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Orthogonal ⊥

ON THE ASYMPTOTIC BEHAVIOUR OF SOME PROBLEMS OF THE CALCULUS OF VARIATIONS

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Abstract. In this note we analyse the asymptotic behaviour of the solution of some class of calculus of variation problems set in cylindrical domains. A special attention is given to limit the assumptions on the functional at stake to a minimum.

1. Monotonicity properties

We collect in this part some results which are perhaps known but for which we have been unable to find proper references. We refer the reader to [14], [15], [3], [4] for an introduction to the Sobolev spaces used here.
Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function such that

$$\mathcal{F}(v) = F(x, v(x), \nabla v(x)) \in L^1(\Omega) \ \forall v \in W_0^{1,q}(\Omega), \ q > 1. \tag{1.1}$$

For $f \in W^{-1,q'}(\Omega)$ the dual of $W_0^{1,q}(\Omega)$ we set

$$E_\Omega(v) = E_{\Omega,f}(v) = \int_\Omega \mathcal{F}(v)\ dx - \langle f, v\rangle \ \ \forall v \in W_0^{1,q}(\Omega) \tag{1.2}$$

where $\langle\ ,\ \rangle$ denotes the duality bracket between $W^{-1,q'}(\Omega)$ and $W_0^{1,q}(\Omega)$. We consider the minimization problem of E_Ω on $W_0^{1,q}(\Omega)$ i.e. we denote by $u = u_f$ a function in $W_0^{1,q}(\Omega)$ such that

$$E_\Omega(u) \le E_\Omega(v) \ \forall v \in W_0^{1,q}(\Omega). \tag{1.3}$$

Then we have:

Theorem 1. *Let $u_1 = u_{f_1}$ and $u_2 = u_{f_2}$ be two minimizers of E_Ω on $W_0^{1,q}(\Omega)$ corresponding to f_1 and f_2 respectively. One has*

$$\left\langle f_1 - f_2, (u_1 - u_2)^+ \right\rangle \ge 0. \tag{1.4}$$

$(\ )^+$ *denotes the positive part of a function.*

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Proof. Set

$$A = \{x \in \Omega \mid u_2(x) > u_1(x)\}. \tag{1.5}$$

Consider

$$v = u_1 \wedge u_2 = u_1 - (u_1 - u_2)^+.$$

One has - if A^C denotes the set $\Omega \backslash A$

$$E_{\Omega,f_1}(v) = \int_\Omega \mathcal{F}(v)\ dx - \langle f_1, v\rangle = \int_A \mathcal{F}(u_1)\ dx + \int_{A^C} \mathcal{F}(u_2)\ dx$$
$$- \langle f_1, u_1\rangle + \left\langle f_1, (u_1 - u_2)^+\right\rangle.$$

We consider next

$$w = u_1 \vee u_2 = u_2 + (u_1 - u_2)^+.$$

One has

$$E_{\Omega,f_2}(w) = \int_\Omega \mathcal{F}(w)\ dx - \langle f_2, w\rangle = \int_A \mathcal{F}(u_2)\ dx + \int_{A^C} \mathcal{F}(u_1)\ dx$$
$$- \langle f_2, u_2\rangle - \left\langle f_2, (u_1 - u_2)^+\right\rangle.$$

Adding up the last two equalities we get

$$\begin{aligned} &E_{\Omega,f_1}(v) + E_{\Omega,f_2}(w) \\ &\quad = E_{\Omega,f_1}(u_1) + E_{\Omega,f_2}(u_2) + \left\langle f_1 - f_2, (u_1 - u_2)^+\right\rangle. \end{aligned} \tag{1.6}$$

Thus (1.4) follows since u_1, u_2 are minimizers of E_{Ω,f_1}, E_{Ω,f_2} respectively. This completes the proof of the theorem.

□

Remark 1. Exchanging the role of f_1 and f_2 we have

$$\begin{aligned} \left\langle f_2 - f_1, (u_2 - u_1)^+\right\rangle &= \left\langle f_2 - f_1, (u_1 - u_2)^-\right\rangle \\ &= \left\langle f_1 - f_2, -(u_1 - u_2)^-\right\rangle \geq 0 \end{aligned}$$

and thus adding to (1.4)

$$\langle f_1 - f_2, u_1 - u_2\rangle \geq 0 \tag{1.7}$$

which means that the operator $f \to u_f$ is monotone.

Definition 1. We say that $f_1 \geq f_2$ when this inequality holds in the distributional sense i.e. when we have

$$\langle f_1, \varphi\rangle \geq \langle f_2, \varphi\rangle\ \ \forall \varphi \in \mathcal{D}(\Omega),\ \ \varphi \geq 0,$$

which by density of $\mathcal{D}(\Omega)$ in $W_0^{1,q}(\Omega)$ means

$$\langle f_1, v\rangle \geq \langle f_2, v\rangle\ \ \forall v \in W_0^{1,q}(\Omega),\ \ v \geq 0. \tag{1.8}$$

Then we have:

Corollary 1. *Let $u_1 = u_{f_1}$ and $u_2 = u_{f_2}$ be two minimizers of E_Ω on $W_0^{1,q}(\Omega)$ corresponding to f_1 and f_2 respectively. If*

$$f_1 \leq f_2 \tag{1.9}$$

one has

$$\left\langle f_1 - f_2, (u_1 - u_2)^+\right\rangle = 0. \tag{1.10}$$

Proof. This follows immediately from the definition above and (1.4). □

Remark 2. In the case where f_1 and f_2 are functions satisfying (1.9) one has $u_1 \leq u_2$ on the set where $f_1 < f_2$.

Corollary 2. *Let $u_1 = u_{f_1}$ and $u_2 = u_{f_2}$ be two minimizers of E_Ω on $W_0^{1,q}(\Omega)$ corresponding to f_1 and f_2 respectively. If* (1.9) *holds and either u_1 or u_2 is unique then*

$$u_1 \leq u_2. \tag{1.11}$$

Proof. Indeed in this case from (1.6) one derives

$$E_{\Omega,f_1}(v) - E_{\Omega,f_1}(u_1) = E_{\Omega,f_2}(u_2) - E_{\Omega,f_2}(w) = 0.$$

Then either $v = u_1$ or $w = u_2$ which in both cases means $u_1 \leq u_2$.

