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TRAVELING SPOTS ON MULTI-DIMENSIONAL EXCITABLE MEDIA

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Abstract. In this paper, the existence of a traveling spot on multi-dimensional excitable media is studied. First, the equations for the front and the back of a traveling spot including the interface equation are derived from the singular limit analysis of the reaction-diffusion system of the FitzHugh-Nagumo type. Then, the existence and uniqueness for the front and the back are shown when the speed of the traveling spot is less than the one of a planar wave. In addition, the non-convexity of the traveling spots is shown depending on the speed. Finally, the traveling spot converges to the planar wave as its speed tends to the one of the planar wave, which means that a traveling spot connects a stationary solution and a planar wave.

1. Introduction

Self-organized pattern formation has attracted considerable attention in recent years. In particular, various spatio-temporal patterns in excitable media have been reported in the last three decades, e.g., [12, 17, 20, 21]. Excitable media possess two states: exciting state and resting state. Localized patterns are produced by the exciting states. To study localized patterns such as a planar wave, a spiral wave, and a spot, several methods have been introduced. Two major methods are bifurcation theory [7, 15, 16] and singular limit analysis [14, 27]. These studies reveal that there is a radially symmetric stationary solution and the asymmetric traveling spot bifurcates from it. Local analysis has been performed theoretically, but the global branches were studied numerically. Namely, the global bifurcation of the singular limit problem remains difficult to address mathematically. New methods in multi-dimensional space have been demanded. Before we state the main results, we review more details of the singular limit problem. The singular limit analysis is a powerful tool to capture patterns. Essentially, by taking an appropriate singular limit of the Allen-Cahn equation, the solution converges to a characteristic function of a moving domain. The moving boundary is called *interface*. It is well known that the interface is ruled by a mean curvature equation with a driving force. This method is applicable to the FitzHugh-Nagumo

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system

$$(1.1)\qquad \begin{cases} u_t = \varepsilon\Delta u + \dfrac{1}{\varepsilon}(f_0(u) - \beta v), & x \in \mathbb{R}^n,\ t > 0, \\ v_t = d\Delta v + g_0(u, v), & x \in \mathbb{R}^n,\ t > 0, \end{cases}$$

where $\varepsilon > 0$, $d > 0$, $f_0(u) = u(u - 1/2)(1 - u)$, $g_0(u, v) = g_1 u - g_2 v + g_3$, and n is an integer greater than two. For the FitzHugh-Nagumo system, the limiting system consists of the interface equation and the equation for the inhibitor v:

$$(1.2)\qquad \begin{cases} V = W(v), & x \in \mathbb{R}^n,\ t > 0, \\ v_t = d\Delta v + g_0(h_\pm(v), v), & x \in \mathbb{R}^n,\ t > 0, \end{cases}$$

where V is a normal velocity of the interface, W is a function of v determined by the speed of a one-dimensional traveling wave, and $h_-(v), h_+(v)$ are the roots of $f_0(u) = v$ with $h_-(v) \le h_+(v)$. See [14, 27] for details. In some studies [4, 14], the mean curvature κ of the interface with small coefficient was added to (1.2). To derive the mean curvature term, let us consider the other parameter settings as follows:

$$(1.3)\qquad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f_\varepsilon(u) - \varepsilon\beta v), & x \in \mathbb{R}^n,\ t > 0, \\ v_t = d\Delta v + g_0(u, v), & x \in \mathbb{R}^n,\ t > 0, \end{cases}$$

where $f_\varepsilon(u) := u(u - 1/2 + \varepsilon\alpha)(1 - u)$. Then, the corresponding limiting system includes a mean curvature in the first equation of (1.2), namely,

$$(1.4)\qquad \begin{cases} V = W(v) - (n-1)\kappa, & x \in \partial\Omega(t)\ t > 0, \\ v_t = d\Delta v + g_0(\chi_\Omega, v), & x \in \mathbb{R}^n,\ t > 0, \end{cases}$$

where $\chi_{\Omega(t)}$ is a characteristic function of the domain Ω(t) surrounded by the interface $\partial\Omega(t)$, i.e.,

$$\chi_{\Omega(t)} = \begin{cases} 1, & x \in \Omega(t), \\ 0, & \text{otherwise.} \end{cases}$$

The convergence of the solutions of (1.3) to those of (1.4) is shown in [1] when $d > 0$. However, it is difficult to investigate the behavior of the solution of this limiting problem even in one spatial dimension. It is partially studied in [11] through the aid of the numerical computation.

