplify the notations, we write $f(r) = \sin r$ and f' is the derivative of f with respect to r. The Laplace operator is

$$\triangle_g u = \partial_r^2 + \frac{3f'}{f} \partial_r u + \frac{1}{f^2} \triangle_{S^3} u.$$

By using $\partial_t = f \partial_r$, we may compute

$$\triangle_g u = f^{-2} \left(\partial_t^2 + 2f' \partial_t + \triangle_{S^3} \right) u.$$

Writing $\triangle_g u = f^{-2}w$, we obtain

$$\begin{aligned}\triangle_g^2 u &= f^{-2}\left(\partial_t^2 + 2f'\partial_t + \triangle_{S^3}\right)(f^{-2}w)\\ &= f^{-4}\left(\partial_t^2 + 2f'\partial_t + \triangle_{S^3}\right)w\\ &\quad + f^{-2}\left(\partial_t^2(f^{-2})w + 2\partial_t(f^{-2})\partial_t w + 2f'\partial_t(f^{-2})w\right)\\ &= f^{-4}\left(\partial_t^2 - 2f'\partial_t + \triangle_{S^3}\right)w - 2\frac{f''}{f^3}w.\end{aligned}$$

By the definition of w and $f'' = -f$, we have

$$\begin{aligned}\triangle_g^2 u &= f^{-4}\left(\partial_t^2 - 2f'\partial_t + \triangle_{S^3}\right)\left(\partial_t^2 + 2f'\partial_t + \triangle_{S^3}\right)u + 2f^{-2}w\\ &= f^{-4}\left((\partial_t^2 + \triangle_{S^3})^2 - 4(f'\partial_t)(f'\partial_t)\right)u\\ &\quad + f^{-4}\left((\partial_t^2 + \triangle_{S^3})(2f'\partial_t) - (2f'\partial_t)(\partial_t^2 + \triangle_{S^3})\right)u + 2f^{-2}w.\end{aligned}$$

In comparison with the case of flat metric, f causes some extra terms. It is the primary goal here to show that we can handle these extra terms properly.

$$4(f'\partial_t)(f'\partial_t) = 4(f')^2\partial_t^2 - 4f^2 f'\partial_t,$$

where we have used $\partial_t = f'\partial_r$ and $f'' = -f$ because $f(r) = \sin r$.

Note that $\triangle_{S^3}$ commutes with $f'\partial_t$ and we compute

$$\begin{aligned}&\partial_t^2(2f'\partial_t) - (2f'\partial_t)\partial_t^2\\ =\ &\partial_t^2(2f')\partial_t + 2\partial_t(2f')\partial_t^2\\ =\ &-4f^2 f'\partial_t - 4f^2\partial_t^2.\end{aligned}$$

In summary, we have

$$\triangle_g^2 u = f^{-4}\left((\partial_t^2 + \triangle_{S^3})^2 - 4\partial_t^2\right)u + 2f^{-2}w, \tag{2.23}$$

where we have used $(f')^2 + f^2 = 1$.

Remark 10. The first term in the above formula is almost the same as the flat case. The importance of the computation is to show the error caused by the round metric is just $f^{-2}w$. Since w involves only first and second order derivatives, it can be controlled by the energy. *If* there is a third order derivative term here, then the proof below would fail.

By the definition of τ, we have

$$\int_{S^3} f^4\triangle_g^2 u\cdot\partial_t u d\theta = \int_{S^3} f^4\tau(u)\cdot\partial_t u d\theta.$$

By (2.23), the above is equivalent to

$$\int_{S^4}\left((\partial_t^2 + \triangle_{S^3})^2 - 4\partial_t^2\right)u\partial_t u d\theta = \int_{S^3}(f^4\tau(u) - 2f^2 w)\partial_t u d\theta.$$

The left hand side is now completely identical to the form which is dealt with in Section 5 of [10]. For simplicity, we set

$$\tilde\tau(u) = \tau(u) - 2f^{-2}w = \tau(u) - 2\triangle_g u.$$

Since u has finite energy, $\tilde\tau(u)$ is also uniformly bounded in L^p for $p\in[4/3,2]$.