□

As an immediate consequence one has

Corollary 3. *Suppose that* 0 *is solution to* (1.3) *for $f = 0$ and that* (1.3) *has a unique solution u_f for $f \geq 0$. Then one has $u_f \geq 0$.*

As a remarkable property we have also

Proposition 1. *The following assertions are equivalent:*

(i) The minimization problem (1.3) *admits a unique solution.*

(ii) If $u = u_f$ denotes a minimizer of E_Ω, the mapping $f \to u_f$ is monotone in the sense that

$$f_1 \leq f_2 \Longrightarrow u_{f_1} \leq u_{f_2}.$$

Proof.

•$(ii) \Longrightarrow (i)$.

Indeed if u_1 and u_2 are two solutions corresponding to the same f the monotonicity property implies $u_1 \geq u_2$ and $u_2 \geq u_1$ i.e. $u_1 = u_2$.

•$(i) \Longrightarrow (ii)$ follows directly from Corollary 2.

□

Remark 3. The same results hold if the minimisation on $W_0^{1,q}(\Omega)$ is replaced by a minimisation on a subset K of $W_0^{1,q}(\Omega)$ having the property that

$$u_1, u_2 \in K \implies u_1 \wedge u_2,\ u_1 \vee u_2 \in K.$$

Note also that our results do not involve any convexity on F or K.

One has also the following additional monotonicity property with respect to the domain Ω.

Theorem 2. *Suppose that $f \geq 0$ in the distributional sense and that* (1.3) *admits a unique solution for two open subsets Ω and Ω' such that*

$$\Omega \subset \Omega'. \tag{1.12}$$

If u, u' denote the solutions to (1.3) *corresponding to Ω, Ω' respectively and if* 0 *is solution to* (1.3) *for $f = 0$ for Ω, Ω' one has*

$$u' \geq u \geq 0. \tag{1.13}$$

Proof. $u, u' \geq 0$ is a consequence of Corollary 3. Suppose that u is extended by 0 outside Ω. Consider then $v = u \wedge u' = u' - (u' - u)^+$ and

$$A = \{x \in \Omega' \mid u(x) > u'(x)\}.$$

Note that A is included in Ω. One has

$$\int_\Omega \mathcal{F}(v)\ dx - \langle f, v\rangle = \int_{A^C} \mathcal{F}(u)\ dx + \int_A \mathcal{F}(u')\ dx - \langle f, v\rangle\,.$$

If

$$\int_A \mathcal{F}(u')\ dx < \int_A \mathcal{F}(u)\ dx - \langle f, u\rangle + \langle f, v\rangle$$

one has

$$\begin{aligned}\int_\Omega \mathcal{F}(v)\ dx - \langle f, v\rangle &< \int_{A^C} \mathcal{F}(u)\ dx + \int_A \mathcal{F}(u)\ dx - \langle f, u\rangle + \langle f, v\rangle - \langle f, v\rangle \\ &= \int_\Omega \mathcal{F}(u)\ dx - \langle f, u\rangle\,.\end{aligned}$$

and a contradiction with the definition of u which is not a minimizer. Thus it holds

$$\int_A \mathcal{F}(u)\ dx \leq \int_A \mathcal{F}(u')\ dx + \langle f, u\rangle - \langle f, v\rangle\,.$$

If now one considers $w = u \vee u' = u + (u' - u)^+$ it comes

$$\int_{\Omega'} \mathcal{F}(w)\ dx - \langle f, w\rangle = \int_{A^C} \mathcal{F}(u')\ dx + \int_A \mathcal{F}(u)\ dx - \langle f, w\rangle$$

and thus

$$\begin{aligned}\int_{\Omega'} \mathcal{F}(w)\ dx - \langle f, w\rangle &\leq \int_{\Omega'} \mathcal{F}(u')\ dx + \langle f, u\rangle - \langle f, v\rangle - \langle f, w\rangle \\ &= \int_{\Omega'} \mathcal{F}(u')\ dx - \langle f, u'\rangle\,.\end{aligned}$$

(Recall that $v = u' - (u' - u)^+$, $w = u + (u' - u)^+$). By uniqueness of the solution one has $w = u'$ which implies that $u' \geq u$. This completes the proof of the theorem.

□

2. A convergence result

We denote by Ω_ℓ the open subset of $\mathbb{R}^n$ defined as

$$\Omega_\ell = \ell\omega_1 \times \omega_2$$

where

(2.1) ω_1 is a bounded convex open subset of $\mathbb{R}^p$ containing 0,

ω_2 being a bounded open subset of $\mathbb{R}^{(n-p)}$.

Thus for every $\ell_0 \leq \ell - 1$ there exists a function $\rho = \rho_{\ell_0}(X_1)$ such that

(2.2) $0 \leq \rho \leq 1,\ \rho = 1 \text{ on } \ell_0\omega_1,\ \rho = 0 \text{ outside } (\ell_0 + 1)\omega_1,\ |\nabla\rho| \leq C$

where C is a constant independent of ℓ_0. We consider then a function $F : \mathbb{R}^n \to \mathbb{R}$ such that

(2.3) F is convex,

and such that there exist positive constants $\lambda, \Lambda, \Lambda'$ such that

$$\lambda|\xi|^q - \Lambda' \leq F(\xi) \leq \Lambda|\xi|^q + \Lambda' \ \forall \xi \in \mathbb{R}^n, \ F(0) = -\Lambda' \tag{2.4}$$

(q is a fixed number such that $q > 1$).

For such a function we have for some constant $C = C(q, \Lambda, \Lambda')$ (see [13]):

$$|F(P) - F(Q)| \leq C\{1 + |P|^{q-1} + |Q|^{q-1}\}|P - Q| \ \forall P, Q \in \mathbb{R}^n. \tag{2.5}$$

Then for

$$f = f(X_2) \in L^{q'}(\omega_2), \ \frac{1}{q} + \frac{1}{q'} = 1, \tag{2.6}$$

we consider the minimisation of the functional

$$E_{\Omega_\ell}(v) = \int_{\Omega_\ell} F(\nabla v) - fv \ dx \tag{2.7}$$

on $W_0^{1,q}(\Omega_\ell)$. We have:

Theorem 3. *There exists u_ℓ solution to*

$$u_\ell \in W_0^{1,q}(\Omega_\ell), \ E_{\Omega_\ell}(u_\ell) \leq E_{\Omega_\ell}(v) \ \forall v \in W_0^{1,q}(\Omega_\ell). \tag{2.8}$$

Moreover if F is strictly convex the solution of the problem above is unique.

