JEPE Vol 1, 2015, p. 335-362

Orthogonal ⊥

POSITIVE SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION WITH SINGULAR DIRICHLET BOUNDARY DATA

MAREK FILA, KAZUHIRO ISHIGE, AND TATSUKI KAWAKAMI

Abstract. The purpose of this paper is to construct positive solutions of the semilinear elliptic equation $-\Delta u = u^p$ in $\mathbb{R}^N_+$ with a singular Dirichlet boundary condition. We show that for $p > (N+1)/(N-1)$ there exists a positive singular solution which behaves like $|x|^{-2/(p-1)}$ as $|x| \to 0$ and like the Poisson kernel as $|x| \to \infty$.

1. Introduction

We consider the problem

$$
\begin{cases} -\Delta u = u^p & \text{in} \quad \mathbb{R}^N_+, \\ u > 0 & \text{in} \quad \mathbb{R}^N_+, \\ u = \varphi \geq 0 & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases} \tag{1.1}
$$

where $N \geq 2$, $\mathbb{R}^N_+ := \{x = (x', x_N) : x' \in \mathbb{R}^{N-1},\ x_N > 0\}$, $p > 1$ and φ is a nonnegative measurable function in $\mathbb{R}^{N-1}$.

For problem (1.1), the exponent

$$
p_* := (N+1)/(N-1)
$$

plays an important role. Namely, it was shown in [2] that for $p \leq p_*$ there is no classical solution of (1.1), no matter which boundary data φ we impose. As we shall see below, this result is sharp.

If $\varphi \equiv 0$ then there are Liouville-type theorems saying that no classical solution of (1.1) exists if $(N-2)p < N+2$ (see [12]) and no classical bounded solution of (1.1) exists if $p < p_c$

2010 *Mathematics Subject Classification.* Primary: 35J61; Secondary: 35B40, 35J25.

Key words and phrases. Semilinear elliptic equation, singular Dirichlet boundary data, asymptotic behavior of solutions.

Received 18/01/2016, accepted 20/01/2016.

The first author was supported by the Slovak Research and Development Agency under the contract No. APVV-14-0378 and by the VEGA grant 1/0711/12. The second author was supported by the Grant-in-Aid for Scientific Research (A)(No. 15H02058), from Japan Society for the Promotion of Science. The third author was supported by the Grant-in-Aid for Young Scientists (B) (No. 24740107) from Japan Society for the Promotion of Science and by the JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers "Mathematical Science of Symmetry, Topology and Moduli, Evolution of International Research Network based on OCAMI".

(see [7]), here

$$p_c := \begin{cases} \infty, & N \le 11, \\ \dfrac{(N-3)^2 - 4N + 4 + 8\sqrt{N-2}}{(N-3)(N-11)}, & N > 11. \end{cases}$$

If $p > p_*$ then a solution of (1.1) exists when $\varphi \not\equiv 0$ is small enough. More precisely, if ψ is bounded, $\psi \not\equiv 0$ and

$$\limsup_{|x'|\to\infty} |x'|^{2/(p-1)} \psi(x') < \infty$$

then there exists a solution (in the sense of Definition 1.1 below) when $\varphi = k\psi$ and $k > 0$ is small enough (see Corollary 1.3 in [11]). Also, when $p > p_*$, a different condition for $\varphi \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, $\varphi \not\equiv 0$, which guarantees the existence of a solution of (1.1), follows from Theorem 1.4 in [11] and Theorem 1.2 in [8]. Namely, it is sufficient if

$$\|\varphi\|_{L^1(\mathbb{R}^{N-1})} \|\varphi\|_{L^\infty(\mathbb{R}^{N-1})}^{(Np-p-N-1)/2}$$

is small enough.

Singular solutions of the problem

$$\begin{cases} -\Delta u = u^p & \text{in} \quad \mathbb{R}^N_+, \qquad p > 1, \\ u > 0 & \text{in} \quad \mathbb{R}^N_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^N_+ \setminus \{0\}, \end{cases} \tag{1.2}$$

were studied in [3] and [6]. It was shown in [3] that a solution of the form

$$u_*(x) := |x|^{-\frac{2}{p-1}} \omega\left(\frac{x}{|x|}\right)$$

exists if and only if

$$p_* < p < p^* := \begin{cases} \infty, & N \le 3, \\ \dfrac{N+1}{N-3}, & N > 3. \end{cases}$$

The existence of a different singular solution U of (1.2) was established in [6] for $p \in (p_*, p_* + \varepsilon)$ where $\varepsilon > 0$. This solution U behaves like u_* as $x \to 0$ and like the Poisson kernel as $|x| \to \infty$.

In this paper we prove that for $p > p_*$ and suitable singular boundary data φ there are singular solutions of (1.1) which behave like $|x|^{-2/(p-1)}$ as $|x| \to 0$ and like the Poisson kernel as $|x| \to \infty$.

For $p > N/(N-2)$, $N > 2$, there is an explicit singular solution

$$u_\infty := c_{p,N}|x|^{-\frac{2}{p-1}}, \qquad c_{p,N} := \left(\frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right)\right)^{\frac{1}{p-1}},$$

and for $N/(N-2) < p < (N+2)/(N-2)$, $N > 2$, there is a family of radial singular solutions u_α, $\alpha > 0$, such that

$$\lim_{|x|\to\infty} |x|^{N-2} u_\alpha(x) = \alpha,$$

see [5]. The solutions we find behave differently from u_α, $0 < \alpha \le \infty$, as $|x| \to \infty$.

For other works on boundary singularities of solutions of semilinear elliptic equations we refer to [4, 13, 15, 16, 17, 18].

To formulate our results we introduce some notation. For any $x' \in \mathbb{R}^{N-1}$ and $\lambda > 0$, let P be the $(N-1)$-dimensional Poisson kernel, that is,

$$P(x', \lambda) := c_N \lambda \left(\lambda^2 + |x'|^2\right)^{-\frac{N}{2}}, \tag{1.3}$$

for $x' \in \mathbb{R}^{N-1}$ and $\lambda > 0$, where c_N is a constant chosen so that

$$\int_{\mathbb{R}^{N-1}} P(x', \lambda)\, dx' = 1, \qquad \lambda > 0. \tag{1.4}$$

Throughout this paper, we often identify $\mathbb{R}^{N-1}$ with $\partial\mathbb{R}^N_+$. Next we set $y_* := (y', -y_N)$ for $y = (y', y_N) \in \mathbb{R}^N_+$ and

$$G(x,y) := \begin{cases} \dfrac{c_N}{2(N-2)} \left(|x-y|^{-(N-2)} - |x-y_*|^{-(N-2)}\right) & \text{if} \quad N \geq 3, \\ \dfrac{1}{4\pi} \log\left(1 + \dfrac{4x_2 y_2}{|x-y|^2}\right) & \text{if} \quad N = 2, \end{cases} \tag{1.5}$$

which is the Green function for $-\Delta_D$. Here Δ_D is the Laplace operator in $\mathbb{R}^N_+$ with the homogeneous Dirichlet boundary condition.

Definition 1.1. Let φ be a nonnegative measurable function in $\mathbb{R}^{N-1}$. Let u be a nonnegative measurable function in $\mathbb{R}^N_+$.
(i) We call the function u a solution of (1.1) if u satisfies

$$u(x', x_N) = \int_{\mathbb{R}^{N-1}} P(x'-y', x_N)\varphi(y')\, dy' + \int_{\mathbb{R}^N_+} G(x,y) u(y)^p\, dy < \infty$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \in [0, \infty)$.
(ii) We call u a minimal solution of (1.1) if, for any solution v of (1.1),

$$u(x', x_N) \leq v(x', x_N)$$

holds for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$.

Now we are ready to state the main result of this paper.

Theorem 1.1. *Let $N \geq 2$ and $p > p_* = (N+1)/(N-1)$. Put*

$$\psi(x') = \min\{|x'|^{-\frac{2}{p-1}}, |x'|^{-N}\} = \begin{cases} |x'|^{-\frac{2}{p-1}} & \text{if} \quad |x'| \leq 1, \\ |x'|^{-N} & \text{if} \quad |x'| > 1, \end{cases} \tag{1.6}$$

for $x' \in \mathbb{R}^{N-1}$. Then there exist constants $0 < c < k < 1$ such that if

$$c\psi(x') \leq \varphi(x') \leq k\psi(x'), \qquad x' \in \mathbb{R}^{N-1}, \tag{1.7}$$

then problem (1.1) possesses a minimal solution u. Furthermore,

$$K|x|^{-\frac{2}{p-1}} \leq u(x) \leq L|x|^{-\frac{2}{p-1}}, \qquad x \in D_{\text{in}} := B(0,1) \cap \mathbb{R}^N_+, \tag{1.8}$$

and

$$K(1+x_N)|x|^{-N} \leq u(x) \leq L(1+x_N)|x|^{-N}, \qquad x \in D_{\text{out}} := \mathbb{R}^N_+ \setminus \overline{B(0,1)}, \tag{1.9}$$

for some positive constants K and L with $K < L$.

The rest of this paper is organized as follows. In Section 2 we recall some preliminary inequalities. In Section 3 we give pointwise estimates of $S(x_N)\psi$. In Section 4 we obtain upper estimates of an integral associated with the Green function G and prove Theorem 1.1.

2. Preliminaries

We recall some properties of the semigroup associated with the Poisson kernel P and a proposition on the Green function G. Here and in the rest of the paper, by c and C we denote generic positive constants which may have different values also within the same line.

We introduce some notation. For any $R > 0$, let

$$B(0,R) := \{x \in \mathbb{R}^N \,:\, |x| < R\} \quad \text{and} \quad B'(0,R) := \{x' \in \mathbb{R}^{N-1} \,:\, |x'| < R\}.$$

For any $1 \le r \le \infty$, we denote by $\|\cdot\|_r$ the usual norm of $L^r := L^r(\mathbb{R}^{N-1})$. For any measurable set E in $\mathbb{R}^{N-1}$, we denote by $|E|$ the $N-1$ dimensional Lebesgue measure of E. For any measurable function f in $\mathbb{R}^{N-1}$, let

$$\mu_f(\lambda) := |\{x \,:\, |f(x)| > \lambda\}| \qquad (\lambda \ge 0)$$

be the distribution function of f. We define the non-increasing rearrangement of f by

$$f^*(s) := \inf\{\lambda > 0 \,:\, \mu_f(\lambda) \le s\}.$$

For any $1 \le r \le \infty$, we define the $L^{r,\infty}$ space by

$$L^{r,\infty} := \{f \,:\, f \text{ is measurable in } \mathbb{R}^{N-1},\ \|f\|_{r,\infty} < \infty\},$$

where

$$\|f\|_{r,\infty} := \sup_{s>0} s^{\frac{1}{r}} f^{**}(s), \qquad f^{**}(s) := \frac{1}{s}\int_0^s f^*(r)dr.$$

For any measurable function φ in $\mathbb{R}^{N-1}$, let

$$[S(x_N)\varphi](x') := \int_{\mathbb{R}^{N-1}} P(x'-y', x_N)\varphi(y')\,dy' \tag{2.1}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \ge 0$. Then, for any $\lambda > 0$,

$$[S(\lambda + x_N)\varphi](x') = [S(\lambda)(S(x_N)\varphi)](x'),$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \ge 0$ if either φ is a nonnegative measurable function in $\mathbb{R}^{N-1}$ or $\varphi \in L^{q,\infty}$ for some $q \in [1,\infty]$. In particular, for any $\varphi \in L^{q,\infty}$ and $1 \le q \le \infty$, there exists a constant C such that

$$\|S(2\lambda)\varphi\|_\infty \le \|P(\lambda)\|_p \|S(\lambda)\varphi\|_r \le C\|P(\lambda)\|_p\|P(\lambda)\|_s\|\varphi\|_{q,\infty}, \qquad \lambda > 0, \tag{2.2}$$

where $1 < p, r, s < \infty$ satisfy

$$\frac{1}{p} + \frac{1}{r} = 1, \qquad \frac{1}{s} + \frac{1}{q} = 1 + \frac{1}{r}$$

(see [14, Section 2]). By (1.3) and (1.4), for any $1 \le r \le \infty$, we have

$$\|P(\lambda)\|_r \le C\lambda^{-(N-1)(1-\frac{1}{r})}, \qquad \lambda > 0.$$

This together with (2.2) yields

$$\|S(\lambda)\varphi\|_\infty \le C\lambda^{-\frac{N-1}{q}}\|\varphi\|_{q,\infty}, \qquad \lambda > 0, \qquad q \in [1,\infty]. \tag{2.3}$$

Next we recall some estimates of the Green function G (see [1, Proposition 1] and [9]).