The same computation as in [10] gives

$$\begin{aligned}(2.24)\qquad & \partial_t \int_{S^3} \partial_t u \partial_t^2 u d\theta - \int_{S^3} \frac{3}{2} \left|\partial_t^2 u\right|^2 + 2\left|\partial_t u\right|^2 d\theta \\ = & \int_{S^3} -\frac{1}{2} \left|\triangle_{S^3} u\right|^2 + \left|\partial_t \nabla_{S^3} u\right|^2 + \int_{-\infty}^{t} \int_{S^3} f^4 \tilde{\tau}(u) \cdot \partial_t u ds d\theta.\end{aligned}$$

We will integrate the above inequality from $t(\lambda_i R)$ to $t(\delta)$. We estimate the right hand side first. Thanks to (2.22), we have

$$\int_{t(\lambda_i R)}^{t(\delta)} \int_{S^3} -\frac{1}{2} \left|\triangle_{S^3} u\right|^2 + \left|\partial_t \nabla_{S^3} u\right|^2 d\theta \leq C\varepsilon^2.$$

Transforming back to $x-$coordinates by $\partial_t = f\partial_r$ and $d\sigma = f^3 d\theta$, we get

$$\begin{aligned} & \left|\int_{t(\lambda_i R)}^{t(\delta)} \int_{-\infty}^{\tilde{t}} \int_{S^3} f^4 \tilde{\tau}(u) \cdot \partial_t u d\theta ds d\tilde{t}\right| \\ \leq & \int_{t(\lambda_i R)}^{t(\delta)} \int_{B_{r(\tilde{t})}} \left|\tilde{\tau}(u)\right| \left|f\partial_r u\right| dx d\tilde{t} \\ \leq & \int_{\lambda_i R}^{\delta} \int_{B_r} \left|\tilde{\tau}(u)\right| \left|\nabla u\right| dx dr \\ \leq & C\delta \left\|\tilde{\tau}(u)\right\|_{L^{4/3}(B_1)} \left\|\nabla u\right\|_{L^4(B_1)}.\end{aligned}$$

In summary, the integration of (2.24) yields (by taking σ small with respect to ε)

$$\begin{aligned}(2.25)\qquad & \int_{t(\lambda_i R)}^{t(\delta)} \int_{S^3} \frac{3}{2} \left|\partial_t^2 u\right|^2 + 2\left|\partial_t u\right|^2 d\theta dt \\ \leq & C\left(\int_{\{t(\delta)\}\times S^3} |\partial_t u \partial_t^2 u| d\theta + \int_{\{t(\lambda_{iR})\}\times S^3} |\partial_t u \partial_t^2 u| d\theta + \varepsilon^2\right) \\ \leq & C\left\|\partial_t u\right\|_{L^2(\{t(\delta)\}\times S^3)} \left\|\partial_t^2 u\right\|_{L^2(\{t(\delta)\}\times S^3)} \\ & + C\left\|\partial_t u\right\|_{L^2(\{t(\lambda_i R)\}\times S^3)} \left\|\partial_t^2 u\right\|_{L^2(\{t(\lambda_i R)\}\times S^3)} + C\varepsilon^2 \\ \leq & C\varepsilon^2,\end{aligned}$$

where the last inequality comes from the (2.15) and Sobolev embedding and trace theorem. In fact, we have $W^{4,p}(\Omega)$ embeds into $W^{3,2}(\Omega)$, which in turn embeds into $W^{2,2}(\partial\Omega)$.

Remark 11. We remark that in fact, the argument above gives an independent proof of the energy identity in the blow up analysis of biharmonic maps with tension field in L^p for some $p \geq \frac{4}{3}$.

For some fixed $t_0 \in [t(\lambda_i R), t(\delta)]$, set

$$F(t) = \int_{t_0-t}^{t_0+t} \int_{S^3} \frac{3}{2} \left|\partial_t^2 u\right|^2 + 2\left|\partial_t u\right|^2 d\theta dt.$$

F is defined for $0 \leq t \leq \min\{t_0 - t(\lambda_i)R, t(\delta) - t_0\}$. Integrating (2.24) from $t_0 - t$ to $t_0 + t$, we obtain

$$\begin{aligned} F(t) &\leq \frac{1}{2\sqrt{3}} \left(\int_{\{t_0-t\}\times S^3} + \int_{\{t_0+t\}\times S^3} \right) \frac{3}{2} \left|\partial_t^2 u\right|^2 + 2\left|\partial_t u\right|^2 d\theta \\ &\quad + \int_{t_0-t}^{t_0+t} \left(\int_{S^3} -\frac{1}{2}\left|\triangle_{S^3} u\right|^2 + \left|\partial_t \nabla_{S^3} u\right|^2 d\theta + \int_{-\infty}^{\tilde{t}} \int_{S^3} f^4 \tilde{\tau}(u)\cdot\partial_t u ds d\theta \right) d\tilde{t}. \end{aligned}$$