Proof. First note that by (2.4) and (2.6), $E_{\Omega_\ell}(v)$ is well defined for any $v \in W_0^{1,q}(\Omega_\ell)$. Moreover by the Hölder and Poincaré inequalities one has

$$\int_{\Omega_\ell} fv \ dx \leq |f|_{q',\Omega_\ell}|v|_{q,\Omega_\ell} \leq C|f|_{q',\Omega_\ell}\big||\nabla v|\big|_{q,\Omega_\ell} \tag{2.9}$$

and thus -see (2.4)-

$$E_{\Omega_\ell}(v) \geq \lambda\big||\nabla v|\big|_{q,\Omega_\ell}^q - C|f|_{q',\Omega_\ell}\big||\nabla v|\big|_{q,\Omega_\ell} - \Lambda'|\Omega_\ell|.$$

This implies that E_{Ω_ℓ} is coercive or that every minimizing sequence is bounded in $W_0^{1,q}(\Omega_\ell)$. To conclude it is enough to show that E_{Ω_ℓ} is lower semicontinuous for the weak topology of $W_0^{1,q}(\Omega_\ell)$. E_{Ω_ℓ} being convex it is enough to show the lower semicontinuity of E_{Ω_ℓ} for the strong topology of $W_0^{1,q}(\Omega_\ell)$. If

$$v_n \to v \text{ in } W_0^{1,q}(\Omega_\ell)$$

up to a subsequence one has

$$v_n \to v \text{ in } L^q(\Omega_\ell), \ \nabla v_n \to \nabla v \text{ a.e. in } \Omega_\ell.$$

By Fatou's lemma one deduces

$$\begin{aligned} \liminf_{n\to\infty} E_{\Omega_\ell}(v_n) &= \liminf_{n\to\infty} \int_{\Omega_\ell} F(\nabla v_n) \ dx - \int_{\Omega_\ell} fv \ dx \\ &\geq \int_{\Omega_\ell} \liminf_{n\to\infty} F(\nabla v_n) \ dx - \int_{\Omega_\ell} fv \ dx = E_{\Omega_\ell}(v). \end{aligned}$$

This completes the proof of the theorem. □

We are interested here in the behaviour of u_ℓ when $\ell \to +\infty$. The issue has been considered before for a certain number of problems and we refer for instance to [1], [2], [5], [6], [7], [9], [10], [11] to have a good spectrum of the techniques involved.

If 0 denotes the 0 vector in $\mathbb{R}^p$ and ∇_{X_2} the gradient in X_2 i.e.

$$\nabla_{X_2} = (\partial_{x_{p+1}}, \cdots, \partial_{x_n})$$

one sets

$$E_{\omega_2}(v) = \int_{\omega_2} F(0, \nabla_{X_2} v) - fv \, dX_2 \ \forall v \in W_0^{1,q}(\omega_2). \tag{2.10}$$

Clearly $E_{\omega_2}(v)$ is well defined and with the same proof as above one has

Theorem 4. *There exists u_∞ solution to*

$$u_\infty \in W_0^{1,q}(\omega_2), \ E_{\omega_2}(u_\infty) \le E_{\omega_2}(v) \ \forall v \in W_0^{1,q}(\omega_2). \tag{2.11}$$

Moreover if $F(0,\cdot)$ is strictly convex the solution of the problem above is unique.

Our goal is now to show:

Theorem 5. *Suppose that $f \ge 0$ and F strictly convex then one has*

$$u_\ell \to u_\infty \tag{2.12}$$

when $\ell \to +\infty$.

Remark 4. For the time being we are not more precise on the way the convergence above occurs. This result completes the results of [8]. See also [5].

We will need several lemmas.

One denotes by $W_{lat}^{1,q}(\Omega_\ell)$ the space

$$W_{lat}^{1,q}(\Omega_\ell) = \{v \in W^{1,q}(\Omega_\ell) \mid v = 0 \text{ on } \ell\omega_1 \times \partial\omega_2\} \tag{2.13}$$

and by $V^{1,q}(\Omega_\ell)$ the space

$$V^{1,q}(\Omega_\ell) = \{v \in W_{lat}^{1,q}(\Omega_\ell) \mid \fint_{\ell\omega_1} \nabla_{X_1} v \, dX_1 = 0 \text{ a.e. } X_2 \in \omega_2\}. \tag{2.14}$$

($\fint_{\ell\omega_1} = \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1}$ where $|\ell\omega_1|$ denotes the measure of $\ell\omega_1$).

Lemma 1. *u_∞ is the unique solution to*

$$u_\infty \in V^{1,q}(\Omega_\ell), \ E_{\Omega_\ell}(u_\infty) \le E_{\Omega_\ell}(v) \ \forall v \in V^{1,q}(\Omega_\ell). \tag{2.15}$$

(We suppose u_∞ extended to Ω_ℓ being taken constant in X_1).

Proof. First notice that $u_\infty \in V^{1,q}(\Omega_\ell)$. Indeed one has since u_∞ is independent of X_1

$$\fint_{\ell\omega_1} \nabla_{X_1} u_\infty \, dX_1 = 0.$$

Next let $u \in V^{1,q}(\Omega_\ell)$. Set

$$v(X_2) = \fint_{\ell\omega_1} u(., X_2) \, dX_1.$$

It is easy to see that $v \in W_0^{1,q}(\omega_2)$. Thus

$$E_{\omega_2}(u_\infty) \le E_{\omega_2}(v) = \int_{\omega_2} F\left(0, \nabla_{X_2} v\right) - fv \; dX_2.$$

Now using the fact that $u \in V^{1,q}(\Omega_\ell)$ one has

$$0 = \fint_{\ell\omega_1} \nabla_{X_1} u \; dX_1.$$

Also by differentiation under the integral, we get

$$\nabla_{X_2} v = \fint_{\ell\omega_1} \nabla_{X_2} u \; dX_1.$$

Therefore we have by Jensen's inequality

$$\begin{aligned} E_{\omega_2}(u_\infty) \le & \int_{\omega_2} \left\{ F\left(\frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} \nabla_{X_1} u dX_1, \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} \nabla_{X_2} u dX_1 \right) \right\} dX_2 \\ & - \int_{\omega_2} f \left\{ \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} u dX_1 \right\} dX_2 \\ & \le \int_{\omega_2} \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} \left\{ F(\nabla u) - fu \right\} dX_1 dX_2 = \frac{E_{\Omega_\ell}(u)}{|\ell\omega_1|}. \end{aligned}$$

Clearly equality holds in the above inequality if $u = u_\infty$. Then the claim follows from the uniqueness of the minimizer.