Proposition 2.1. *Let G be as in* (1.5)*.*

(i) *If $N \geq 3$ then there exists a positive constant C such that*

$$G(x,y) \leq C\frac{x_N y_N}{|x-y|^{N-2}(|x-y|^2+4x_N y_N)}, \qquad x,y\in\mathbb{R}^N_+, \quad x\neq y. \tag{2.4}$$

(ii) *If $N=2$ then, for any $\alpha\in(0,1]$, there exists a positive constant C such that*

$$G(x,y) \leq C\left(\frac{x_2 y_2}{|x-y|^2}\right)^{\alpha}, \qquad x,y\in\mathbb{R}^2_+, \quad x\neq y. \tag{2.5}$$

3. Pointwise estimates of harmonic extensions

In this section we obtain some pointwise estimates of $S(x_N)\psi$. In what follows, we set

$$m := \frac{2}{p-1}$$

for simplicity. Then

$$m < N-1 \quad \text{if} \quad p > p_*. \tag{3.1}$$

Furthermore, let $D_{\rm in}$ and $D_{\rm out}$ are as in (1.8) and (1.9), respectively.

Lemma 3.1. *Let $p>p_*$ and ψ be as in* (1.6)*. Then there exists a positive constant C such that*

$$[S(x_N)\psi](x') \geq C|x|^{-m}, \quad x\in D_{\rm in}, \tag{3.2}$$

$$[S(x_N)\psi](x') \geq C(1+x_N)|x|^{-N}, \quad x\in D_{\rm out}. \tag{3.3}$$

Proof. It follows from (1.3) and (2.1) that

$$\begin{aligned}
[S(x_N)\psi](x') &= c_N x_N^{-(N-1)}\int_{\mathbb{R}^{N-1}}\left(1+\left|\frac{y'}{x_N}\right|^2\right)^{-\frac{N}{2}}\psi(x'-y')\,dy' \\
&= c_N\int_{\mathbb{R}^{N-1}}\left(1+|z'|^2\right)^{-\frac{N}{2}}\psi(x'-x_N z')\,dz', \quad x\in\mathbb{R}^N_+.
\end{aligned} \tag{3.4}$$

We prove (3.2). For any $x\in D_{\rm in}$, since

$$|x'-x_N z'| \leq |x'|+x_N|z'| \leq |x'|+x_N \leq \sqrt{2}, \qquad z'\in B'(0,1),$$

by (1.6) and (3.4) we have

$$\begin{aligned}
[S(x_N)\psi](x') &\geq C\int_{\mathbb{R}^{N-1}}(1+|z'|)^{-N}\psi(x'-x_N z')dy' \\
&\geq C\int_{B'(0,1)}(1+|z'|)^{-N}|x'-x_N z'|^{-m}\,dy' \\
&\geq C(|x'|+x_N)^{-m}\int_0^1(1+r)^{-N}r^{N-2}dr \geq C|x|^{-m}.
\end{aligned}$$

This implies (3.2).

We prove (3.3). Set

$$D_1 := \left\{ x \in D_{\rm out} : x_N \le \frac{1}{4} \right\}, \quad D_2 := \left\{ x \in D_{\rm out} : x_N > \frac{1}{4} \right\}.$$

For any $x \in D_1$, since

$$\frac{\sqrt{3}-1}{2} \le |x'| - 2x_N \le |x'| - x_N|z'| \le |x' - x_N z'| \le |x'| + x_N|z'| \le 2(|x'| + x_N)$$

for all $z' \in \mathbb{R}^{N-1}$ with $1 \le |z'| \le 2$, it follows from (1.6) and (3.4) that

$$\begin{aligned}[S(x_N)\psi](x') &\ge C \int_{\mathbb{R}^{N-1}} (1+|z'|)^{-N} \psi(x' - x_N z') dy' \\ &\ge C \int_{1 \le |z'| \le 2} (1+|z'|)^{-N} |x' - x_N z'|^{-N} dy' \\ &\ge C|x|^{-N} \ge C(1+x_N)|x|^{-N}, \qquad x \in D_1. \end{aligned} \tag{3.5}$$

On the other hand, for any $x \in D_2$, since $5x_N > 1 + x_N$, by (1.3), (1.6) and (2.1) we have

$$\begin{aligned}[S(x_N)\psi](x') &\ge C x_N \int_{1 \le |y'| \le 2} (x_N + |x'| + |y'|)^{-N} |y'|^{-N} dy' \\ &\ge C(1+x_N)|x|^{-N} \int_1^2 r^{-2} dr \ge C(1+x_N)|x|^{-N}, \qquad x \in D_2. \end{aligned}$$

This together with (3.5) implies (3.3), and the proof is complete. ☐

Lemma 3.2. *Let $p > p_*$ and ψ be as in* (1.6). *Then there exists a positive constant C such that*

$$[S(x_N)\psi](x') \le C|x|^{-m}, \quad x \in D_{\rm in}, \tag{3.6}$$

$$[S(x_N)\psi](x') \le C(1+x_N)|x|^{-N}, \quad x \in D_{\rm out}. \tag{3.7}$$

Proof. We first prove (3.6). Let $x \in D_{\rm in}$ and set

$$B_1(x') := \left\{ y' \in \mathbb{R}^{N-1} : |x' - y'| \le \frac{|x'|}{2} \right\}, \quad B_2(x') := \left\{ y' \in \mathbb{R}^{N-1} : |x' - y'| > \frac{|x'|}{2} \right\}.$$

It follows from (1.3) and (2.1) that

$$\begin{aligned}[S(x_N)\psi](x') &= c_N x_N \int_{\mathbb{R}^{N-1}} \left(x_N^2 + |y'|^2\right)^{-\frac{N}{2}} \psi(x' - y')\, dy' \\ &\le C\biggl(\int_{B_1(x')} + \int_{B_2(x')} \biggr) x_N (x_N + |y'|)^{-N} \psi(x' - y')\, dy'. \end{aligned} \tag{3.8}$$

By (3.1) we can find $\varepsilon \in (0,1)$ such that

$$-(N-1) + m + \varepsilon < 0. \tag{3.9}$$

For any $y' \in B_1(x')$, since $|x'|/2 \le |y'| \le (3|x'|)/2$, we see that

$$x_N (x_N + |y'|)^{-N} \le C|x'|^{-(N-1)} \le C|x'|^{-m-\varepsilon} |x' - y'|^{-(N-1)+m+\varepsilon}. \tag{3.10}$$

By (1.6) and (3.10) we have

$$\begin{aligned}\int_{B_1(x')} x_N(x_N+|y'|)^{-N}\psi(x'-y')\,dy' &\le \int_{B_1(x')} x_N(x_N+|y'|)^{-N}|x'-y'|^{-m}\,dy'\\ &\le C|x'|^{-m-\varepsilon}\int_{B_1(x')}|x'-y'|^{-(N-1)+\varepsilon}\,dy'\\ &= C|x'|^{-m-\varepsilon}\int_0^{\frac{|x'|}{2}} r^{-1+\varepsilon}\,dr \le C|x'|^{-m}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_{B_2(x')} x_N(x_N+|y'|)^{-N}\psi(x'-y')\,dy' &= \int_{B_2(x')} x_N(x_N+|y'|)^{-N}|x'-y'|^{-m}\,dy'\\ &\le C|x'|^{-m}\int_{\mathbb{R}^{N-1}}(1+|z'|)^{-N}\,dz' \le C|x'|^{-m}.\end{aligned}$$

These together with (3.8) yield

$$[S(x_N)\psi](x') \le C|x'|^{-m}, \qquad x\in D_{\rm in}. \tag{3.11}$$

On the other hand, it follows from (1.6) and (3.1) that $\psi\in L^{r_*,\infty}$ with $r_*=(N-1)/m>1$. This together with (2.3) implies that

$$[S(x_N)\psi](x') \le \|S(x_N)\psi\|_\infty \le Cx_N^{-m}\|\psi\|_{r_*,\infty}, \quad x\in\mathbb{R}^N_+. \tag{3.12}$$

Therefore, by (3.11) and (3.12) we obtain

$$[S(x_N)\psi](x') \le C\min\left\{|x'|^{-m}, x_N^{-m}\right\}, \qquad x\in D_{\rm in}. \tag{3.13}$$

On the other hand, the following inequality holds (see e.g., [10, Section 4]): For any $q>0$, there exists a positive constant C such that

$$\min\left\{a^{-q}, b^{-q}\right\} \le C(a+b)^{-q} \quad \text{for} \quad a,b>0. \tag{3.14}$$

Then, by (3.13) and (3.14) we have (3.6).

Next we prove (3.7). Let $x\in D_{\rm out}$ with $|x'|\le 1/2$ and set

$$B_3(x') := \left\{y'\in\mathbb{R}^{N-1} : |x'-y'|\le 2\right\}, \quad B_4(x') := \{y'\in\mathbb{R}^{N-1} : |x'-y'|>2\}.$$

Similarly to (3.8), we have

$$[S(x_N)\psi](x') \le Cx_N\left(\int_{B_3(x')}+\int_{B_4(x')}\right)(x_N+|y'|)^{-N}\psi(x'-y')\,dy'. \tag{3.15}$$

Since $2x_N\ge\sqrt{3}>1\ge|x'|$, we see that $|x|\le Cx_N$. It follows from (1.6) and (3.1) that

$$\begin{aligned}x_N\int_{B_3(x')}(x_N+|y'|)^{-N}\psi(x'-y')\,dy' &\le x_N\int_{B_3(x')}(x_N+|y'|)^{-N}|x'-y'|^{-m}\,dy'\\ &\le x_N^{-N+1}\int_{B_3(x')}|x'-y'|^{-m}\,dy'\\ &\le C(1+x_N)|x|^{-N}\int_0^2 r^{-m+N-2}\,dr \le C(1+x_N)|x|^{-N}.\end{aligned} \tag{3.16}$$

Furthermore, since $|x'|\le 1/2$ and

$$|y'| \ge |x'-y'|-|x'| \ge |x'-y'|-1 \ge 1, \qquad y'\in B_4(x'),$$

by (1.6) we obtain

$$
\begin{aligned}
x_N \int_{B_4(x')} (x_N + |y'|)^{-N} \psi(x'-y')\, dy' &\le x_N \int_{B_4(x')} (x_N+|y'|)^{-N} |x'-y'|^{-N}\, dy' \\
&\le (1+x_N)^{-N+1} \int_{B_4(x')} |x'-y'|^{-N}\, dy' \\
&\le C(1+x_N)|x|^{-N} \int_2^\infty r^{-2}\, dr \le C(1+x_N)|x|^{-N}.
\end{aligned} \tag{3.17}
$$

Then, by (3.15), (3.16) and (3.17) we have (3.7) for the case $|x'| \le 1/2$.