With the help of (2.22), we can have

$$\int_{t_0-t}^{t_0+t} \int_{S^3} \frac{-1}{2}\left|\triangle_{S^3} u\right|^2 + \left|\partial_t \nabla_{S^3} u\right|^2 d\theta ds \leq C\varepsilon^2 \left(e^{-(t(\delta)-t_0)} + e^{-(t_0 - t(\lambda_i R))} \right) e^t.$$

On the other hand,

$$\begin{aligned} & \left| \int_{t_0-t}^{t_0+t} \int_{-\infty}^{\tilde{t}} \int_{S^3} f^4 \tilde{\tau}(u)\cdot\partial_t u ds d\theta d\tilde{t} \right| \\ \leq\ & \int_{t_0-t}^{t_0+t} \int_{B_{r(\tilde{t})}} \left|\tilde{\tau}(u)\right| \left|\nabla u\right| \left|f\right| dx d\tilde{t} \\ \leq\ & C e^{\frac{1}{2}(t_0+t)} \left\|\tilde{\tau}(u)\right\|_{L^{4/3}(B_1)} \left\|\nabla u\right\|_{L^4(B_1)} \\ \leq\ & C\delta^{1/2} e^{-1/2(\log\delta - t_0)} e^{t/2} \\ \leq\ & C\delta^{1/2} e^{-1/2(t(\delta)-t_0)} e^t \end{aligned}$$

Remark 12. Note that since $r'(t) = \sin r$ and $\frac{1}{2} r \leq \sin r \leq r$ for $r < 1$, we have

$$e^t < r(t) < e^{t/2}$$

for $t < 0$.

Hence, if δ is small, we obtain

$$F(t) \leq \frac{1}{2}\partial_t F(t) + C\varepsilon^2 \left(e^{-\frac{1}{2}(t(\delta)-t_0)} + e^{-\frac{1}{2}(t_0 - t(\lambda_i R))} \right) e^t.$$

Multiplying e^{-2t} to both sides of the inequality, we have

$$(e^{-2t}F(t))' \geq -C\varepsilon^2 \left(e^{-\frac{1}{2}(t(\delta)-t_0)} + e^{-\frac{1}{2}(t_0 - t(\lambda_i R))} \right) e^{-t}.$$

We assume without loss of generality that $t(\delta) - t_0 \leq t_0 - t(\lambda_i R)$. Then, we integrate the above inequality from $t = 1$ to $t = t(\delta) - t_0$ to get

$$\begin{aligned} F(1) &\leq e^{-2(t(\delta)-t_0)+2} F(t(\delta) - t_0) + C\varepsilon^2 \left(e^{-\frac{1}{2}(t(\delta)-t_0)} + e^{-\frac{1}{2}(t_0 - t(\lambda_i R))} \right) \\ &\leq C\varepsilon^2 \left(e^{-\frac{1}{2}(t(\delta)-t_0)} + e^{-\frac{1}{2}(t_0 - t(\lambda_i R))} \right). \end{aligned}$$

Here we used (2.25).

Together with (2.22), we obtain

$$\int_{t_0-1}^{t_0+1} \int_{S^3} |\tilde{\nabla}^2 u|^2 + |\tilde{\nabla} u|^2 d\theta dt \leq C\varepsilon^2 \left(e^{-\frac{1}{2}(t(\delta)-t_0)} + e^{-\frac{1}{2}(t_0 - t(\lambda_i R))} \right),$$

Here $\tilde{\nabla}$ is the gradient of $[t(\lambda_i R), t(\delta)] \times S^3$ with the product metric. Recall that $\left|\tilde{t}(r) - t(r)\right|$ is bounded by some universal constant and ∂_t and $\partial_{\tilde{t}}$ are comparable. Hence, we can translate the above decay estimate into a decay with respect to $\tilde{t} = \log r$.