□

One can then show:

Lemma 2. *One has for $\ell' > \ell$*

$$0 \le u_\ell \le u_{\ell'} \le u_\infty. \tag{2.16}$$

Proof. Recall that we are assuming $f \ge 0$. The nonnegativity of all the functions above results from the corollary 3. Indeed for $f = 0$ one has for instance

$$\begin{aligned} E_{\Omega_\ell}(v) = \int_{\Omega_\ell} F(\nabla v) dx \ge & \int_{\Omega_\ell} -\Lambda' dx \\ & = \int_{\Omega_\ell} F(0) dx = E_{\Omega_\ell}(0) \; \forall v \in W_0^{1,q}(\Omega_\ell) \end{aligned}$$

i.e. 0 is a minimizer. The monotonicity property $\ell \to u_\ell$ follows from the theorem 2. To prove that u_ℓ (or $u_{\ell'}$) is less than u_∞ one proceeds as follows. One considers

$$A_\ell = \{ x \in \Omega_\ell \mid u_\ell(x) > u_\infty(x) \}.$$

One has

$$E_{A_\ell}(u_\ell) \le E_{A_\ell}(u_\infty) \qquad \left(E_{A_\ell}(\cdot) = \int_{A_\ell} F(\nabla \cdot) - f \cdot \; dx\right). \tag{2.17}$$

Indeed, if not, setting $v_\ell = u_\ell \wedge u_\infty$ one has $v_\ell \in W_0^{1,q}(\Omega_\ell)$ and

$$E_{\Omega_\ell}(v_\ell) = E_{A_\ell}(v_\ell) + E_{A_\ell^C}(v_\ell) = E_{A_\ell}(u_\infty) + E_{A_\ell^C}(u_\ell) < E_{\Omega_\ell}(u_\ell)$$

and a contradiction with the definition of u_ℓ (A_ℓ^C denotes the set $\Omega \backslash A_\ell$). Assuming then (2.17) and considering $w_\ell = u_\ell \vee u_\infty$ one has $w_\ell \in V^{1,q}(\Omega_\ell)$. Indeed if ν denotes the normal to $\partial(\ell\omega_1)$ this follows from

$$\begin{aligned} ⨍_{\ell\omega_1} \nabla_{X_1} w_\ell(.,X_2)\, dX_1 &= \frac{1}{|\ell\omega_1|} \int_{\partial(\ell\omega_1)} w_\ell(.,X_2)\nu \, d\sigma(X_1) \\ &= \frac{1}{|\ell\omega_1|} \int_{\partial(\ell\omega_1)} u_\infty(.,X_2)\nu \, d\sigma(X_1) \\ &= ⨍_{\ell\omega_1} \nabla_{X_1} u_\infty(.,X_2)\, dX_1 = 0, \end{aligned}$$

(since $u_\infty(X_1, X_2)$ is constant in X_1). Then

$$E_{\Omega_\ell}(w_\ell) = E_{A_\ell}(w_\ell) + E_{A_\ell^C}(w_\ell) = E_{A_\ell}(u_\ell) + E_{A_\ell^C}(u_\infty) \le E_{\Omega_\ell}(u_\infty)$$

and thus $w_\ell = u_\infty$ -i.e. $u_\ell \le u_\infty$.

□

Clearly it follows from the lemma 2 that there exists $\tilde{u}_\infty$ such that

$$u_\ell \le \tilde{u}_\infty \le u_\infty \ \forall \ell, \ u_\ell \to \tilde{u}_\infty \text{ a.e. in } \mathbb{R}^p \times \omega_2. \tag{2.18}$$

Lemma 3. *The function $\tilde{u}_\infty$ is independent of X_1.*

Proof. For $i = 1, \cdots, p$ we set

$$\tau_h^i v(x) = v(x - he_i), \ h > 0$$

where e_i denotes the i-th vector of the canonical basis of $\mathbb{R}^p$. We claim that

$$u_{\ell+h} \ge \tau_{h'}^i u_\ell \ \text{ for } 0 < h' \le \lambda h \tag{2.19}$$

$\lambda \le 1$ being so small that

$$\lambda e_i \in \omega_1. \tag{2.20}$$

Indeed if (2.20) holds we have for $X_1 - h'e_i \in \ell\omega_1$ and some $Y_1 \in \omega_1$

$$X_1 = \ell Y_1 + h'e_i = (\ell + h)\{\frac{\ell}{\ell + h} Y_1 + \frac{h}{\ell + h}\frac{h'}{h} e_i\} \in (\ell + h)\omega_1$$

(since $Y_1, \frac{h'}{h} e_i \in \omega_1$ and ω_1 is convex containing 0). Thus the support of $\tau_{h'}^i u_\ell$ is contained in $\Omega_{\ell+h}$. One defines

$$\mathcal{A} = \{x \in \Omega_{\ell+h} \mid \tau_{h'}^i u_\ell(x) > u_{\ell+h}(x)\}.$$

One has $E_{\mathcal{A}}(u_{\ell+h}) \le E_{\mathcal{A}}(\tau_{h'}^i u_\ell)$. Indeed if not

$$w_\ell = u_{\ell+h} \vee \tau_{h'}^i u_\ell = u_{\ell+h} + (\tau_{h'}^i u_\ell - u_{\ell+h})^+ \in W_0^{1,q}(\Omega_{\ell+h})$$

and

$$\begin{aligned} E_{\Omega_{\ell+h}}(w_\ell) &= E_{\mathcal{A}}(\tau_{h'}^i u_\ell) + E_{\mathcal{A}^C}(u_{\ell+h}) \\ &< E_{\mathcal{A}}(u_{\ell+h}) + E_{\mathcal{A}^C}(u_{\ell+h}) = E_{\Omega_{\ell+h}}(u_{\ell+h}) \end{aligned}$$

and a contradiction with the definition of $u_{\ell+h}$. Let us set

$$\mathcal{A}' = \{x \in \Omega_\ell \mid \tau_{-h'}^i(u_{\ell+h})(x) < u_\ell(x)\}.$$

$$\begin{aligned} x \in \mathcal{A}' &\Leftrightarrow x + h'e_i \in \Omega_{\ell+h} \text{ and } u_{\ell+h}(x+h'e_i) < u_\ell(x) \\ &\Leftrightarrow y = x + h'e_i \in \Omega_{\ell+h} \text{ and } u_{\ell+h}(y) < \tau^i_{h'} u_\ell(y) \\ &\Leftrightarrow x + h'e_i \in \mathcal{A}, \end{aligned}$$

i.e. one has

$$\mathcal{A}' = \mathcal{A} - h'e_i. \tag{2.21}$$

Consider next

$$v_\ell = u_\ell \wedge \tau^i_{-h'}(u_{\ell+h}) = u_\ell - (u_\ell - \tau^i_{-h'}(u_{\ell+h}))^+.$$