It remains to prove (3.7) for the case $|x'| > 1/2$. Let $x \in D_{\rm out}$ with $|x'| > 1/2$ and set

$$
B_5(x') := \left\{ y' \in \mathbb{R}^{N-1} : |x'-y'| \le \frac{1}{8} \right\}, \quad B_6(x') := \left\{ y' \in \mathbb{R}^{N-1} : \frac{1}{8} < |x'-y'| \le \frac{|x'|}{2} \right\},
$$
$$
B_7(x') := \left\{ y' \in \mathbb{R}^{N-1} : |x'-y'| > \frac{|x'|}{2} \right\}.
$$

Similarly to (3.8), we have

$$
[S(x_N)\psi](x') \le C x_N \left(\int_{B_5(x')} + \int_{B_6(x')} + \int_{B_7(x')} \right) (x_N+|y'|)^{-N} \psi(x'-y')\, dy'. \tag{3.18}
$$

Since $|y'| \ge |x'| - 1/8$ for any $y' \in B_5(x')$, by (1.6) and (3.1) we have

$$
\begin{aligned}
x_N \int_{B_5(x')} (x_N + |y'|)^{-N} \psi(x'-y')\, dy' &\le x_N \int_{B_5(x')} (x_N+|y'|)^{-N} |x'-y'|^{-m}\, dy' \\
&\le C(1+x_N) \int_0^{\frac{1}{8}} \left(x_N + |x'| - \frac{1}{8} \right)^{-N} r^{-m+N-2}\, dr \le C(1+x_N)|x|^{-N}.
\end{aligned} \tag{3.19}
$$

Since $|x'|/2 \le |y'| \le 3|x'|/2$ for any $y' \in B_6(x')$, it follows from (1.6) that

$$
\begin{aligned}
x_N \int_{B_6(x')} (x_N + |y'|)^{-N} \psi(x'-y')\, dy' &\le x_N \int_{B_6(x')} (x_N+|y'|)^{-N} |x'-y'|^{-N}\, dy' \\
&\le C(1+x_N)|x|^{-N} \int_{B_6(x')} |x'-y'|^{-N}\, dy' \\
&\le C(1+x_N)|x|^{-N} \int_{\frac{1}{8}}^\infty r^{-2}\, dr \le C(1+x_N)|x|^{-N}.
\end{aligned} \tag{3.20}
$$

Furthermore, since $|x'-y'| > |x'|/2 > 1/4$ for any $y' \in B_7(x')$, by (1.6) we obtain

$$
\begin{aligned}
x_N \int_{B_7(x')} (x_N + |y'|)^{-N} \psi(x'-y')\, dy' &\le x_N^{1-N} \int_{B_7(x')} \left(1 + \frac{|y'|}{x_N} \right)^{-N} |x'-y'|^{-N}\, dy' \\
&\le C|x'|^{-N} \int_{\mathbb{R}^{N-1}} (1+|z'|)^{-N}\, dz' \le C|x'|^{-N} \le C(1+x_N)|x'|^{-N}.
\end{aligned} \tag{3.21}
$$

Similarly, we see that

$$
\begin{aligned}
x_N \int_{B_7(x')} (x_N + |y'|)^{-N} \psi(x' - y')\, dy' &\le x_N^{1-N} \int_{B_7(x')} \left(1 + \frac{|y'|}{x_N}\right)^{-N} |x' - y'|^{-N}\, dy' \\
&\le (1 + x_N) x_N^{-N} \int_{B_7(x')} |x' - y'|^{-N}\, dy' \\
&\le (1 + x_N) x_N^{-N} \int_{\frac{1}{4}}^{\infty} r^{-2}\, dr \le C(1 + x_N) x_N^{-N}.
\end{aligned} \tag{3.22}
$$

Then (3.14), (3.21) and (3.22) yield

$$
\begin{aligned}
& x_N \int_{B_7(x')} (x_N + |y'|)^{-N} \psi(x' - y')\, dy' \\
& \le C(1 + x_N) \min\{|x'|^{-N}, x_N^{-N}\} \le C(1 + x_N)|x|^{-N}.
\end{aligned} \tag{3.23}
$$

Then we deduce from (3.18), (3.19), (3.20) and (3.23) that (3.7) holds for the case $|x'| > 1/2$. Thus Lemma 3.2 follows. □

4. Proof of Theorem 1.1

In this section we obtain two pointwise estimates of

$$
[(-\Delta_D)^{-1} f^p](x) := \int_{\mathbb{R}^N_+} G(x, y) f(y)^p\, dy, \qquad x \in \mathbb{R}^N_+, \tag{4.1}
$$

and prove Theorem 1.1.

Lemma 4.1. *Let $p > p_*$. Assume that*

$$
0 \le f(x) \le |x|^{-m}, \qquad x \in \mathbb{R}^N_+. \tag{4.2}
$$

Then there exists a positive constant C such that

$$
[(-\Delta_D)^{-1} f^p](x) \le C x_N |x|^{-m-1}, \qquad x \in \mathbb{R}^N_+.
$$

Proof. Let $x = (x', x_N) \in \mathbb{R}^N_+$. We divide $\mathbb{R}^{N-1}$ into the following three sets:

$$
\begin{aligned}
& D_1 := \left\{ y' \in \mathbb{R}^{N-1} : |x' - y'| \le \frac{|x'|}{2} \right\}, \ D_2 := \left\{ y' \in \mathbb{R}^{N-1} : \frac{|x'|}{2} < |x' - y'| \le 2|x'| \right\}, \\
& D_3 := \{ y' \in \mathbb{R}^{N-1} : |x' - y'| > 2|x'| \}.
\end{aligned}
$$

Since $mp = m + 2$, by (4.1) and (4.2) it suffices to prove that

$$
\int_0^\infty \int_{D_k} G(x, y) |y|^{-(m+2)}\, dy'\, dy_N \le C x_N |x|^{-m-1}, \quad k = 1, 2, 3. \tag{4.3}
$$

Here C is independent of $x \in \mathbb{R}^N_+$.

Proof in the case $k = 1$. In the case $|x'| = 0$, we immediately obtain (4.3) in the case $k = 1$. So it suffices to consider the case $|x'| \ne 0$.

For any $y' \in D_1$ we see that $|y'| \ge |x'|/2$. In the case $N \ge 3$, by (2.4) we have

$$
\begin{aligned}
&\int_{\frac{x_N}{2}}^{\infty}\int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\le C\int_{\frac{x_N}{2}}^{\infty}\int_{|x'-y'|\le\frac{|x'|}{2}} |x'-y'|^{-(N-2)}|y|^{-(m+2)}\,dy'\,dy_N \\
&\le C\int_{\frac{x_N}{2}}^{\infty}(|x'|+y_N)^{-(m+2)}\,dy_N\int_{|x'-y'|\le\frac{|x'|}{2}} |x'-y'|^{-(N-2)}\,dy' \\
&\le C(|x'|+x_N)^{-(m+1)}|x'| \le C(|x'|+x_N)^{-m} \le Cx_N^{-m}.
\end{aligned}
\tag{4.4}
$$

In the case $N = 2$, by (2.5) with $\alpha = 1/4$ we have

$$
\begin{aligned}
&\int_{\frac{x_2}{2}}^{\infty}\int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\le C\int_{\frac{x_2}{2}}^{\infty}\int_{|x_1-y_1|\le\frac{|x_1|}{2}} \left(\frac{x_2y_2}{|x_1-y_1|^2}\right)^{\frac{1}{4}}(|y_1|+y_2)^{-(m+2)}\,dy_1\,dy_2 \\
&\le Cx_2^{\frac{1}{4}}\int_{\frac{x_2}{2}}^{\infty}(|x_1|+y_2)^{-m-\frac{7}{4}}\,dy_2\int_{|x_1-y_1|\le\frac{|x_1|}{2}}|x_1-y_1|^{-\frac{1}{2}}\,dy_1 \\
&\le Cx_2^{\frac{1}{4}}(|x_1|+x_2)^{-m-\frac{3}{4}}|x_1|^{\frac{1}{2}} \le Cx_2^{\frac{1}{4}}(|x_1|+x_2)^{-m-\frac{1}{4}} \le Cx_2^{-m}.
\end{aligned}
\tag{4.5}
$$

On the other hand, in the case $N \ge 2$, by (2.4), (2.5) and (3.1) we see that

$$
G(x,y) \le C\frac{x_Ny_N}{|x_N-y_N|^{1+m}|x'-y'|^{N-1-m}} \le Cx_N^{-m}\frac{y_N}{|x'-y'|^{N-1-m}} \tag{4.6}
$$

for all $y \in \mathbb{R}^N_+$ with $0 \le y_N \le x_N/2$. Then we have

$$
\begin{aligned}
&\int_0^{\frac{x_N}{2}}\int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\le Cx_N^{-m}\int_0^{\frac{x_N}{2}}\int_{|x'-y'|\le\frac{|x'|}{2}} \frac{y_N}{|x'-y'|^{N-1-m}}|y|^{-(m+2)}\,dy'\,dy_N \\
&\le Cx_N^{-m}\int_0^{\frac{x_N}{2}}(|x'|+y_N)^{-(m+1)}\,dy_N\int_{|x'-y'|\le\frac{|x'|}{2}}|x'-y'|^{m+1-N}\,dy' \\
&\le Cx_N^{-m}|x'|^{-m}|x'|^m \le Cx_N^{-m}.
\end{aligned}
$$

This together with (4.4) and (4.5) implies that

$$
\int_0^{\infty}\int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \le Cx_N^{-m}. \tag{4.7}
$$

Next we prove that

$$
\int_0^{\infty}\int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \le Cx_N|x'|^{-m-1}. \tag{4.8}
$$

In the case $N \geq 3$, similarly to (4.4), by (2.4) we obtain

$$\begin{aligned}
&\int_0^{\frac{3x_N}{2}} \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\leq C \int_0^{\frac{3x_N}{2}} \int_{|x'-y'|\leq \frac{|x'|}{2}} |x'-y'|^{-(N-2)}(|y'|+y_N)^{-(m+2)}\,dy'\,dy_N \qquad (4.9)\\
&\leq C|x'| \int_0^{\frac{3x_N}{2}} (|x'|+y_N)^{-(m+2)}\,dy_N \leq Cx_N|x'|^{-m-1}.
\end{aligned}$$