$$\int_{\tilde{t}_0-1}^{\tilde{t}_0+1} \int_{S^3} |\tilde{\nabla}^2 u|^2 + |\tilde{\nabla} u|^2 d\theta d\tilde{t} \le C\varepsilon^2 \left(e^{-\frac{1}{2}(\log(\delta)-\tilde{t}_0)} + e^{-\frac{1}{2}(\tilde{t}_0-\log(\lambda_i R))}\right),$$

Direct computation shows that

$$\begin{aligned}\int_{B_{e^{\tilde{t}_0+1}} \setminus B_{e^{\tilde{t}_0-1}}} |\nabla^2 u|^2 + \frac{1}{|x|^2}|\nabla u|^2 dx &\le C\int_{\tilde{t}_0-1}^{\tilde{t}_0+1} \int_{S^3} |\tilde{\nabla}^2 u|^2 + |\tilde{\nabla} u|^2 d\theta d\tilde{t} \\ &\le C\varepsilon^2 \left(e^{-\frac{1}{2}(\log\delta-\tilde{t}_0)} + e^{-\frac{1}{2}(\tilde{t}_0-\log\lambda_i R)}\right).\end{aligned}$$

Then by Sobolev embedding and the ε_0-regularity (Theorem 3), we have

$$\begin{aligned}& osc_{((\tilde{t}_0-1/2,\tilde{t}_0+1/2)\times S^3)} u \\ \le\ & C\Big(\int_{B_{e^{\tilde{t}_0+1}} \setminus B_{e^{\tilde{t}_0-1}}} |\nabla^2 u|^2 + \frac{1}{|x|^2}|\nabla u|^2 dx\Big)^{1/2} + e^{4\tilde{t}_0(1-1/p)}\|\tau(u)\|_{L^p(B_{e^{\tilde{t}_0+1}} \setminus B_{e^{\tilde{t}_0-1}})} \\ \le\ & C\varepsilon \left(e^{-\frac{1}{4}(\log\delta-\tilde{t}_0)} + e^{-\frac{1}{4}(\tilde{t}_0-\log\lambda_i R)}\right).\end{aligned}$$

It is easy to derive the no neck estimate from here. Hence, we complete the proof of Theorem.

3. BOUNDING WIDTH BY ENERGY

Let N be the manifold in the Theorem 1 and g be any Riemannian metric on N. Since $N = N'\#T^m$, there is an embedded sphere S of dimension $m-1$ in N which separates N into N_1 and N_2 and N/N_2 is homeomorphic to N' and N/N_1 is homeomorphic to T^m. Here N/N_i is the quotient topology space by identifying all points in N_i as one point.

Let $\tilde{N}$ be a cover of N and $\tilde{g}$ be the lift of g. For a continuous map $u : S^4 \to N$, we define the width of u as

$$W(u) = \max_{x,y\in S^4} d_{(\tilde{N},\tilde{g})}(\tilde{u}(x), \tilde{u}(y))$$

for a lift $\tilde{u}$ of u. Since the lift is unique up to the action of the deck transformation of $\tilde{N}$, the definition is independent of the choice of $\tilde{u}$.

Remark 13. It is perhaps more natural to use the universal cover. Theoretically, any cover will make the proof work. Since the main purpose is to construct examples, we use the definition which is convenient for our purpose. Of cause, the width depends on the choice of the cover.

Similarly, we can define the width of u from $\mathbb{R}^4$ to N by

$$W(u) = \sup_{x,y\in\mathbb{R}^4} d_{(\tilde{N},\tilde{g})}(\tilde{u}(x), \tilde{u}(y))$$

for a lift $\tilde{u}$.

Remark 14. Since $\mathbb{R}^4$ is non-compact, it is possible that $W(u)$ is not finite. For application in this paper, we shall only be interested in the bubble map $u : \mathbb{R}^4 \to N$. There are several ways to see that for a bubble map with finite energy this width is finite. First, one can compose u with the stereographic projection and prove a removable singularity theorem for a PDE system similar but not identical to the biharmonic map equation as Wang did for quasi-biharmonic maps in Lemma 3.4 [16]. Second, the proof of removable singularity theorem in [10] can be applied in this case. Finally, since all such bubble maps come from the limit of some biharmonic map sequence, as remarked near the end of Section 2 of [10], this is a consequence of the main theorem in [10].