When $d = 0$, the equation for v is easier to be handled. One-dimensional traveling pulses have been studied by many researchers, for example, [3, 10, 25] and so on. Karma [13] and Pelcé - Sun [24] studied spiral waves in two-dimensional excitable media (see also Scheel [26] and the references therein). They also provide some numerical results of the existence of rotating spiral waves. This system is often called a *wave front interaction model*. In [28], Zykov and Showalter extended this system to the traveling spot. Along these lines, we consider the multi-dimensional excitable media. However, (1.3) becomes a bistable system because there are three intersection points between the graph $v = f_\varepsilon(u)/(\varepsilon\beta)$ and the nullcline $g_0(u, v) = 0$ for small $\varepsilon > 0$. Therefore (1.4) is not suitable for the excitable

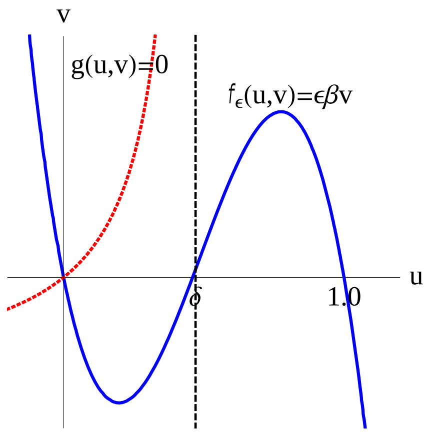


FIGURE 1. Nullclines of nonlinear functions.

media when ε is small. Chen, Kohsaka, and Ninomiya [5] introduced the FitzHugh-Nagumo type reaction-diffusion system with small parameter ε as follows:

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f_\varepsilon(u) - \varepsilon\beta v), & x \in \mathbb{R}^n,\ t > 0, \\ v_t = g(u, v), & x \in \mathbb{R}^n,\ t > 0, \end{cases} \tag{1.5}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and

$$f_\varepsilon(u) := u(1-u)\left(u - \frac{1}{2} + \varepsilon\alpha\right), \quad g(u, v) = g_1 u - \frac{g_2 v}{g_4 + g_3 v} \tag{1.6}$$

with positive constants $g_1, \cdots, g_4$ and α. The system (1.5) is a modified version of the FitzHugh-Nagumo system (1.3). Under the assumption

$$\frac{g_2}{g_3} < \frac{g_1}{2},$$

the ordinary differential equations corresponding to (1.5) is monostable for any $\varepsilon > 0$. Hence (1.5) is an excitable system for any $\varepsilon > 0$ (see Figure 1). In [5], the following free boundary problem has been formally derived from the singular limit problem of (1.5):

$$\begin{cases} V = W(v) - (n-1)\kappa, & x \in \partial\Omega(t), t > 0, \\ v_t = g(\chi_{\Omega(t)}, v), & x \in \mathbb{R}^n, t > 0 \end{cases} \tag{1.7}$$

where $\Omega(t)$, V, $\chi_{\Omega(t)}$, and κ are defined as above. In the process to derive the singular limit problem, we obtain

$$W(v) = a - bv$$

where $a = \sqrt{2}\alpha$ and $b = 6\sqrt{2}\beta$. This system (1.7) looks similar to (1.4) with $d = 0$. As stated before, even though (1.7) can be formally derived from (1.3), (1.3) becomes bistable for small $\varepsilon > 0$ and (1.4) is anymore realistic as the limit problem of the excitable system.