In the case $N = 2$, it follows from (2.5) with $\alpha = 1$ that

$$G(x,y) \leq C\frac{x_2 y_2}{|x_2-y_2|^2} \leq C \qquad (4.10)$$

for all $y \in \mathbb{R}^2_+$ with $0 \leq y_2 \leq x_2/2$. This together with (4.2) yields

$$\begin{aligned}
\int_0^{\frac{x_2}{2}} \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy &\leq C \int_0^{\frac{x_2}{2}} \int_{|x_1-y_1|\leq \frac{|x_1|}{2}} (|y_1|+y_2)^{-(m+2)}\,dy_1\,dy_2 \qquad (4.11)\\
&\leq C|x_1| \int_0^{\frac{x_2}{2}} (|x_1|+y_2)^{-(m+2)}\,dy_2 \leq Cx_2|x_1|^{-m-1}.
\end{aligned}$$

Furthermore, it follows from (1.5) that

$$\begin{aligned}
&\int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\leq C \int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} \int_{|x_1-y_1|\leq \frac{|x_1|}{2}} \log\left(1+\frac{4x_2y_2}{|x_1-y_1|^2}\right)(|y_1|+y_2)^{-(m+2)}\,dy_1\,dy_2 \\
&\leq C \int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} (|x_1|+y_2)^{-(m+2)}\,dy_2 \int_0^{\frac{|x_1|}{2}} \log\left(1+\frac{6x_2^2}{r^2}\right)dr \\
&\leq C \int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} (|x_1|+y_2)^{-(m+2)}\,dy_2 \left(\frac{|x_1|}{2}\log\left(1+\frac{24x_2^2}{|x_1|^2}\right) + \int_0^{\frac{|x_1|}{2}} \frac{12x_2^2}{r^2+6x_2^2}\,dr\right).
\end{aligned}$$

Since $x \geq \log(1+x)$ for $x \geq 0$, we deduce in the case $N = 2$ that

$$\begin{aligned}
&\int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} \int_{D_1} G(x,y)f(y)^p\,dy \qquad (4.12)\\
&\leq Cx_2^2|x_1|^{-1}(|x_1|+x_2)^{-(m+1)} + Cx_2|x_1|(|x_1|+x_2)^{-(m+2)} \leq Cx_2|x_1|^{-m-1}.
\end{aligned}$$

On the other hand, in the case $N \geq 2$, by (2.4) and (2.5) with $\alpha = 1$ we have

$$G(x,y) \leq C\frac{x_N y_N}{|x-y|^N} \qquad (4.13)$$

for all $y \in \mathbb{R}^N_+$. Since $|y'| \ge |x'|/2$ for $y' \in D_1$, by (4.13) we have

$$\begin{aligned}
&\int_{\frac{3x_N}{2}}^{\infty}\int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\le C\int_{\frac{3x_N}{2}}^{\infty}\int_{|x'-y'|\le\frac{|x'|}{2}} \frac{x_N y_N}{|x-y|^N}(|y'|+y_N)^{-(m+2)}\,dy'\,dy_N \\
&\le Cx_N\int_{\frac{3x_N}{2}}^{\infty}\int_{|x'-y'|\le\frac{|x'|}{2}} y_N(|x'|+y_N)^{-(m+2)}(|x'-y'|+y_N-x_N)^{-N}\,dy'\,dy_N \\
&\le Cx_N\int_{\frac{3x_N}{2}}^{\infty}\int_0^{\infty} y_N(|x'|+y_N)^{-(m+2)}(r+y_N-x_N)^{-N}r^{N-2}\,dr\,dy_N.
\end{aligned}$$

So we obtain

$$\begin{aligned}
&\int_{\frac{3x_N}{2}}^{\infty}\int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\le Cx_N\int_{\frac{3x_N}{2}}^{\infty}\int_0^{\infty} y_N(|x'|+y_N)^{-(m+2)}(r+y_N-x_N)^{-2}\,dr\,dy_N \\
&\le Cx_N\int_{\frac{3x_N}{2}}^{\infty}(|x'|+y_N)^{-(m+2)}\frac{y_N}{y_N-x_N}\,dy_N \\
&\le Cx_N\int_0^{\infty}(|x'|+y_N)^{-(m+2)}\,dy_N \le Cx_N|x'|^{-m-1}.
\end{aligned}$$

This together with (4.9), (4.11) and (4.12) implies (4.8).

We deduce from (4.7) and (4.8) that

$$\int_0^{\infty}\int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \le \ Cx_N \min\left\{|x'|^{-m-1}, x_N^{-m-1}\right\}.$$

This together with (3.14) implies that (4.3) with $k=1$.

Proof in the case $k=2$. Similarly to the case $k=1$, it suffices to consider the case $|x'|\neq 0$. We have

$$|x-y|^2+2x_Ny_N \ge \sqrt{|x-y|^2+2x_Ny_N}\sqrt{2x_Ny_N} \ge |x'-y'|x_N$$

if $y_N \ge x_N/2$ and

$$|x-y|^2+2x_Ny_N \ge |x-y|^2 \ge |x'-y'||x_N-y_N| \ge \frac{1}{2}|x'-y'|x_N$$

if $0 \le y_N < x_N/2$. Then, in the case $N\ge 3$, it follows from (2.4) that

$$G(x,y) \le C\frac{y_N}{|x'-y'|^{N-1}} \le C\frac{y_N}{|x'|^{N-1}}, \qquad y' \in D_2.$$

Since $D_2 \subset \{y' \in \mathbb{R}^{N-1} : |y'| \le 3|x'|\}$, we have

$$
\begin{aligned}
&\int_{\frac{x_N}{2}}^{\infty} \int_{D_2} G(x,y)|y|^{-(m+2)}\,dy \\
&\le C|x'|^{-(N-1)} \int_{\frac{x_N}{2}}^{\infty} \int_{\frac{|x'|}{2}<|x'-y'|\le 2|x'|} y_N |y|^{-(m+2)}\,dy'\,dy_N \\
&\le C|x'|^{-(N-1)} \int_{\frac{x_N}{2}}^{\infty} \int_0^{3|x'|} (r+y_N)^{-(m+1)} r^{N-2}\,dr\,dy_N \\
&\le C \int_{\frac{x_N}{2}}^{\infty} y_N^{-(m+1)}\,dy_N \le C x_N^{-m}.
\end{aligned}
\tag{4.14}
$$

In the case $N = 2$, it follows from (2.5) with $\alpha = 1/2$ that

$$
\begin{aligned}
&\int_{\frac{x_2}{2}}^{\infty} \int_{D_2} G(x,y)|y|^{-(m+2)}\,dy \\
&\le C x_2^{\frac{1}{2}} \int_{\frac{x_2}{2}}^{\infty} \int_{\frac{|x_1|}{2}<|x_1-y_1|\le 2|x_1|} y_2^{\frac{1}{2}} |x_1-y_1|^{-1} |y|^{-(m+2)}\,dy_1\,dy_2 \\
&\le C x_2^{\frac{1}{2}} |x_1|^{-1} \int_{\frac{x_2}{2}}^{\infty} \int_0^{3|x_1|} (r+y_2)^{-m-\frac{3}{2}}\,dr\,dy_2 \\
&\le C x_2^{\frac{1}{2}} \int_{\frac{x_2}{2}}^{\infty} y_2^{-m-\frac{3}{2}}\,dy_2 \le C x_2^{-m}.
\end{aligned}
\tag{4.15}
$$

On the other hand, in the case $N \ge 2$, by (3.1) and (4.6) we have

$$
\begin{aligned}
&\int_0^{\frac{x_N}{2}} \int_{D_2} G(x,y)|y|^{-(m+2)}\,dy \\
&\le C x_N^{-m} \int_0^{\frac{x_N}{2}} \int_{D_2} \frac{y_N}{|x'-y'|^{N-1-m}} |y|^{-(m+2)}\,dy'\,dy_N \\
&\le C x_N^{-m} |x'|^{-(N-1)+m} \int_0^{3|x'|} \int_0^{\infty} r^{N-2} (r+y_N)^{-(m+1)}\,dy_N\,dr \\
&\le C x_N^{-m} |x'|^{-(N-1)+m} \int_0^{3|x'|} r^{N-m-2}\,dr \\
&\le C x_N^{-m} |x'|^{-(N-1)+m} |x'|^{N-1-m} \le C x_N^{-m}.
\end{aligned}
$$

This together with (4.14) and (4.15) implies that

$$
\int_0^{\infty} \int_{D_2} G(x,y)|y|^{-(m+2)}\,dy \le C x_N^{-m}. \tag{4.16}
$$

Furthermore, it follows from (4.13) that

$$
G(x,y) \le C \frac{x_N y_N}{|x'-y'|^N} \le C \frac{x_N y_N}{|x'|^N}, \qquad y' \in D_2,
$$

and we obtain

$$\begin{aligned}
&\int_0^\infty \int_{D_2} G(x,y)|y|^{-(m+2)}\,dy \\
&\le Cx_N|x'|^{-N}\int_0^\infty\int_0^{3|x'|} y_N(r+y_N)^{-(m+2)}r^{N-2}\,dr\,dy_N \\
&\le Cx_N|x'|^{-N}\int_0^\infty\int_0^{3|x'|} (r+y_N)^{-(m+1)}r^{N-2}\,dr\,dy_N \\
&\le Cx_N|x'|^{-N}\int_0^{3|x'|} r^{N-m-2}\,dr \le Cx_N|x'|^{-m-1-\varepsilon}\int_0^{3|x'|}|x'|^{-N+m+1+\varepsilon}r^{N-m-2}\,dr \\
&\le Cx_N|x'|^{-m-1-\varepsilon}\int_0^{3|x'|} r^{-1+\varepsilon}\,dr \le Cx_N|x'|^{-m-1},
\end{aligned}$$

where ε is as in (3.9). This together with (4.16) yields

$$\int_0^\infty \int_{D_2} G(x,y)|y|^{-(m+2)}\,dy \le Cx_N \min\left\{|x'|^{-m-1}, x_N^{-m-1}\right\}.$$

Therefore, by (3.14) we obtain (4.3) with $k=2$.