The main result of this section is

Lemma 5. *For any $C_1 > 0$, there is another constant C_2 depending on C_1 and the geometry of N such that any biharmonic map u from $\mathbb{R}^4$ (or S^4) to N with $\mathcal{E}(u) < C_1$ satisfies that $W(u) < C_2$.*

The proof uses the compactness properties of biharmonic maps (taking the bubbling into account). The non-compactness of $\mathbb{R}^4$ causes some technical problem. We need the following lemma to control the energy decay at the infinity.

Lemma 6. *There is a constant $\varepsilon_2 > 0$ depending on N. If $u : \mathbb{R}^4 \to N$ is a biharmonic map satisfying*

$$\int_{\mathbb{R}^4\setminus B_1} \left|\nabla^2 u\right|^2 + |\nabla u|^4 \, dx < \varepsilon_2,$$

then u is uniformly continuous at the infinity in the sense that for any $\varepsilon > 0$, there is $R > 0$ independent of u such that

$$osc_{\mathbb{R}^4\setminus B_R} u < \varepsilon.$$

Proof. The proof is just another version of Section 6 of [10]. The only difference is that for a removable singularity theorem, we study $B_1 \setminus \{0\}$, which is

$$B_1 \setminus \{0\} = \bigcup_{i=1}^{\infty} A_i$$

where

$$A_i = B_{e^{-(i-1)L}} \setminus B_{e^{-iL}},$$

while in this lemma, we study the asymptotic behavior of u on

$$\mathbb{R}^4 \setminus B_1 = \bigcup_{i=-\infty}^{0} A_i.$$

In the proof of the removable singularity theorem, we prove exponential decay as $i \to \infty$ ($|x| \to 0$), while here we prove exponential decay as $i \to -\infty$ ($|x| \to \infty$). We need ε_2 to be small, so that we can use Theorem 4 on $A_{i-1} \cup A_i \cup A_{i+1}$ for $i = -1, -2, \cdots$.

This lemma follows from the exponential decay of $|\nabla_{S^3} u|$ and $|\partial_t u|$. □

Proof of Lemma 5. We only prove the case for $\mathbb{R}^4$ and the case for S^4 is simpler. Recall that the width of biharmonic map u from $\mathbb{R}^4$ is finite as discussed in Remark 14. If the lemma is not true, we can find a sequence of biharmonic maps $u_k : \mathbb{R}^4 \to N$ with $\mathcal{E}(u_k) \leq C_1$, but

$$\lim_{k\to\infty} W(u_k) = +\infty.$$

Since $\mathcal{E}(u_k)$ and $W(u_k)$ are invariant under the scaling, we may assume without loss of generality that

$$\int_{\mathbb{R}^4\setminus B_1} \left|\nabla^2 u_k\right|^2 + |\nabla u_k|^4 \, dx < \varepsilon_2. \tag{3.1}$$

(3.1) implies that the bubble points are restricted to $\bar{B}_1$.

Let u_∞ be the weak limit. Since there is no bubble outside $\bar{B}_1$, u_k converges to u_∞ on $B_R \setminus B_2$ uniformly for fixed R. Together with Lemma 6 and (3.1), the convergence is uniform on $\mathbb{R}^4 \setminus B_2$.

The bubbles are described as follows. Assume that there are l bubbles (including ghost bubbles, which is just trivial map), $\omega_i (i = 1, \cdots, l)$ and there are $m (m \leq l)$ blow-up points $p_i (i = 1, \cdots, m)$ with $p_i \subset B_2$. Each ω_i is the limit of

$$w_{i,k}(x) := u_k(\lambda_{i,k} x + x_{i,k}).$$

Since there could be bubbles on top of ω_i, the convergence is strong on the domain

$$\Omega_{i,k} = B_R \setminus \left(\bigcup_s B_\delta(y_{k,s}) \right),$$

where we use s to parameterize the bubbles on top of ω_i. Moreover, for each bubble ω_i, there is a neck region of the form $B_{r_2}(*) \setminus B_{r_1}(*)$, which we denote by $N_{i,k}$. There is no need to be precise about r_1, r_2 and the centers of the balls, it suffices to notice that the no neck theorem implies that

$$\lim_{k\to\infty} \operatorname{osc}_{N_{i,k}} u_k = o(\delta, R), \tag{3.2}$$

where $o(\delta, R)$ goes to zero when $\delta \to 0$ and $R \to \infty$.