Clearly $v_\ell \in W^{1,q}_0(\Omega_\ell)$ (recall that $\tau^i_{-h'}(u_{\ell+h}) \geq 0$). Then

$$E_{\Omega_\ell}(v_\ell) = E_{\mathcal{A}'}(\tau^i_{-h'}(u_{\ell+h})) + E_{\mathcal{A}'^C}(u_\ell).$$

Using (2.21) one has

$$\begin{aligned} E_{\mathcal{A}'}(\tau^i_{-h'}(u_{\ell+h})) &= \int_{\mathcal{A}'} \{F(\nabla u_{\ell+h}(x+h'e_i)) - f(X_2)u_{\ell+h}(x+h'e_i)\}dx \\ &= \int_{\mathcal{A}} \{F(\nabla u_{\ell+h}(y)) - f(X_2)u_{\ell+h}(y)\}dy = E_{\mathcal{A}}(u_{\ell+h}) \end{aligned}$$

and

$$\begin{aligned} E_{\mathcal{A}}(\tau^i_{h'}(u_\ell)) &= \int_{\mathcal{A}} \{F(\nabla u_\ell(x-h'e_i)) - f(X_2)u_\ell(x-h'e_i)\}\ dx \\ &= \int_{\mathcal{A}'} \{F(\nabla u_\ell(y)) - f(X_2)u_\ell(y)\}dy = E_{\mathcal{A}'}(u_\ell). \end{aligned}$$

Thus we obtain from above – recall that $E_{\mathcal{A}}(u_{\ell+h}) \leq E_{\mathcal{A}}(\tau^i_{h'}u_\ell)$

$$\begin{aligned} E_{\Omega_\ell}(v_\ell) =& E_{\mathcal{A}}(u_{\ell+h}) + E_{\mathcal{A}'^C}(u_\ell) \leq E_{\mathcal{A}}(\tau^i_{h'}u_\ell) + E_{\mathcal{A}'^C}(u_\ell) \\ &= E_{\mathcal{A}'}(u_\ell) + E_{\mathcal{A}'^C}(u_\ell) = E_{\Omega_\ell}(u_\ell). \end{aligned}$$

This is only possible if $v_\ell = u_\ell$ i.e. $(u_\ell - \tau^i_{-h'}(u_{\ell+h}))^+ = 0$ which is also

$$\tau^i_{h'}(u_\ell) \leq u_{\ell+h}.$$

Similary one would get

$$\tau^i_{-h'}(u_\ell) \leq u_{\ell+h}.$$

Thus, passing to the limit in ℓ in the inequalities above one derives

$$\tilde{u}_\infty(x - h'e_i) \leq \tilde{u}_\infty(x),\ \tilde{u}_\infty(x + h'e_i) \leq \tilde{u}_\infty(x) \ \text{ a.e. } x,$$

which implies

$$\tilde{u}_\infty(x) \leq \tilde{u}_\infty(x - h'e_i) \leq \tilde{u}_\infty(x) \ \text{ a.e. } x,\ \forall i = 1, \cdots, p,\ \forall h' \text{ small}.$$

This completes the proof of the lemma.

□

Lemma 4. *The function $\tilde{u}_\infty$ belongs to $W^{1,q}_0(\omega_2)$.*

Proof. Let $\rho = \rho_{\ell_0}$ be the function defined in (2.2). Since $(1-\rho)u_\ell \in W_0^{1,q}(\Omega_\ell)$ one derives from (2.8)

$$\int_{\Omega_\ell} \{F(\nabla u_\ell) - f u_\ell\} dx \le \int_{\Omega_\ell} \{F(\nabla((1-\rho)u_\ell)) - f(1-\rho)u_\ell\} dx.$$

Since $(1-\rho) = 1$ outside of Ω_{ℓ_0+1} and 0 on Ω_{ℓ_0} we get

$$\begin{aligned}\int_{\Omega_{\ell_0+1}} &\{F(\nabla u_\ell) - f u_\ell\} dx \\ &\le \int_{\Omega_{\ell_0}} F(0) dx + \int_{D_{\ell_0}} \{F(\nabla((1-\rho)u_\ell)) - f(1-\rho)u_\ell\}\ dx \\ &\le \int_{D_{\ell_0}} \{F(\nabla((1-\rho)u_\ell)) - f(1-\rho)u_\ell\} dx\end{aligned}$$

(with $D_{\ell_0} = \Omega_{\ell_0+1} \backslash \Omega_{\ell_0}$) and thus

$$\begin{aligned}\int_{\Omega_{\ell_0+1}} F(\nabla u_\ell) dx &\le \int_{D_{\ell_0}} F(\nabla((1-\rho)u_\ell)) dx + \int_{\Omega_{\ell_0+1}} |f||u_\ell| dx \\ &\le \int_{D_{\ell_0}} F(\nabla((1-\rho)u_\ell)) dx + |f|_{q',\Omega_{\ell_0+1}} |u_\ell|_{q,\Omega_{\ell_0+1}} \\ &\le \int_{D_{\ell_0}} F(\nabla((1-\rho)u_\ell)) dx + C|f|_{q',\Omega_{\ell_0+1}} \big||\nabla u_\ell|\big|_{q,\Omega_{\ell_0+1}},\end{aligned}$$

by the Poincaré inequality. Using the assumptions on F and Young's inequality it comes

$$\begin{aligned}\lambda \int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q\ dx - \Lambda'|\Omega_{\ell_0+1}| &\le \Lambda \int_{D_{\ell_0}} |\nabla((1-\rho)u_\ell)|^q\ dx \\ &\quad + \epsilon\big||\nabla u_\ell|\big|^q_{q,\Omega_{\ell_0+1}} + C(\epsilon)|f|^{q'}_{q',\Omega_{\ell_0+1}} + \Lambda'|\Omega_{\ell_0+1}|.\end{aligned}$$

Choosing $\epsilon = \frac{\lambda}{2}$ we get

$$\begin{aligned}\frac{\lambda}{2} &\int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q\ dx \\ &\le \Lambda \int_{D_{\ell_0}} |-u_\ell \nabla\rho + (1-\rho)\nabla u_\ell|^q\ dx + C(\lambda)|f|^{q'}_{q',\Omega_{\ell_0+1}} + 2\Lambda'|\Omega_{\ell_0+1}| \\ &\le C \int_{D_{\ell_0}} |\nabla u_\ell|^q\ dx + C(\lambda) \int_{(\ell_0+1)\omega_1} \int_{\omega_2} |f|^{q'}\ dX_2 dX_1 + 2\Lambda'|\Omega_{\ell_0+1}|\end{aligned}$$