Proof in the case $k=3$. Since

$$|y'| \ge |x'-y'| - |x'| > 0, \qquad y \in D_3, \tag{4.17}$$

by (4.13) we have

$$\begin{aligned}
&\int_0^\infty \int_{D_3} G(x,y)|y|^{-(m+2)}\,dy \\
&\le Cx_N\int_0^\infty\int_{|x'-y'|>2|x'|} y_N|x'-y'|^{-N}(|y'-x'|-|x'|+y_N)^{-(m+2)}\,dy'\,dy_N \\
&\le Cx_N\int_0^\infty\int_{2|x'|}^\infty r^{-2}(r-|x'|+y_N)^{-(m+1)}\,dr\,dy_N \\
&\le Cx_N\int_0^\infty (|x'|+y_N)^{-(m+1)}\left(\int_{2|x'|}^\infty r^{-2}\,dr\right)dy_N \le Cx_N|x'|^{-m-1}.
\end{aligned} \tag{4.18}$$

It follows from (2.4), (2.5) and (4.17) that

$$\begin{aligned}
&\int_{\frac{x_N}{2}}^\infty \int_{D_3} G(x,y)|y|^{-(m+2)}\,dy \\
&\le C\int_{\frac{x_N}{2}}^\infty\int_{|x'-y'|>2|x'|} |x'-y'|^{-(N-2)}(|y'-x'|-|x'|+y_N)^{-(m+2)}\,dy'\,dy_N \\
&\le C\int_{\frac{x_N}{2}}^\infty\int_{2|x'|}^\infty (r-|x'|+y_N)^{-(m+2)}\,dr\,dy_N \\
&\le C\int_{\frac{x_N}{2}}^\infty (|x'|+y_N)^{-(m+1)}\,dy_N \le C|x|^{-m} \le Cx_N^{-m} \quad \text{if} \quad N\ge 3,
\end{aligned} \tag{4.19}$$

and

$$
\begin{aligned}
&\int_{\frac{x_2}{2}}^{\infty}\int_{D_3} G(x,y)|y|^{-(m+2)}\,dy \\
&\le C\int_{\frac{x_2}{2}}^{\infty}\int_{|x_1-y_1|>2|x_1|}\left(\frac{x_2y_2}{|x_2-y_2|^2}\right)^{\alpha}(|y_1-x_1|-|x_1|+y_2)^{-(m+2)}\,dy_1\,dy_2 \\
&\le C\int_{\frac{x_2}{2}}^{\infty}\int_{2|x_1|}^{\infty}\left(\frac{x_2}{|x_2-y_2|^2}\right)^{\alpha}(r-|x_1|+y_2)^{-(m+2)+\alpha}\,dr\,dy_2 \\
&\le C\biggl(\int_{\frac{x_2}{2}}^{2x_2}+\int_{2x_2}^{\infty}\biggr)\left(\frac{x_2}{|x_2-y_2|^2}\right)^{\alpha}(|x_1|+y_2)^{-(m+1)+\alpha}\,dy_2 \\
&\le Cx_2^{\alpha}|x|^{-(m+1)+\alpha}\int_{\frac{x_2}{2}}^{2x_2}|x_2-y_2|^{-2\alpha}\,dy_2+Cx_2^{-\alpha}\int_{2x_2}^{\infty}(|x_1|+y_2)^{-(m+1)+\alpha}\,dy_2 \\
&\le Cx_2^{1-\alpha}|x|^{-(m+1)+\alpha}+Cx_2^{-\alpha}|x|^{-m+\alpha}\le Cx_2^{-m}\quad\text{if}\quad N=2,
\end{aligned}
\tag{4.20}
$$

where $\alpha\in(0,1/2)$ with $\alpha<m$.

On the other hand, it follows from (2.4) and (2.5) with $\alpha=1$ that

$$
G(x,y)\le C\frac{x_Ny_N}{|x_N-y_N|^{1+m+\varepsilon}|x'-y'|^{N-1-m-\varepsilon}}\le Cx_N^{-m-\varepsilon}\frac{y_N}{|x'-y'|^{N-1-m-\varepsilon}}
$$

for all $y\in\mathbb{R}^N_+$ with $0\le y_N\le x_N/2$, where ε is as (3.9). This together with (4.17) yields

$$
\begin{aligned}
&\int_0^{\frac{x_N}{2}}\int_{D_3}G(x,y)|y|^{-(m+2)}\,dy'\,dy_N \\
&\le Cx_N^{-m-\varepsilon}\int_0^{\frac{x_N}{2}}\int_{|x'-y'|>2|x'|}\frac{y_N}{|x'-y'|^{N-1-m-\varepsilon}}(|x'-y'|-|x'|+y_N)^{-(m+2)}\,dy' \\
&\le Cx_N^{-m-\varepsilon}\int_0^{\frac{x_N}{2}}\int_{2|x'|}^{\infty}y_Nr^{-1+m+\varepsilon}(r-|x'|+y_N)^{-(m+2)}\,dr\,dy_N \\
&\le Cx_N^{-m-\varepsilon}\int_0^{\frac{x_N}{2}}\biggl(\int_{2|x'|}^{2|x'|+y_N}+\int_{2|x'|+y_N}^{\infty}\biggr)y_Nr^{-1+\varepsilon}(r-|x'|+y_N)^{-2}\,dr\,dy_N \\
&=: Cx_N^{-m-\varepsilon}\int_0^{\frac{x_N}{2}}(I_1(y_N)+I_2(y_N))\,dy_N.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
I_1(y_N)&\le C(|x'|+y_N)^{-1}\int_{2|x'|}^{2|x'|+y_N}r^{-1+\varepsilon}\,dr\le C(|x'|+y_N)^{-1+\varepsilon}, \\
I_2(y_N)&\le C\int_{2|x'|+y_N}^{\infty}(r-|x'|+y_N)^{-2+\varepsilon}\,dr\le C(|x'|+y_N)^{-1+\varepsilon}.
\end{aligned}
$$

These imply that

$$\begin{aligned}&\int_0^{\frac{x_N}{2}}\int_{D_3} G(x,y)|y|^{-(m+2)}\,dy'\,dy_N\\&\le Cx_N^{-m-\varepsilon}\int_0^{\frac{x_N}{2}}(|x'|+y_N)^{-1+\varepsilon}\,dy_N\le Cx_N^{-m-\varepsilon}\int_0^{\frac{x_N}{2}}y_N^{-1+\varepsilon}\,dy_N\le Cx_N^{-m}.\end{aligned}\tag{4.21}$$

Therefore, by (4.19), (4.20) and (4.21) we obtain

$$\int_0^\infty\int_{D_3} G(x,y)|y|^{-(m+2)}\,dy_1\,dy_2\le Cx_N^{-m}.\tag{4.22}$$

We deduce from (3.14), (4.18) and (4.22) that

$$\int_0^\infty\int_{D_3} G(x,y)|y|^{-(m+2)}\,dy_1\,dy_2\le Cx_N\min\{|x'|^{-m-1},x_N^{-m-1}\}\le Cx_N|x|^{-m-1},$$

which implies (4.3) with $k=3$. Thus (4.3) holds for $k=1,2,3$, and the proof of Lemma 4.1 is complete. □

Lemma 4.2. *Let $p>p_*$. Assume that*

$$0\le f(x)\le\begin{cases}|x|^{-m}, & x\in D_{\rm in},\\ (1+x_N)|x|^{-N}, & x\in D_{\rm out}.\end{cases}\tag{4.23}$$

Then there exists a positive constant C such that

$$[(-\Delta_D)^{-1}f^p](x)\le Cx_N|x|^{-N},\qquad x\in D_{\rm out}.$$

Proof. Let $x=(x',x_N)\in D_{\rm out}$. We divide $\mathbb{R}^N_+$ into the following seven sets:

$$\begin{aligned}D_1&:=\left\{y\in\mathbb{R}^N_+:|y'|\le\frac14,\quad y_N\le\frac{1+x_N}{4}\right\},\\
D_2&:=\left\{y\in\mathbb{R}^N_+:|y'|\le\frac14,\quad |y'|\le\frac{|x'|}{2},\quad y_N\ge\frac{1+x_N}{4}\right\},\\
D_3&:=\left\{y\in\mathbb{R}^N_+:|y'|\le\frac14,\quad \frac{|x'|}{2}\le|y'|\le\frac{3|x'|}{2},\quad y_N\ge\frac{1+x_N}{4}\right\},\\
D_4&:=\left\{y\in\mathbb{R}^N_+:|y'|\le\frac14,\quad |y'|\ge\frac{3|x'|}{2},\quad y_N\ge\frac{1+x_N}{4}\right\},\\
D_5&:=\left\{y\in\mathbb{R}^N_+:\frac14\le|y'|\le\frac{|x'|}{2}\right\},\\
D_6&:=\left\{y\in\mathbb{R}^N_+:|y'|\ge\frac14,\quad \frac{|x'|}{2}\le|y'|\le\frac{3|x'|}{2}+1+x_N,\right\},\\
D_7&:=\left\{y\in\mathbb{R}^N_+:|y'|\ge\frac{3|x'|}{2}+1+x_N\right\}.\end{aligned}$$

Then, by (4.1) it suffices to prove that

$$\int_{D_k} G(x,y)f(y)^p\,dy\le Cx_N|x|^{-N},\quad k=1,\dots,7.\tag{4.24}$$

Here we recall that $mp=m+2$.

Proof in the case $k = 1$. By (3.1) and (4.23) we have

$$0 \le f(x) \le C|x|^{-m}, \qquad x \in \mathbb{R}^N_+. \tag{4.25}$$

For any $y \in D_1$, since

$$|x-y| \ge |x| - |y| \ge |x| - \left(\frac{1}{2} + \frac{x_N}{4}\right) \ge \frac{3|x|}{4} - \frac{1}{2} \ge \frac{|x|}{4},$$

by (4.13) we see that

$$G(x,y) \le C\frac{x_N y_N}{|x-y|^N} \le Cx_N|x|^{-N} y_N, \qquad y \in D_1. \tag{4.26}$$

Then it follows from (3.1), (4.25) and (4.26) that

$$\begin{aligned}
\int_{D_1} G(x,y) f(y)^p \, dy &\le Cx_N|x|^{-N} \int_{D_1} y_N |y|^{-(m+2)} \, dy \\
&\le Cx_N|x|^{-N} \int_{|y'| \le \frac{1}{4}} \int_0^{\frac{1+x_N}{4}} (|y'| + y_N)^{-(m+1)} \, dy_N \, dy' \\
&\le Cx_N|x|^{-N} \int_{|y'| \le \frac{1}{4}} |y'|^{-m} \, dy' \le Cx_N|x|^{-N} \int_0^{\frac{1}{4}} r^{-m+N-2} \, dr \le Cx_N|x|^{-N}.
\end{aligned}$$

This implies (4.24) with $k = 1$.

Proof in the case $k = 2$. For any $y \in D_2$, since

$$|x' - y'| \ge |x'| - |y'| \ge \frac{|x'|}{2},$$

by (4.13) we have

$$G(x,y) \le C\frac{x_N y_N}{|x-y|^N} \le C\frac{x_N y_N}{|x'-y'|^N} \le Cx_N y_N |x'|^{-N}, \qquad y \in D_2. \tag{4.27}$$

On the other hand, by (3.1) and (4.23) we have

$$0 \le f(y) \le C(1+y_N)|y|^{-N} \le Cy_N|y|^{-N}, \qquad y \in D_2 \cup D_3 \cup D_4. \tag{4.28}$$

Furthermore, it follows from $p > p_*$ that

$$-Np + p + N + 1 < 0. \tag{4.29}$$

Then, by (4.27), (4.28) and (4.29) we obtain

$$\begin{aligned}
\int_{D_2} G(x,y) f(y)^p \, dy &\le Cx_N|x'|^{-N} \int_{D_2} y_N^{p+1} |y|^{-Np} \, dy \\
&\le Cx_N|x'|^{-N} \int_{|y'| \le \frac{|x'|}{2}} \int_{\frac{1+x_N}{4}}^{\infty} (|y'| + y_N)^{-Np+p+1} \, dy_N \, dy' \\
&\le Cx_N|x'|^{-N} \int_{|y'| \le \frac{|x'|}{2}} (1 + x_N + |y'|)^{-Np+p+2} \, dy' \\
&\le Cx_N|x'|^{-N} \int_0^{\frac{|x'|}{2}} (1+r)^{-Np+p+N} \, dr \le Cx_N|x'|^{-N}.
\end{aligned} \tag{4.30}$$