By definition, if $\tilde{u}_k$ is a lift of u_k, we have

$$\begin{aligned} W(u_k) &= \sup_{y,z\in\mathbb{R}^4} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)) \\ &\leq \sum_{i=1}^{l} \sup_{y,z\in\Omega_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(\lambda_{i,k} y + x_{i,k}), \tilde{u}_k(\lambda_{i,k} z + x_{i,k})) \\ &\quad + \sup_{y,z\in N_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)) \\ &\quad + \sup_{y,z\in\mathbb{R}^4\setminus\bigcup_{i=1}^m B_\delta(p_i)} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)). \end{aligned} \tag{3.3}$$

Now we give an upper bound for the left hand side of the above equation. For the first line, since $w_{i,k}$ converges strongly to ω_i on $\Omega_{i,k}$, we have

$$\max_{y\in\Omega_{i,k}} d_{(N,g)}(w_{i,k}(y), \omega_i(y)) \leq o(1).$$

Here $o(1)$ goes to zero as $k \to \infty$. Noticing that $\tilde{u}_k(\lambda_{i,k} x + x_{i,k})$ is a lift of $w_{i,k}(x)$ (defined on $\Omega_{i,k}$), we can find a lift of ω_i, denoted by $\tilde{w}_i$ such that

$$\max_{y\in\Omega_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(\lambda_{i,k} y + x_{i,k}), \tilde{\omega}_i(y)) \leq o(1).$$

Therefore, we have

$$\limsup_{k\to\infty} \sup_{y,z\in\Omega_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(\lambda_{i,k} y + x_{i,k}), \tilde{u}_k(\lambda_{i,k} z + x_{i,k})) \leq W(\omega_i). \tag{3.4}$$

For the second line, we need some general fact from Riemannian geometry as follows. There is some small $\sigma > 0$ depending on both (N, g) and $(\tilde{N}, \tilde{g})$ such that for any geodesic ball $B \subset N$ of radius σ and its lift $\tilde{B} \subset \tilde{N}$, we have that $(B, d_{(N,g)})$ is isometric to $(\tilde{B}, d_{(\tilde{N},\tilde{g})})$ as metric spaces.

Thanks to (3.2), for small δ and large R so that the image $u_k(N_{i,k})$ lies in a geodesic ball of radius σ, we have

$$\limsup_{k\to\infty} \sup_{y,z\in N_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)) \leq Co(\delta, R). \tag{3.5}$$

To bound the last line in (3.3), it suffices to note that u_k converges uniformly on $\mathbb{R}^4 \setminus \bigcup_m B_\delta(p_i)$ to u_∞. To see this, we note that u_k converges strongly on $B_2 \setminus \cup B_\sigma(p_i)$ and u_k converges strongly on $\mathbb{R}^4 \setminus B_2$ as remarked earlier. Hence,

$$\limsup_{k\to\infty} \sup_{y,z\in\mathbb{R}^4\setminus\bigcup_{i=1}^m B_\delta(p_i)} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)) \leq W(u_\infty). \tag{3.6}$$

(3.4), (3.5) and (3.6) add up to give an upper bound for $W(u_k)$, which contradicts the assumption that $\lim_{k\to\infty} W(u_k) = \infty$ and hence proves the lemma. ☐

4. PROOF OF THE MAIN THEOREM

Let $u(t)$ be a solution to (1.1) with $u(0) = u_0$. Along the flow,

$$\frac{d}{dt}E(u) \leq 0.$$

Hence, $E(u)$ is uniformly bounded (before the possible blow-up at least). Since the target manifold is compact, u is bounded and hence $\mathcal{E}(u)$ is also uniformly bounded.

The key observation to the proof is that for some $C_1 > 0$ and arbitrarily large C_3, we can choose u_0 with $\mathcal{E}(u_0) < C_1$ and any smooth u' homotopic to u_0 satisfies $W(u') > C_3$.

Assuming that such u_0 is found, we claim that $u(t)$ must blow-up in finite time and hence Theorem 1 is proved. If otherwise, the solution exists for any $t > 0$. Since

$$\int_0^\infty \int_{S^4} |\partial_t u|^2 \, dv dt < \infty,$$

we may choose a sequence of t_k going to ∞ such that

$$\lim_{k\to\infty} \|\partial_t u\|_{L^2}(t_k) \to 0.$$

For simplicity, we denote $u(t_k)$ by u_k.