To look for the traveling spot solution of (1.7), we assume that the exciting region $\Omega(t)$ and the function v do not change their shapes and translate along the x_1-direction with constant speed c. Namely, the traveling spot of (1.7) with speed c satisfies

$$\begin{cases} \Omega(t) = \{(x_1 + ct, x') \in \Omega(0)\}, \\ v(x,t) = v(x_1 - ct, x', 0), \end{cases}$$

after an appropriate translation and rotation where $x' := (x_2, x_3, \cdots, x_n)$. In addition, we assume that $\Omega(t)$ and v are rotationally symmetric with respect to the x_1-axis. Then, we rewrite the free boundary problem (1.7) as

$$\begin{cases} c\cos\theta = W(v) - (n-1)\kappa, & x \in \partial\Omega(0), \\ -cv_{x_1} = g(\chi_{\Omega(0)}, v), & x \in \mathbb{R}^n, \end{cases} \tag{1.8}$$

where θ is the angle of the outer normal vector as measured from the positive x_1-axis, and V is equal to $c\cos\theta$. Therefore, if there exists a solution (Ω, v, c) of (1.8), then $\Omega(t) = \{(x_1 + ct, x') \mid (x_1, x') \in \Omega\}$ and $v(x_1 - ct, x')$ satisfy (1.7).

Theorem 1. *For any $c \in (0, a)$, there exist a unique constant b and a solution (Ω, v, c) of (1.8) such that*

$$\begin{aligned} & v \in C^1(\mathbb{R}^n \setminus \partial\Omega) \cap C^0(\mathbb{R}^n), \qquad \partial\Omega \in C^2, \\ & v(x_1, x') = 0 \quad \text{if } |x'| \geq r_0, \\ & \lim_{R\to\infty} \sup_{|(x_1,x')| \geq R} v(x_1, x') = 0, \end{aligned} \tag{1.9}$$

where r_0 is the distance between the x_1-axis and the boundary $\partial\Omega$ of the rotationally symmetric domain Ω.

Zykov and Showalter [28] studied the two-dimensional traveling spot for (1.4) with $g_0(u,v)$ replaced by $g_0(u,0)$ and $d = 0$. In this case, we note that $g_0(u,0)$ vanishes in the resting region. Thus, for their solutions, the concentration of the inhibitor v must be a constant with respect to x_1 after the back. However, for the excitable media such as the photosensitive Belousov-Zhabotinsky experiment, the inhibitor v must be recovered to 0 after the exciting region. Therefore, our singular limit problem (1.7) and the corresponding reaction-diffusion system (1.5) are improved ones.

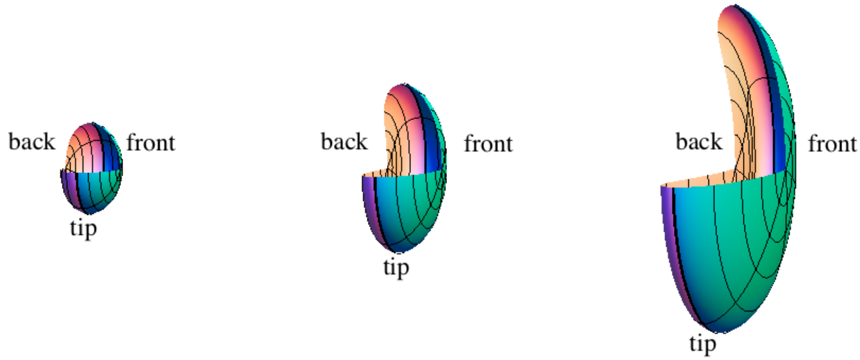
We also remark that the positivity of the diffusion coefficient is essential in the proof of the local existence of solutions to (1.4) and that the local existence of solutions to (1.7) is still open. Recently, Ninomiya and Wu [22] have shown the local existence of solutions of (1.7) near traveling waves in an appropriate functional space.

The traveling spot obtained in Theorem 1 possesses the following properties.