On the other hand, in the case $N \ge 3$, by (2.4) we see that

$$G(x,y) \le C|x'-y'|^{-(N-2)} \le C|x'|^{-(N-2)}, \qquad y \in D_2.$$

Then it follows from (4.28) and (4.29) that

$$
\begin{aligned}
\int_{D_2} G(x,y)f(y)^p\,dy &\le C|x'|^{-(N-2)}\int_{D_2} y_N^p|y|^{-Np}\,dy\\
&\le C|x'|^{-(N-2)}\int_{|y'|\le\frac{|x'|}{2}}\int_{\frac{1+x_N}{4}}^{\infty}(|y'|+y_N)^{-Np+p}\,dy_N\,dy'\\
&\le C|x'|^{-(N-2)}\int_{|y'|\le\frac{|x'|}{2}}(1+|y'|+x_N)^{-Np+p+1}\,dy'\\
&\le C|x'|^{-(N-2)}\int_0^{\frac{|x'|}{2}} r^{N-2}(1+r+x_N)^{-Np+p+1}\,dr\\
&\le C(1+x_N)^{-Np+p+2}\le C(1+x_N)^{-(N-1)}\le Cx_N^{-(N-1)}.
\end{aligned}
\tag{4.31}
$$

Furthermore, in the case $N=2$, since $p>3$, by (2.5) with $\alpha\in(0,1/2)$ and (4.28) we obtain

$$
\begin{aligned}
\int_{D_2} G(x,y)f(y)^p\,dy &\le C\int_{D_2}\left(\frac{x_2y_2}{|x_2-y_2|^2}\right)^{\alpha} y_2^p|y|^{-2p}\,dy\\
&\le Cx_2^{\alpha}\int_{\frac{1+x_2}{4}}^{\infty}\int_0^{\frac{|x_1|}{2}}|x_2-y_2|^{-2\alpha}(r+y_2)^{-p+\alpha}\,dr\,dy_2\\
&\le Cx_2^{\alpha}\left(\int_{\frac{1+x_2}{4}}^{1+2x_2}+\int_{1+2x_2}^{\infty}\right)|x_2-y_2|^{-2\alpha}y_2^{-p+\alpha+1}\,dy_2\\
&\le Cx_2^{\alpha}\Big((1+x_2)^{1-2\alpha}(1+x_2)^{-p+\alpha+1}+(1+x_2)^{-2\alpha}(1+x_2)^{-p+\alpha+2}\Big)\\
&\le C(1+x_2)^{-p+2}\le C(1+x_2)^{-1}\le Cx_2^{-1}.
\end{aligned}
$$

This together with (3.14), (4.30) and (4.31) yields

$$
\int_{D_2} G(x,y)f(y)^p\,dy\le Cx_N\min\{|x'|^{-N},x_N^{-N}\}\le Cx_N|x|^{-N}.
$$

Thus (4.24) holds for $k=2$.

Proof in the case $k=3$. In the case $N\ge 3$, by (2.4) we have

$$
G(x,y)\le C\frac{x_Ny_N}{|x'-y'|^{N-2}(x_N+y_N)^2},\qquad y\in\mathbb{R}^N_+.
\tag{4.32}
$$

Then, since $|x'-y'| \le C|x'|$ for any $y \in D_3$, by (4.28) and (4.29) we see that

$$
\begin{aligned}
&\int_{D_3} G(x,y) f(y)^p\,dy \\
&\le C x_N \int_{\frac{1+x_N}{4}}^{\infty} \int_{\frac{|x'|}{2} \le |y'| \le \frac{3|x'|}{2}} \frac{y_N^{p+1}}{|x'-y'|^{N-2}(x_N+y_N)^2} |y|^{-Np}\,dy'\,dy_N \\
&\le C x_N \int_{\frac{1+x_N}{4}}^{\infty} \int_{|x'-y'| \le C|x'|} y_N^{p-1}(|x'|+y_N)^{-Np}|x'-y'|^{N-2}\,dy'\,dy_N \\
&\le C x_N \int_{\frac{1+x_N}{4}}^{\infty} \int_0^{C|x'|} y_N^{p-1}(|x'|+y_N)^{-Np} r^{-(N-2)} r^{N-2}\,dr\,dy_N \\
&\le C x_N \int_{\frac{1+x_N}{4}}^{\infty} y_N^{p-1}(|x'|+y_N)^{-Np+1}\,dy_N \le C x_N \int_{\frac{1+x_N}{4}}^{\infty} (|x'|+y_N)^{-Np+p}\,dy_N \\
&\le C x_N (1+|x|)^{-Np+p+1} \le C x_N |x|^{-N}.
\end{aligned}
\tag{4.33}
$$

In the case $N=2$, for any $\alpha \in (0,1]$, by (2.5) we obtain

$$
G(x,y) \le C \left(\frac{x_2 y_2}{|x_1-y_1||x_2-y_2|} \right)^{\alpha}, \qquad y \in \mathbb{R}^2_+.
$$

For any $y \in D_3$, since $x \in D_{\mathrm{out}}$ and $|x_1|/2 \le |y_1| \le 1/4$, we see that $x_2 \ge 1-|x_1| \ge 1/2$. This implies that $x_2^\alpha \le 2x_2$ for any $\alpha \in (0,1)$ and $x_2 \ge (1+x_2)/4$. Then, since $p>3$, by (4.28) we have

$$
\begin{aligned}
&\int_{D_3} G(x,y) f(y)^p\,dy \\
&\le C x_2^\alpha \int_{\frac{1+x_2}{4}}^{\infty} \int_{\frac{|x_1|}{2} \le |y_1| \le \frac{3|x_1|}{2}} \left(\frac{y_2}{|x_1-y_1||x_2-y_2|} \right)^{\alpha} y_2^p |y|^{-2p}\,dy_1\,dy_2 \\
&\le C x_2^\alpha \int_{\frac{1+x_2}{4}}^{\infty} \int_{|x_1-y_1| \le C|x_1|} y_2^{p+\alpha}|x_2-y_2|^{-\alpha}(|x_1|+y_2)^{-2p}|x_1-y_1|^{-\alpha}\,dy_1\,dy_2 \\
&\le C x_2^\alpha \left(\int_{\frac{1+x_2}{4}}^{2x_2} + \int_{2x_2}^{\infty} \right) y_2^\alpha |x_2-y_2|^{-\alpha}(|x_1|+y_2)^{-p+1-\alpha}\,dy_2 \\
&\le C x_2^\alpha \left(|x|^{-p+1} \int_{\frac{1+x_2}{4}}^{2x_2} |x_2-y_2|^{-\alpha}\,dy_2 + \int_{2x_2}^{\infty} (|x_1|+y_2)^{-p+1-\alpha}\,dy_2 \right) \\
&\le C x_2 |x|^{-2} \left(|x|^{-p+3} + |x|^{-p+4-\alpha} \right) \le C x_2 |x|^{-2},
\end{aligned}
\tag{4.34}
$$

where $\alpha \in (0,1)$ with $-p+4<\alpha$. This together with (4.33) implies that (4.24) with $k=3$.

Proof in the case $k=4$. For any $y \in D_4$, since

$$
|x'-y'| \ge |y'| - |x'| \ge \frac{|x'|}{2},
$$

by (4.13) we obtain

$$G(x,y) \le C\frac{x_N y_N}{|x'-y'|^N} \le C x_N y_N |x'|^{-N}. \tag{4.35}$$

Then it follows from (4.28) and (4.29) that

$$\begin{aligned}\int_{D_4} G(x,y) f(y)^p\,dy &\le C x_N |x'|^{-N} \int_{\frac{1+x_N}{4}}^{\infty} \int_{\frac{3|x'|}{2}\le|y'|\le\frac{1}{4}} y_N^{p+1}|y|^{-Np}\,dy'\,dy_N \\ &\le C x_N |x'|^{-N} \int_{\frac{1}{4}}^{\infty}\int_0^{\frac{1}{4}} y_N^{p+1}(r+y_N)^{-Np+N-2}\,dr\,dy_N \\ &\le C x_N |x'|^{-N}\int_{\frac{1}{4}}^{\infty} y_N^{-Np+p+N}\,dy_N \le C x_N |x'|^{-N}.\end{aligned} \tag{4.36}$$

For any $y \in D_4$, since $|x'| \le 1/6$, we see that

$$\left\{y' \in \mathbb{R}^{N-1} : \frac{3|x'|}{2} \le |y'| \le \frac{1}{4}\right\} \subset \left\{y' \in \mathbb{R}^{N-1} : |x'-y'| \le 1\right\}. \tag{4.37}$$

Then, in the case $N \ge 3$, by (4.28), (4.29), (4.32) and (4.37) we have

$$\begin{aligned}&\int_{D_4} G(x,y) f(y)^p\,dy \\ &\le C x_N \int_{\frac{1+x_N}{4}}^{\infty}\int_{\frac{3|x'|}{2}\le|y'|\le\frac{1}{4}} \frac{y_N^{p+1}}{|x'-y'|^{N-2}(x_N+y_N)^2}|y|^{-Np}\,dy'\,dy_N \\ &\le C x_N \int_{\frac{1+x_N}{4}}^{\infty}\int_{|x'-y'|\le 1} (|x'|+y_N)^{-Np+p-1}|x'-y'|^{-N+2}\,dy'\,dy_N \\ &\le C x_N \int_{\frac{1+x_N}{4}}^{\infty} (|x'|+y_N)^{-Np+p-1}\,dy_N \le C x_N\left(|x'|+\frac{1+x_N}{4}\right)^{-Np+p} \\ &\le C x_N\left(|x'|+\frac{1+x_N}{4}\right)^{-N} \le C x_N^{-(N-1)}.\end{aligned} \tag{4.38}$$

On the other hand, in the case $N = 2$, for $y \in D_4$, we have $|x_1| \le 1/6$ and it follows from $x \in D_{\rm out}$ that $x_2 \ge 1 - |x_1| \ge 5/6$ and $2x_2 \ge (1+x_2)/4$. Then, similarly to (4.34), by (4.37) we have

$$\begin{aligned}
&\int_{D_4} G(x,y)f(y)^p\,dy \\
&\le Cx_2^\alpha \int_{\frac{1+x_2}{4}}^\infty \int_{\frac{3|x_1|}{2}\le|y_1|\le\frac14} \left(\frac{y_2}{|x_1-y_1||x_2-y_2|}\right)^\alpha y_2^p|y|^{-2p}\,dy_1\,dy_2 \\
&\le Cx_2^\alpha \int_{\frac{1+x_2}{4}}^\infty \int_{|x_1-y_1|\le1} y_2^{-p+\alpha}|x_2-y_2|^{-\alpha}|x_1-y_1|^{-\alpha}\,dy_1\,dy_2 \\
&\le Cx_2^\alpha \left(\int_{\frac{1+x_2}{4}}^{2x_2} + \int_{2x_2}^\infty\right) y_2^{-p+\alpha}|x_2-y_2|^{-\alpha}\,dy_2 \\
&\le Cx_2^\alpha \left(x_2^{-p+\alpha}\int_{\frac{x_2}{4}}^{2x_2}|x_2-y_2|^{-\alpha}\,dy_2 + \int_{2x_2}^\infty y_2^{-p}\,dy_2\right) \\
&\le Cx_2^{-p+1+\alpha} \le Cx_2^{-1},
\end{aligned}$$

where $\alpha\in(0,1)$. This together with (3.14), (4.36) and (4.38) implies that

$$\int_{D_4} G(x,y)f(y)^p\,dy \le Cx_N\min\{|x'|^{-N}, x_N^{-N}\} \le Cx_N|x|^{-N}.$$

Thus (4.24) holds for $k=4$.