(we have used here the Poincaré inequality again). Thus we derive

$$\begin{aligned}\frac{\lambda}{2} &\int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q\ dx \\ &\le C \int_{D_{\ell_0}} |\nabla u_\ell|^q\ dx + (\ell_0+1)^p |\omega_1| \{C(\lambda)|f|^{q'}_{q',\omega_2} + 2\Lambda'|\omega_2|\}.\end{aligned}$$

If we still denote by C the maximum $\frac{2C}{\lambda} \vee \{\frac{2}{\lambda}|\omega_1|(C(\lambda)|f|_{q',\omega_2}^{q'} + 2\Lambda'|\omega_2|)\}$ we obtain

$$\int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q \, dx \le C \int_{D_{\ell_0}} |\nabla u_\ell|^q \, dx + C(\ell_0+1)^p$$

where C is independent of ℓ_0 and ℓ. Thus it comes

$$\int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx \le C \int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q \, dx - C \int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx + C(\ell_0+1)^p$$

and then

$$\int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx \le \gamma \int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q \, dx + \gamma(\ell_0+1)^p$$

with $\gamma = \frac{C}{C+1} < 1$. Iterating this inequality $[\ell - \ell_0]$ times where [ ] denotes the integer part of a number leads to

$$\begin{aligned}\int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx &\le \gamma\{\gamma \int_{\Omega_{\ell_0+2}} |\nabla u_\ell|^q \, dx + \gamma(\ell_0+2)^p\} + \gamma(\ell_0+1)^p \\ &\le \gamma^2 \int_{\Omega_{\ell_0+2}} |\nabla u_\ell|^q \, dx + \gamma^2(\ell_0+2)^p + \gamma(\ell_0+1)^p \\ &\le \gamma^{[\ell-\ell_0]} \int_{\Omega_{\ell_0+[\ell-\ell_0]}} |\nabla u_\ell|^q \, dx + \sum_{k=1}^{[\ell-\ell_0]} \gamma^k(\ell_0+k)^p.\end{aligned}$$

Using the fact that

$$\ell - \ell_0 - 1 < [\ell - \ell_0] \le \ell - \ell_0$$

we derive

$$\int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx \le \frac{1}{\gamma^{\ell_0+1}} \, \gamma^\ell \int_{\Omega_\ell} |\nabla u_\ell|^q \, dx + \sum_{k=1}^{\infty} \gamma^k(\ell_0+k)^p. \tag{2.22}$$

Since

$$\lim_{k\to+\infty} \frac{\gamma^{k+1}(\ell_0+k+1)^p}{\gamma^k(\ell_0+k)^p} = \gamma < 1,$$

the series above converges and has for sum some real number that we will denote by $\Sigma(\ell_0)$. We know on the other hand taking $v = 0$ in (2.8) that

$$\int_{\Omega_\ell} F(\nabla u_\ell) \, dx - \int_{\Omega_\ell} f u_\ell \, dx \le \int_{\Omega_\ell} F(0) \, dx$$

i.e.

$$\begin{aligned}\int_{\Omega_\ell} F(\nabla u_\ell) \, dx - \int_{\Omega_\ell} F(0) \, dx &\le \int_{\Omega_\ell} f u_\ell dx \\ &\le |f|_{q',\Omega_\ell} |u_\ell|_{q,\Omega_\ell} \le C|f|_{q',\Omega_\ell} \big||\nabla u_\ell|\big|_{q,\Omega_\ell}\end{aligned}$$

by Hölder's inequality. Thus

$$\lambda \int_{\Omega_\ell} |\nabla u_\ell|^q \, dx \le C|f|_{q',\Omega_\ell} \big||\nabla u_\ell|\big|_{q,\Omega_\ell}.$$

This leads to

$$\int_{\Omega_\ell} |\nabla u_\ell|^q \, dx \le C \int_{\ell\omega_1} \int_{\omega_2} |f|^{q'} \, dX_2 dX_1 \le C\ell^p.$$

Thus, going back to (2.22) we obtain

$$\int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx \le \frac{1}{\gamma^{\ell_0+1}} \, C\gamma^\ell \ell^p + \Sigma(\ell_0) \le K(\ell_0) \ \forall \ell \tag{2.23}$$

where $K(\ell_0)$ is a constant independent of ℓ. It follows that u_ℓ is bounded in $W^{1,q}_{lat}(\Omega_{\ell_0})$ and thus up to a subsequence one has that there exists $\hat{u}_\infty \in W^{1,q}_{lat}(\Omega_{\ell_0})$ such that

$$u_\ell \rightharpoonup \hat{u}_\infty \text{ in } W^{1,q}_{lat}(\Omega_{\ell_0}), \ u_\ell \to \hat{u}_\infty \text{ in } L^q(\Omega_{\ell_0}).$$

Since by the dominated convergence theorem one has

$$u_\ell \to \tilde{u}_\infty \text{ in } L^q(\Omega_{\ell_0})$$

one obtains

$$\hat{u}_\infty = \tilde{u}_\infty \in W^{1,q}_{lat}(\Omega_{\ell_0})$$

i.e. $\tilde{u}_\infty \in W^{1,q}_0(\omega_2)$. This completes the proof of the lemma.

□

Lemma 5. *One has*

$$\tilde{u}_\infty = u_\infty. \tag{2.24}$$

Proof. First notice that from (2.22), (2.23) one has

$$\begin{aligned} \int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx &\le 1 + \ell_0^p \sum_{k=1}^{+\infty} \gamma^k (1+k)^p \\ &= 1 + C\ell_0^p, \end{aligned} \tag{2.25}$$

for ℓ large enough i.e. $\ell \ge \ell(\ell_0)$, $\ell_0 \ge 1$. If $\rho = \rho_{\ell_0}$ is the function defined in (2.2) one has

$$(1-\rho)u_\ell + \rho u_\infty \in W^{1,q}_0(\Omega_\ell)$$

and thus from (2.8)

$$\begin{aligned} &\int_{\Omega_\ell} \{F(\nabla u_\ell) - f u_\ell\} \, dx \\ &\quad \le \int_{\Omega_\ell} \Big\{ F(\nabla\{(1-\rho)u_\ell + \rho u_\infty\}) - f\{(1-\rho)u_\ell + \rho u_\infty\} \Big\} \, dx. \end{aligned}$$