Since $\mathcal{E}(u_k)$ is bounded and the $\varepsilon-$regularity (Theorem 3) holds, the usual blow-up analysis works. Assume that there are l bubbles $\omega_i (i = 1, \cdots, l)$, which is the limit of $u_k(\lambda_{i,k}x + x_{i,k})$ and $m(m < l)$ blow-up points p_i. Let $\Omega_{i,k}$ and $N_{i,k}$ as before. We still have

$$\begin{aligned} W(u_k) \quad \leq \quad & \sum_{i=1}^{l} \sup_{y,z\in\Omega_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(\lambda_{i,k}y + x_{i,k}), \tilde{u}_k(\lambda_{i,k}z + x_{i,k})) \\ & + \sup_{y,z\in N_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)) \\ & + \sup_{y,z\in\mathbb{R}^4\setminus\bigcup_{i=1}^m B_\delta(p_i)} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)). \end{aligned}$$

By Theorem 2, we can bound the right hand side by

$$\sum_{i=1}^{l} W(\omega_i) + W(u_\infty) + 1.$$

By Lemma 5, each $W(\omega_i)$ and $W(u_\infty)$ is bounded by a constant C_2 depending on C_1. Moreover, the number of bubbles is also bounded by a constant depending on C_1. Hence, there is a constant C_4 such that

$$\limsup_{k\to\infty} W(u_k) < C_4.$$

This would be a contradiction and hence proves Theorem 1 if $C_4 < C_3$.

Now let's show how to construct u_0.

Recall that $N = N'\#T^m$. There is a natural cover of N, which is obtained by modifying $\mathbb{R}^m$. $\mathbb{R}^m$ is the universal cover of T^m, with the deck transformation group $G = \mathbb{Z}^m$. Let p_0 be any point of $\mathbb{R}^m$ and let the orbit of the action of G containing p_0 be $\{p_i\}_{i=0}^{\infty}$. Suppose U_i be a small neighborhood of p_i diffeomorphic to the ball of dimension m and $V \subset N'$ be an open set diffeomorphic to a ball. For each $i = 0, 1, \cdots$, we remove U_i from $\mathbb{R}^m$ and identify the boundary of U_i with the boundary of a copy of $N' \setminus V$, which we denote by W_i. The new complete non-compact manifold is denoted by $\tilde{N}$. G acts on $\tilde{N}$ naturally and the quotient is N. If N is equipped with a Riemannian metric g and $\tilde{g}$ is the pull back metric, then the projection $\pi : \tilde{N} \to N$ is isometric map.

Since $\pi_4(N')$ is not trivial, there is a smooth map $h : S^4 \to N'$, which is not homotopic to constant map. Since $m > 4$ and h is not surjective, assume by deforming it smoothly that

(1) $h(S^4) \subset N' \setminus \bar{V}$;

(2) h maps the entire southern hemisphere to a single point $q \in N' \setminus \bar{V}$.

Let h_i be the copy of h from S^4 to W_i and q_i be the copy of q in W_i.

For any C_3, pick i such that

$$d_{(\tilde{N},\tilde{g})}(W_0, W_i) > C_3.$$

Let $\Psi(\Phi)$ be the stereoprojection from $\mathbb{R}^4$ to S^4, which maps the infinity to the south (north) pole and maps ∂B_1 to the equator. Consider the map $w : \mathbb{R}^4 \to W_i$ defined by

$$w(x) = h_i \circ \Psi(x).$$

w is a constant map outside B_1. Set

$$C_1 = E(w) + E(h) + 1.$$

We claim that for σ very small, we can find smooth u_0 satisfying

(1)

$$u_0 = \begin{cases} \pi \circ h_0(x) & x \in S^4 \setminus B_\sigma(S); \\ \pi \circ w\left(\frac{\Phi^{-1}(x)}{\sigma^2/2}\right) & x \in B_{\sigma^2}(S). \end{cases}$$

(2) $E(u_0) < C_1$.

By the above definition, we observe that $u_0|_{\partial B_\sigma(S)} = q_0$ and $u_0|_{\partial B_{\sigma^2}(S)} = q_i$. The first observation follows trivially from the definition of h_0. For the latter, we notice that $\left|\Phi^{-1}(x)\right|$ is almost σ^2 for every $x \in \partial B_{\sigma^2}(S)$, because Φ is almost an isometry near S and σ is going to be small.