Proof in the case $k=5$. Since $x\in D_{\text{out}}$, we have

$$\begin{aligned}
|x-y| &\ge |x| - (|y'|+y_N) \ge |x| - \frac{|x'|}{2} - \frac{1+x_N}{4} \\
&\ge |x| - \frac{|x'|}{4} - \frac{\sqrt2|x|}{4} - \frac14 \ge \frac{3-\sqrt2}{4}|x| - \frac14 \ge \frac{2-\sqrt2}{4}|x|
\end{aligned}$$

for $y\in D_5$ with $0\le y_N\le(1+x_N)/4$. Then it follows from (4.13) that

$$G(x,y)\le C\frac{x_Ny_N}{|x-y|^N}\le Cx_Ny_N|x|^{-N}$$

for $y\in D_5$ with $0\le y_N\le(1+x_N)/4$. By (4.23) and (4.29) we obtain

$$\begin{aligned}
&\int_{D_5\cap\{0\le y_N\le\frac{1+x_N}{4}\}} G(x,y)f(y)^p\,dy \\
&\le Cx_N|x|^{-N}\int_0^{\frac{1+x_N}{4}}\int_{\frac14\le|y'|\le\frac{|x'|}{2}} y_N(1+y_N)^p|y|^{-Np}\,dy'\,dy_N \\
&\le Cx_N|x|^{-N}\int_0^{\frac{1+x_N}{4}}\int_{\frac14\le r\le\frac{|x'|}{2}} (1+y_N)^{p+1}(y_N+r)^{-Np+N-2}\,dr\,dy_N \\
&\le Cx_N|x|^{-N}\int_0^\infty(1+y_N)^{-Np+N+p}\,dy_N \le Cx_N|x|^{-N}.
\end{aligned}\tag{4.39}$$

On the other hand, since $|x'-y'|\ge|x'|/2$ for any $y\in D_5$, similarly to (4.36), by (4.23), (4.29) and (4.35) we obtain

$$
\begin{aligned}
&\int_{D_5\cap\{y_N\ge\frac{1+x_N}{4}\}} G(x,y)f(y)^p\,dy\\
&\le Cx_N|x'|^{-N}\int_{\frac{1+x_N}{4}}^\infty\int_{\frac14\le|y'|\le\frac{|x'|}{2}} y_N(1+y_N)^p|y|^{-Np}\,dy'\,dy_N\\
&\le Cx_N|x'|^{-N}\int_{\frac14}^\infty\int_{\frac14\le r\le\frac{|x'|}{2}} y_N^{p+1}(r+y_N)^{-Np+N-2}\,dr\,dy_N\\
&\le Cx_N|x'|^{-N}\int_{\frac14}^\infty y_N^{-Np+N+p}\,dy'\,dy_N\le Cx_N|x'|^{-N}.
\end{aligned}
\tag{4.40}
$$

Furthermore, in the case $N\ge3$, by (4.23), (4.29) and (4.32) we have

$$
\begin{aligned}
&\int_{D_5\cap\{y_N\ge\frac{1+x_N}{4}\}} G(x,y)f(y)^p\,dy\\
&\le Cx_N\int_{\frac{1+x_N}{4}}^\infty\int_{\frac14\le|y'|\le\frac{|x'|}{2}}\frac{y_N}{|x'-y'|^{N-2}(x_N+y_N)^2}(1+y_N)^p|y|^{-Np}\,dy'\,dy_N\\
&\le Cx_N(1+x_N)^{-1}\int_{\frac{1+x_N}{4}}^\infty\int_{\frac14}^{\frac{|x'|}{2}}(r+y_N)^{-Np}(1+y_N)^p\,dr\,dy_N\\
&\le Cx_N(1+x_N)^{-1}\int_{\frac{1+x_N}{4}}^\infty(1+y_N)^{-Np+p+1}\,dy_N\\
&\le Cx_N(1+x_N)^{-Np+p+1}\le Cx_N(1+x_N)^{-N}\le Cx_N^{-(N-1)}.
\end{aligned}
\tag{4.41}
$$

In the case where $N=2$ and $x_2\ge1/7$, since $2x_2\ge(1+x_2)/4$ and $p>3$, it follows from (2.5) and (4.23) that

$$
\begin{aligned}
&\int_{D_5\cap\{y_2\ge\frac{1+x_2}{4}\}} G(x,y)f(y)^p\,dy\\
&\le Cx_2^\alpha\int_{\frac{1+x_2}{4}}^\infty\int_{\frac14\le|y_1|\le\frac{|x_1|}{2}}\left(\frac{y_2}{|x_2-y_2|^2}\right)^\alpha(1+y_2)^p|y|^{-2p}\,dy_1\,dy_2\\
&\le Cx_2^\alpha\int_{\frac{1+x_2}{4}}^\infty\int_{\frac14}^{\frac{|x_1|}{2}} y_2^{p+\alpha}|x_2-y_2|^{-2\alpha}(r+y_2)^{-2p}\,dr\,dy_2\\
&\le Cx_2^\alpha\left(\int_{\frac{1+x_2}{4}}^{2x_2}+\int_{2x_2}^\infty\right)y_2^{-p+\alpha+1}|x_2-y_2|^{-2\alpha}\,dy_2\le Cx_2^{-p+2}\le Cx_2^{-1},
\end{aligned}
\tag{4.42}
$$

where $\alpha \in (0, 1/2)$. In the case where $N = 2$ and $x_2 \le 1/7$, since $2x_2 \le (1 + x_2)/4$ and $p > 3$, by (4.13) and (4.23) we have

$$\begin{aligned}
&\int_{D_5 \cap \{y_2 \ge \frac{1+x_2}{4}\}} G(x, y) f(y)^p \, dy \\
&\le C x_2 \int_{\frac{1+x_2}{4}}^{\infty} \int_{\frac{1}{4} \le |y_1| \le \frac{|x_1|}{2}} \frac{y_2 (1 + y_2)^p}{|x_2 - y_2|^2} |y|^{-2p} \, dy_1 \, dy_2 \\
&\le C x_2 \int_{2x_2}^{\infty} \int_{\frac{1}{4}}^{\infty} (y_2 - x_2)^{-2} (r + y_2)^{-p+1} \, dr \, dy_2 \\
&\le C x_2^{-1} \int_{2x_2}^{\infty} (1 + y_2)^{-p+2} \, dy_2 \le C x_2^{-1} (1 + x_2)^{-p+3} \le C x_2^{-1}.
\end{aligned}$$

This together with (3.14), (4.40), (4.41) and (4.42) yields

$$\int_{D_5 \cap \{y_N \ge \frac{1+x_N}{4}\}} G(x, y) f(y)^p \, dy \le C x_N \min\{|x'|^{-N}, x_N^{-N}\} \le C x_N |x|^{-N}. \tag{4.43}$$

Therefore, by (4.39) and (4.43) we obtain (4.24) with $k = 5$.

Proof in the case $k = 6$. We recall that

$$D_6 \subset \left\{ y \in \mathbb{R}^N_+ : 0 \le |x' - y'| \le \frac{5|x'|}{2} + 1 + x_N \right\}.$$

Put

$$A = \max\left\{ \frac{1}{4}, \frac{|x'|}{2} \right\} \ge \frac{1}{4}.$$

Then $2A \ge |x'|$. In the case $N \ge 3$, by (2.4) we have

$$G(x, y) \le C \frac{x_N y_N}{|x' - y'|^{N-2} (|x' - y'| + x_N + y_N)^2}, \qquad y \in \mathbb{R}^N_+.$$

It follows from (4.23) and (4.29) that

$$\begin{aligned}
&\int_{D_6 \cap \{0 \le y_N \le \frac{x_N}{2}\}} G(x, y) f(y)^p \, dy \\
&\le C x_N \int_0^{\frac{x_N}{2}} \int_{A \le |y'| \le \frac{3|x'|}{2} + 1 + x_N} \frac{y_N (1 + y_N)^p |y|^{-Np}}{|x' - y'|^{N-2} (|x' - y'| + x_N + y_N)^2} \, dy' \, dy_N \\
&\le C x_N \int_0^{\frac{x_N}{2}} \int_0^{\frac{5|x'|}{2} + 1 + x_N} y_N (A + y_N)^{-Np+p} (r + y_N)^{-2} \, dr \, dy_N \\
&\le C x_N \int_0^{\frac{x_N}{2}} (A + y_N)^{-Np+p} \, dy_N \le C x_N A^{-Np+p+1} \le C x_N A^{-N} \le C x_N |x'|^{-N}.
\end{aligned} \tag{4.44}$$

In the case $N=2$, since $p>3$, by (4.10) and (4.23) we have

$$
\begin{aligned}
&\int_{D_6\cap\{0\le y_2\le \frac{x_2}{2}\}} G(x,y)f(y)^p\,dy\\
&\le C\int_0^{\frac{x_2}{2}}\int_{A\le |y_1|\le \frac{3|x_1|}{2}+1+x_2}(1+y_2)^p|y|^{-2p}\,dy_1\,dy_2 \qquad (4.45)\\
&\le C\int_0^{\frac{x_2}{2}}(A+y_2)^{-p+1}\,dy_2\le Cx_2A^{-p+1}\le Cx_2A^{-2}\le Cx_2|x_1|^{-2}.
\end{aligned}
$$

Furthermore, by (4.13) we see that

$$
G(x,y)\le C\frac{x_Ny_N}{|x_N-y_N|^N},\qquad y\in\mathbb{R}^N_+.
$$

Then it follows from (4.23) and (4.29) that

$$
\begin{aligned}
&\int_{D_6\cap\{0\le y_N\le \frac{x_N}{2}\}} G(x,y)f(y)^p\,dy\\
&\le Cx_N\int_0^{\frac{x_N}{2}}\int_{A\le |y'|\le \frac{3|x'|}{2}+1+x_N} y_N(1+y_N)^p|y|^{-Np}|x_N-y_N|^{-N}\,dy'\,dy_N\\
&\le Cx_N^{-N+1}\int_0^{\frac{x_N}{2}}\int_{A\le |y'|\le \frac{3|x'|}{2}+1+x_N} y_N(1+y_N)^p(r+y_N)^{-Np+N-2}\,dr\,dy_N\\
&\le Cx_N^{-N+1}\int_0^{\frac{x_N}{2}}(A+y_N)^{-Np+p+N}\,dy_N\le Cx_N^{-N+1}.
\end{aligned}
$$

This together with (3.14), (4.44) and (4.45) implies that

$$
\int_{D_6\cap\{0\le y_N\le \frac{x_N}{2}\}} G(x,y)f(y)^p\,dy\le Cx_N\min\{|x'|^{-N},x_N^{-N}\}\le Cx_N|x|^{-N}. \qquad (4.46)
$$