Since $(1-\rho)u_\ell + \rho u_\infty = u_\ell$ on $\Omega_\ell \backslash \Omega_{\ell_0+1}$ and is equal to u_∞ on Ω_{ℓ_0} we get

$$\begin{aligned} \int_{\Omega_{\ell_0+1}} \{F(\nabla u_\ell) - f u_\ell\} \, dx &\le \int_{\Omega_{\ell_0}} \{F(\nabla u_\infty) - f u_\infty\} \, dx \\ &+ \int_{D_{\ell_0}} \Big\{ F(\nabla\{(1-\rho)u_\ell + \rho u_\infty\}) - f\{(1-\rho)u_\ell + \rho u_\infty\} \Big\} \, dx. \end{aligned} \tag{2.26}$$

Let us denote by I the last integral above. One has

$$\begin{aligned} I = \int_{D_{\ell_0}} \Big\{ & F(\nabla\{(1-\rho)u_\ell + \rho u_\infty\}) - F((1-\rho)\nabla u_\ell + \rho\nabla u_\infty) \\ & + F((1-\rho)\nabla u_\ell + \rho\nabla u_\infty) - f\{(1-\rho)u_\ell + \rho u_\infty\}\Big\}\, dx \\ \le \int_{D_{\ell_0}} & \Big\{F(\nabla\{(1-\rho)u_\ell + \rho u_\infty\}) - F((1-\rho)\nabla u_\ell + \rho\nabla u_\infty)\Big\}\, dx \\ & + \int_{D_{\ell_0}} (1-\rho)\{F(\nabla u_\ell) - f u_\ell\}\, dx + \rho\{F(\nabla u_\infty) - f u_\infty\}\, dx \end{aligned}$$

due to the convexity of F. It follows from (2.26) that

$$\begin{aligned} \int_{\Omega_{\ell_0+1}} \rho\{F(\nabla u_\ell) - f u_\ell\} dx \le & \int_{\Omega_{\ell_0+1}} \rho\{F(\nabla u_\infty) - f u_\infty\} dx \\ + \int_{D_{\ell_0}} & \Big\{F(\nabla\{(1-\rho)u_\ell + \rho u_\infty\}) - F((1-\rho)\nabla u_\ell + \rho\nabla u_\infty)\Big\} dx. \end{aligned} \tag{2.27}$$

Let J be the last integral above. Since

$$\nabla\{(1-\rho)u_\ell + \rho u_\infty\} - \{(1-\rho)\nabla u_\ell + \rho\nabla u_\infty\} = -(u_\ell - u_\infty)\nabla\rho$$

we deduce from (2.5)

$$\begin{aligned} J \le C \int_{D_{\ell_0}} & \Big(1 + |\nabla\{(1-\rho)u_\ell + \rho u_\infty\}|^{q-1} \\ & + |\{(1-\rho)\nabla u_\ell + \rho\nabla u_\infty\}|^{q-1}\Big)|u_\ell - u_\infty| dx \\ \le C \int_{D_{\ell_0}} & \Big(1 + |\nabla u_\ell|^{q-1} + |u_\ell|^{q-1} + |\nabla u_\infty|^{q-1} + |u_\infty|^{q-1}\Big)|u_\ell - u_\infty| dx \end{aligned}$$

(we used the inequality $(a+b)^{q-1} \le C(a^{q-1}+b^{q-1})$ valid for $q \ge 1,\ a, b \ge 0$). Using Hölder's inequality we derive

$$\begin{aligned} J \le C\Big(\int_{D_{\ell_0}} |u_\ell - u_\infty|^q\, dx\Big)^{\frac{1}{q}} & \\ \Big\{(\int_{D_{\ell_0}} 1\, dx)^{\frac{1}{q'}} + (\int_{D_{\ell_0}} |\nabla u_\ell|^q\, dx)^{\frac{1}{q'}} & + (\int_{D_{\ell_0}} |u_\ell|^q\, dx)^{\frac{1}{q'}} \\ + (\int_{D_{\ell_0}} |\nabla u_\infty|^q\, dx)^{\frac{1}{q'}} & + (\int_{D_{\ell_0}} |u_\infty|^q\, dx)^{\frac{1}{q'}}\Big\}. \end{aligned}$$

Due to the Poincaré inequality it comes

$$\begin{aligned} J \le C\Big(\int_{D_{\ell_0}} |u_\ell - u_\infty|^q\, dx\Big)^{\frac{1}{q}} & \\ \Big\{|D_{\ell_0}|^{1-\frac{1}{q}} + (\int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q\, dx)^{1-\frac{1}{q}} & + (\int_{\Omega_{\ell_0+1}} |\nabla u_\infty|^q\, dx)^{1-\frac{1}{q}}\Big\}. \end{aligned}$$

Using (2.25) we deduce that for ℓ large enough

$$J = J_\ell \le C\Big(\int_{D_{\ell_0}} |u_\ell - u_\infty|^q \, dx\Big)^{\frac{1}{q}} (\ell_0+1)^{p(1-\frac{1}{q})}.$$

Thus, for large ℓ, (2.27) can be written as

$$\begin{aligned}\int_{\Omega_{\ell_0+1}} \rho\{F(\nabla u_\ell) - f u_\ell\}\, dx \le \int_{\Omega_{\ell_0+1}} \rho\{F(\nabla u_\infty) - f u_\infty\}\, dx \\ + C\Big(\int_{D_{\ell_0}} |u_\ell - u_\infty|^q \, dx\Big)^{\frac{1}{q}} (\ell_0+1)^{p(1-\frac{1}{q})}.\end{aligned} \tag{2.28}$$

Passing to the lim inf and using the weak lower semicontinuity of the left hand side (recall that F is convex) we get

$$\begin{aligned}\int_{\Omega_{\ell_0+1}} \rho\{F(\nabla \tilde u_\infty) - f \tilde u_\infty\}\, dx \le \int_{\Omega_{\ell_0+1}} \rho\{F(\nabla u_\infty) - f u_\infty\}\, dx \\ + C\Big(\int_{D_{\ell_0}} |\tilde u_\infty - u_\infty|^q \, dx\Big)^{\frac{1}{q}} (\ell_0+1)^{p(1-\frac{1}{q})}.\end{aligned}$$

Considering each of the terms above one has

$$\begin{aligned}\int_{\Omega_{\ell_0+1}} \rho\{F(\nabla \tilde u_\infty) - f\tilde u_\infty\}dx &= \int_{(\ell_0+1)\omega_1}\int_{\omega_2} \rho\{F(\nabla \tilde u_\infty) - f\tilde u_\infty\}dx \\ &= E_{\omega_2}(\tilde u_\infty)\int_{(\ell_0+1)\omega_1} \rho dX_1.\end{aligned}$$