Since the energy is scaling invariant and Φ is almost isometric in small neighborhood of the south pole, we have

$$\left(\int_{S^4\setminus B_\sigma(S)}+\int_{B_{\sigma^2}(S)}\right)|\triangle u_0|^2\,dv<E(h)+E(w)+\frac{1}{2}.$$

It suffices to show that we can define u_0 on $B_\sigma(S)\setminus B_{\sigma^2}(S)$ so that u_0 is smooth and the contribution to the energy on this part is smaller than $\frac{1}{2}$. By choosing σ small, the metric of S^4 on B_σ is close to the flat metric. Hence, it suffices to check this with flat metric.

Let $\gamma:[0,1]\to\tilde{N}$ be the shortest geodesic in $\tilde{N}$ connecting q_0 to q_i. Let $\varphi:[0,1]\to[0,1]$ be a smooth function satisfying

(1) $\varphi'\geq 0$;

(2) $\varphi(x)=0$ for all $0\leq x\leq\frac{1}{8}$ and $\varphi(x)=1$ for all $\frac{7}{8}\leq x\leq 1$;

(3) $|\varphi'|+|\varphi''|\leq C$ for some universal constant C.

Set

$$u_0(x)=\pi\circ\gamma\circ\varphi\left(\frac{\log\sigma-\log|x|}{-\log\sigma}\right).$$

For simplicity, we write L for $d_{(\tilde{N},\tilde{g})}(q_0,q_i)$. Note that

$$|(\pi\circ\gamma)'|=L.$$

Since γ and $\pi\circ\gamma$ are geodesics, we have

$$(\pi\circ\gamma)''+B(\pi\circ\gamma)((\pi\circ\gamma)',(\pi\circ\gamma)')=0$$

where B is the second fundamental form of N. Therefore,

$$|(\pi\circ\gamma)''|=CL^2.$$

We estimate the derivative of u_0 as follows.

$$|\partial_r u_0|\leq\frac{CL}{r(-\log\sigma)}$$

and

$$\left|\partial_r^2 u_0\right|\leq\frac{CL}{r^2(-\log\sigma)}.$$

Hence,

$$\begin{aligned}&\int_{B_\sigma\setminus B_{\sigma^2}}|\triangle u_0|^2\,dx\\ \leq\;& C\int_{\sigma^2}^{\sigma}\left|\partial_r^2 u_0+\frac{3}{r}\partial_r u_0\right|^2 r^3dr\\ \leq\;&\frac{CL^2}{(\log\sigma)^2}\int_{\sigma^2}^{\sigma}\frac{1}{r}dr\\ \leq\;&\frac{CL^2}{(-\log\sigma)}.\end{aligned}$$

For any L, we can choose σ so that the above is as small as we want. Hence, we check that u_0 satisfies $E(u_0)<C_1$. It remains to check that for any map u' homotopic to u_0,

$W(u') > C_3$. Let $\tilde{u}'$ be the lift of u', which is homotopic to the following lift of u_0,

$$\tilde{u}_0 = \begin{cases} h_0(x) & x \in S^4 \setminus B_\sigma(S); \\ \gamma \circ \varphi(\frac{\log\sigma - \log|x|}{-\log\sigma}) & x \in B_\sigma(S) \setminus B_{\sigma^2}(S); \\ w(\frac{\Phi^{-1}(x)}{\sigma^2/2}) & x \in B_{\sigma^2}(S). \end{cases}$$

We claim that $\widetilde{u'} \cap W_0 \neq \emptyset$ and $\widetilde{u'} \cap W_i \neq \emptyset$. To see this, consider a continuous map $\tilde{\pi}$ from $\tilde{N}$ to N' (precisely, a manifold homeomorphic to N'), which maps any point in $\tilde{N} \setminus W_0$ to one point. If $\tilde{u}' \cap W_0$ is empty, then $\tilde{\pi} \circ \tilde{u}'$ is a constant map. However, $\tilde{\pi} \circ \tilde{u}_0$ is homotopic to h_0 and hence is nontrivial. The proof for $\widetilde{u'} \cap W_i \neq \emptyset$ is the same.

In summary, we have constructed a map u_0 such that $E(u_0) < C_1$ and $W(u') > C_3$ for any u' homotopic to u_0. This finishes the proof of Theorem 1.

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