On the other hand, in the case $N\ge 3$, similarly to (4.44), we have

$$
\begin{aligned}
&\int_{D_6\cap\{y_N\ge \frac{x_N}{2}\}} G(x,y)f(y)^p\,dy\\
&\le Cx_N\int_{\frac{x_N}{2}}^\infty\int_{A\le |y'|\le \frac{3|x'|}{2}+1+x_N}\frac{y_N(1+y_N)^p|y|^{-Np}}{|x'-y'|^{N-2}(|x'-y'|+x_N+y_N)^2}\,dy'\,dy_N \qquad (4.47)\\
&\le Cx_N\int_{\frac{x_N}{2}}^\infty(A+y_N)^{-Np+p}\,dy_N\le Cx_N(A+x_N)^{-Np+p+1}\le Cx_N|x|^{-N}.
\end{aligned}
$$

Furthermore, in the case $N = 2$, since $p > 3$, by (2.5) and (4.23) we obtain

$$
\begin{aligned}
&\int_{D_6\cap\{\frac{x_2}{2}\le y_2\le 2x_2\}} G(x,y)f(y)^p\,dy \\
&\le Cx_2^\alpha \int_{\frac{x_2}{2}}^{2x_2}\int_{A\le|y_1|\le\frac{|x_1|}{3}+1+x_2}\left(\frac{y_2}{|x_2-y_2|^2}\right)^\alpha (1+y_2)^p|y|^{-2p}\,dy_1\,dy_2 \\
&\le Cx_2^\alpha \int_{\frac{x_2}{2}}^{2x_2} y_2^\alpha|x_2-y_2|^{-2\alpha}(A+y_2)^{-p+1}\,dy_2 \\
&\le Cx_2\left(A+\frac{x_2}{2}\right)^{-p+1} \le Cx_2\left(A+\frac{x_2}{2}\right)^{-2} \le Cx_2|x|^{-2},
\end{aligned}
\tag{4.48}
$$

where $\alpha \in (0, 1/2)$. In addition, since it follows from (4.13) that

$$
G(x,y) \le C\frac{x_2y_2}{|x-y|^2} \le C\frac{x_2y_2}{(|x_1-y_1|+y_2)^2}
$$

for any $y_2 \ge 2x_2$, by (4.23) we see that

$$
\begin{aligned}
&\int_{D_6\cap\{y_2\ge 2x_2\}} G(x,y)f(y)^p\,dy \\
&\le Cx_2\int_{2x_2}^\infty\int_{A\le|y_1|\le\frac{|x_1|}{3}+1+x_2}\frac{y_2}{(|x_1-y_1|+y_2)^2}(1+y_2)^p|y|^{-2p}\,dy_1\,dy_2 \\
&\le Cx_2\int_{2x_2}^\infty\int_0^{\frac{5|x_1|}{2}+1+x_2} y_2(r+y_2)^{-2}(A+y_2)^{-p}\,dr\,dy_2 \\
&\le Cx_2\int_{2x_2}^\infty (A+y_2)^{-p}\,dy_2 \le Cx_2(A+2x_2)^{-p+1} \le Cx_2(A+2x_2)^{-2} \le Cx_2|x|^{-2}.
\end{aligned}
$$

This together with (4.47) and (4.48) implies that

$$
\int_{D_6\cap\{y_N\ge\frac{x_N}{2}\}} G(x,y)f(y)^p\,dy \le Cx_N|x|^{-N}. \tag{4.49}
$$

Combining (4.46) and (4.49), we obtain (4.24) with $k = 6$.

Proof in the case $k = 7$. For any $y \in D_7$, by (4.13) we have

$$
G(x,y) \le C\frac{x_Ny_N}{|x'-y'|^N} \le Cx_Ny_N|x|^{-N}.
$$

Since $x \in D_{\mathrm{out}}$, by (4.23) and (4.29) we see that

$$
\begin{aligned}
&\int_{D_7} G(x,y)f(y)^p\,dy \\
&\le Cx_N|x|^{-N}\int_0^\infty\int_{|y'|\ge\frac{3|x'|}{2}+1+x_N} y_N(1+y_N)^p|y|^{-Np}\,dy'\,dy_N \\
&\le Cx_N|x|^{-N}\int_0^\infty\int_{\frac{3|x'|}{2}+1+x_N}^\infty (r+y_N)^{-Np+p+N-1}\,dr\,dy_N \\
&\le Cx_N|x|^{-N}\int_0^\infty (|x|+y_N)^{-Np+p+N}\,dy_N \le Cx_N|x|^{-N}.
\end{aligned}
$$

This implies (4.24) with $k = 7$. Thus (4.24) holds for $k = 1, \dots, 7$, and the proof of Lemma 4.2 is complete. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We proceed similarly as in the proof of Theorem 1.4 in [10]. Define $\{u_n\}$ inductively by

$$\begin{aligned} &u_1(x) := 0,\\ &u_{n+1}(x) := [S(x_N)\varphi](x') + \int_{\mathbb{R}^N_+} G(x,y)u_n(y)^p\,dy, \quad n = 1, 2, \dots, \end{aligned}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$. Then we can prove by induction that

$$0 \leq u_{n-1}(x) \leq u_n(x)$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$. This means that the limit function

$$u(x) := \lim_{n\to\infty} u_n(x) \in [0, \infty] \tag{4.50}$$

can be defined for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$.

Let c and k be sufficiently small positive constants such that $c < k$. Assume that

$$\varphi(x') \geq c\psi(x'), \qquad x' \in \mathbb{R}^{N-1},$$

where ψ is the function given in (1.6). It follows from Lemma 3.1 that

$$u(x) \geq [S(x_N)\varphi](x') \geq c[S(x_N)\psi](x') \geq C_1 c \begin{cases} |x|^{-m}, & x \in D_{\mathrm{in}},\\ (1 + x_N)|x|^{-N}, & x \in D_{\mathrm{out}}, \end{cases} \tag{4.51}$$

where C_1 is a positive constant independent of c. On the other hand, if φ satisfies

$$\varphi(x') \leq k\psi(x'), \qquad x' \in \mathbb{R}^{N-1},$$

then, by Lemma 3.2 we see that

$$u_2(x) = [S(x_N)\varphi](x) \leq k[S(x_N)\psi](x') \leq C_2 k \begin{cases} |x|^{-m}, & x \in D_{\mathrm{in}},\\ (1 + x_N)|x|^{-N}, & x \in D_{\mathrm{out}}, \end{cases} \tag{4.52}$$

where C_2 is a positive constant independent of k. If

$$u_n(x) \leq 2C_2 k \begin{cases} |x|^{-m}, & x \in D_{\mathrm{in}},\\ (1 + x_N)|x|^{-N}, & x \in D_{\mathrm{out}}, \end{cases}$$

for some $n \in \{2, 3, \dots\}$, then, by Lemmas 4.1 and 4.2 we see that

$$\begin{aligned} u_{n+1}(x) &= [S(x_N)\varphi](x') + [(-\Delta_D)^{-1}u_n^p](x)\\ &\leq [C_2 k + 2C_3(2C_2 k)^p] \begin{cases} |x|^{-m}, & x \in D_{\mathrm{in}},\\ (1 + x_N)|x|^{-N}, & x \in D_{\mathrm{out}}, \end{cases} \end{aligned}$$

where C_3 is a positive constant independent of k. Therefore, taking a sufficiently small constant k if necessary, we have

$$u_{n+1}(x) \leq 2C_2 k \begin{cases} |x|^{-m}, & x \in D_{\mathrm{in}},\\ (1 + x_N)|x|^{-N}, & x \in D_{\mathrm{out}}. \end{cases}$$

This together with (4.50) and (4.52) implies that

$$u(x) \le 2C_2k \begin{cases} |x|^{-m}, & x \in D_{\rm in}, \\ (1+x_N)|x|^{-N}, & x \in D_{\rm out}. \end{cases} \tag{4.53}$$

Therefore, if φ satisfies (1.7) with sufficiently small constants c and k, then (4.51) and (4.53) imply that u satisfies (1.8) and (1.9). Then, similarly to [11], we see that u is a minimal solution of (1.1). Thus Theorem 1.1 follows. □

References

[1] I. Bachar and H. Mâagli, *Estimates on the Green's function and existence of positive solutions of nonlinear singular elliptic equations in the half space,* Positivity, **9** (2005), 153–192.

[2] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, *Superlinear indefinite elliptic problems and nonlinear Liouville theorems,* Topol. Methods Nonlinear Anal., **4** (1994), 59–78.

[3] M.-F. Bidaut-Véron, A. Ponce and L. Véron, *Isolated boundary singularities of semilinear elliptic equations,* Calc. Var. Partial Differential Equations, **40** (2011), 183–221.

[4] M.-F. Bidaut-Véron and L. Vivier, *An elliptic semilinear equation with source term involving boundary measures: the subcritical case,* Rev. Mat. Iberoam., **16** (2000), 477–513.

[5] C.-C. Chen and C.-S. Lin, *Existence of positive weak solutions with a prescribed singular set of semilinear elliptic equation,* J. Geom. Anal., **9** (1999), 221–246.

[6] M. del Pino, M. Musso and F. Pacard, *Boundary singularities for weak solutions of semilinear elliptic problems,* J. Funct. Anal., **253** (2007), 241–272.

[7] A. Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of* $\mathbb{R}^N$, J. Math. Pures Appl., **87** (2007), 537–561.

[8] M. Fila, K. Ishige and T. Kawakami, *Large-time behavior of solutions of a semilinear elliptic equation with a dynamical boundary condition,* Adv. Differential Equations, **18** (2013), 69–100.

[9] M. Fila, K. Ishige and T. Kawakami, *Large-time behavior of small solutions of a two-dimensional semilinear elliptic equation with a dynamical boundary condition,* Asymptotic Analysis, **85** (2013), 107–123.

[10] M. Fila, K. Ishige and T. Kawakami, *Existence of positive solutions of a semilinear elliptic equation with a dynamical boundary condition,* Calc. Var. Partial Differential Equations, **54** (2015), 2059–2078.

[11] M. Fila, K. Ishige and T. Kawakami, *Minimal solutions of a semilinear elliptic equation with a dynamical boundary condition,* J. Math. Pures Appl., to appear, doi:10.1016/j.matpur.2015.11.014.

[12] B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations,* Comm. Partial Differential Equations, **6** (1981), 883–901.

[13] A. Gmira and L. Véron, *Boundary singularities of solutions of some nonlinear elliptic equations,* Duke Math. J., **64** (1991), 271–324.

[14] K. Ishige, K. Kawakami and K. Kobayashi, *Global solutions for a nonlinear integral equation with a generalized heat kernel,* Discrete Contin. Dyn. Syst. Ser. S., **7** (2014), 767–783.

[15] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case,* Arch. Ration. Mech. Anal., **144** (1998), 201–231.

[16] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case,* J. Math. Pures Appl., **77** (1998), 481–524.

[17] M. Marcus and L. Véron, *Removable singularities and boundary traces,* J. Math. Pures Appl., **80** (2001), 879–900.

[18] Ph. Souplet, *Optimal regularity conditions for elliptic problems via L^p_δ-spaces,* Duke Math. J., **127** (2005), 175–192.

Department of Applied Mathematics and Statistics, Comenius University, 84248 Bratislava, Slovakia
E-mail address: fila@fmph.uniba.sk

Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan
E-mail address: ishige@math.tohoku.ac.jp

Department of Mathematical Sciences, Osaka Prefecture University, Sakai 599-8531, Japan
E-mail address: kawakami@ms.osakafu-u.ac.jp