Similarly

$$\int_{\Omega_{\ell_0+1}} \rho\{F(\nabla u_\infty) - f u_\infty\}\, dx = E_{\omega_2}(u_\infty)\int_{(\ell_0+1)\omega_1} \rho dX_1.$$

Finally

$$\begin{aligned}\Big(\int_{D_{\ell_0}} |\tilde u_\infty - u_\infty|^q \, dx\Big)^{\frac{1}{q}} &= \Big(\int_{(\ell_0+1)\omega_1\setminus \ell_0\omega_1}\int_{\omega_2} |\tilde u_\infty - u_\infty\}|^q \, dx\Big)^{\frac{1}{q}} \\ &= |\tilde u_\infty - u_\infty|_{q,\omega_2} |(\ell_0+1)\omega_1 \setminus \ell_0\omega_1|^{\frac{1}{q}} \\ &= |\tilde u_\infty - u_\infty|_{q,\omega_2} (|(\ell_0+1)\omega_1| - |\ell_0\omega_1|)^{\frac{1}{q}} \\ &\le C\ell_0^{(p-1)\frac{1}{q}}.\end{aligned}$$

where C is independent of ℓ_0. Thus we have obtained

$$\begin{aligned}E_{\omega_2}(\tilde u_\infty) &\le E_{\omega_2}(u_\infty) + C\ell_0^{(p-1)\frac{1}{q}} \ell_0^{p(1-\frac{1}{q})} \Big/ \int_{\Omega_{\ell_0+1}} \rho \, dX_1 \\ &\le E_{\omega_2}(u_\infty) + C\frac{\ell_0^{(p-1)\frac{1}{q}} \ell_0^{p(1-\frac{1}{q})}}{\ell_0^p}\end{aligned}$$

for some constant C, i.e.

$$E_{\omega_2}(\tilde u_\infty) \le E_{\omega_2}(u_\infty) + C\ell_0^{-\frac{1}{q}}.$$

Letting $\ell_0 \to +\infty$ we obtain $E_{\omega_2}(\tilde u_\infty) \le E_{\omega_2}(u_\infty)$. By definition of u_∞ this implies that $\tilde u_\infty = u_\infty$ and the result follows.

□

Remark 5. We have now proved the theorem 5. Note that the convergence (2.12) takes place pointwise and also in $L^q(\Omega_{\ell_0})$-strong and $W^{1,q}_{lat}(\Omega_{\ell_0})$-weak for any ℓ_0.

Remark 6. The function

$$F(\xi) = \frac{1}{q}|\xi|^q$$

satisfies the assumptions (2.3), (2.4). The solution of the minimisation problem (2.8) is also weak solution of the q-Laplace Dirichlet problem

$$\begin{cases} -\nabla \cdot (|\nabla u_\ell|^{q-2}\nabla u_\ell) = f \text{ in } \Omega_\ell, \\ u_\ell = 0 \text{ on } \partial\Omega_\ell. \end{cases}$$

One recovers the results of [12] i.e. the convergence of u_ℓ toward the weak solution u_∞ to

$$\begin{cases} -\nabla_{X_2} \cdot (|\nabla_{X_2} u_\infty|^{q-2}\nabla_{X_2} u_\infty) = f \text{ in } \omega_2, \\ u_\infty = 0 \text{ on } \partial\omega_2, \end{cases}$$

with a clear meaning for $\nabla_{X_2}\cdot$ i.e. the divergence in X_2.

To conclude this section we would like to consider the asymptotic behavior of

$$\frac{E_{\Omega_\ell}(u_\ell)}{|\ell\omega_1|}$$

when $\ell \to \infty$.

In particular we would like to prove (Cf. [8]):

Theorem 6. *(***Convergence of the energy***) One has for some constant $C > 0$ and sufficiently large ℓ,*

$$E_{\omega_2}(u_\infty) \leq \frac{E_{\Omega_\ell}(u_\ell)}{|\ell\omega_1|} \leq E_{\omega_2}(u_\infty) + \frac{C}{\ell}$$

where u_ℓ, u_∞ are the solutions to (2.8) *and* (2.11) *respectively.*

Proof. Since $u_\ell \in W^{1,q}_0(\Omega_\ell) \subset V^{1,q}(\Omega_\ell)$ the left hand side inequality is an immediate consequence of (2.15) which can be written for $v = u_\ell$

$$|\ell\omega_1|E_{\omega_2}(u_\infty) = E_{\Omega_\ell}(u_\infty) \leq E_{\Omega_\ell}(u_\ell).$$

To prove the right hand side inequality, one considers $\rho = \rho_{\ell-1}(X_1)$, as defined in (2.2). Since $\rho u_\infty \in W^{1,q}_0(\Omega_\ell)$ from (2.8), we have

$$E_{\Omega_\ell}(u_\ell) \leq E_{\Omega_\ell}(\rho u_\infty).$$

Thus from (2.2), (2.4) we deduce for some constant C

$$\begin{aligned}
E_{\Omega_\ell}(\rho u_\infty) &= E_{\Omega_{\ell-1}}(u_\infty) + \int_{\Omega_\ell\setminus\Omega_{\ell-1}} F\big(\nabla(\rho u_\infty)\big) - f u_\infty \rho dx \\
&\le E_{\Omega_\ell}(u_\infty) + \int_{\Omega_\ell\setminus\Omega_{\ell-1}} F\big(\nabla(\rho u_\infty)\big) - F\big(\nabla u_\infty\big) - f u_\infty(\rho-1) dx \\
&\le |\ell\omega_1| E_{\omega_2}(u_\infty) + \int_{\Omega_\ell\setminus\Omega_{\ell-1}} 2\Lambda' + \Lambda|\nabla(\rho u_\infty)|^q + \Lambda|\nabla u_\infty|^q + |f||u_\infty| dx \\
&\le |\ell\omega_1| E_{\omega_2}(u_\infty) + C\int_{\Omega_\ell\setminus\Omega_{\ell-1}} 1 + |\nabla u_\infty|^q + |u_\infty|^q + |f||u_\infty| dx \\
&\le |\ell\omega_1| E_{\omega_2}(u_\infty) + C\{|\ell\omega_1| - |(\ell-1)\omega_1|\}\int_{\omega_2} 1 + |\nabla u_\infty|^q + |u_\infty|^q + |f||u_\infty| dx.
\end{aligned}$$

Dividing by $|\ell\omega_1|$ the result follows, i.e. one has for some other constant C

$$\frac{E_{\Omega_\ell}(u_\ell)}{|\ell\omega_1|} \le E_{\omega_2}(u_\infty) + \frac{C}{\ell}.$$

This completes the proof of the theorem. □

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