Theorem 2. *Let (Ω_c, v, c) be a traveling spot in Theorem 1. Then, the following hold:*

(i) *there is a c_* such that for $c \in [0, c_*)$, Ω_c is convex;*
(ii) *there is a c^* such that for $c \in (c^*, a)$, Ω_c is non-convex;*
(iii) *as $c \searrow +0$, Ω_c converges to the stationary disk $\{x \in \mathbb{R}^n \mid |x| < 1/a\}$;*
(iv) *as $c \nearrow a$, Ω_c converges to the planar wave $\{x = (x_1, x') \in \mathbb{R}^n \mid -aG^{-1}(2a/b) < x_1 < 0\}$ locally uniformly in x'.*

To treat a multi-dimensional traveling spot, we divide it into three part: a front, a back, and a tip. The interface where $V > 0$ (resp. $V < 0$, $V = 0$) is called a *front* (resp. a *back*,



(a) $c = 0.6,\ b = 0.31685$ (b) $c = 0.85,\ b = 0.63448$ (c) $c = 0.95,\ b = 0.8003178$

FIGURE 2. Profiles of the boundaries of Ω_c of 3D traveling spots when $n = 3,\ g_1 = 1,\ g_2 = 44,\ g_3 = 100,\ g_4 = 1$, and $a = 1$.

a *tip*). Compared with the work [5] when $n = 2$, a singularity appears in this case (see more detail in (2.2)). It makes the analysis become more complicated even for the front (see Section 3). Since the proof of the existence of the back is based on the continuous dependence for the solution with respect to the parameter b, the method is not directly applicable to our case due to the singularity. To overcome this difficulty, we construct a series of backs on the approximate intervals. Then we show the existence of the back as its limit. In addition, we prove the convergence of the traveling spots to the planar traveling wave as c tends to a. This result implies that the traveling spots connect a stationary solution and the planar traveling wave.

The remainder of this paper is organized as follows. First, we derive the equations to describe the front and the back of the traveling spot in Section 2. Next, in Section 3, we show the existence and uniqueness of the front. Then, we prove the existence and uniqueness of the back in Section 4. Finally, we study the relationship between the shape of the spot and the speed c in Section 5. We show the convexity of the traveling spot for small c and that the traveling spot is close to the planer wave when c is close to a.

2. Problem setting

First we change the Euclidean coordinates (x_1, x') to cylindrical coordinates (x_1, r, ω) by the relation $r = |x'|$ and $x' = r\omega$. We assume that $\Omega(t)$ and $v(x_1, x', t)$ are radially symmetric in x'. Moreover, we assume

$$\begin{aligned} &\Omega(t) = \{(x_1, r\omega) \in \mathbb{R}^n \mid w_-(r) + ct \le x_1 \le w_+(r) + ct,\ \omega \in S^{n-2}\}, \\ &v(x_1, x') = \widetilde{v}(x_1, r), \end{aligned} \tag{2.1}$$

with some functions $w_\pm$. If the interface $\partial\Omega$ is represented by the graph $x_1 = w(r)$, then the normal vector of the curve $x_1 = w(r)$ in the (x_1, r)-plane is given by $\pm(-1, w_r)/\sqrt{w_r^2 + 1}$. We denote the angle between the outer normal vector and the x_1-axis in the (x_1, r)-plane by θ. The unit normal vector N in the (x_1, x')-plane can be written as

$$N = \left(\cos\theta, \frac{x'}{|x'|} \sin\theta \right).$$

Notice that θ only depends on r. We compute that

$$\begin{aligned} \mathrm{div} N &= \frac{\partial}{\partial x_1} \cos\theta + \sum_{j=2}^{n} \frac{\partial}{\partial x_j} \left(\frac{x_j}{|x'|} \sin\theta \right) \\ &= \frac{n-1}{r} \sin\theta - \sum_{j=2}^{n} \frac{x_j^2}{r^3} \sin\theta + \sum_{j=2}^{n} \frac{x_j^2}{r^2} \cos\theta \frac{d\theta}{dr} \\ &= \frac{n-2}{r} \sin\theta + \cos\theta \frac{d\theta}{dr}. \end{aligned}$$

By the definition of a mean curvature, we have

$$\kappa = \frac{1}{n-1} \mathrm{div} N = \frac{1}{n-1} \left((n-2) \frac{\sin\theta}{r} + \frac{d\theta}{dr} \cos\theta \right).$$

Thus we derive

$$\frac{d\theta}{dr} = \frac{(n-1)\kappa - (n-2)\dfrac{\sin\theta}{r}}{\cos\theta}.$$

Introducing the arc length s of the curve in the (x_1, r)-plane and combining (1.8), we obtain the following system to describe traveling spots:

$$\begin{cases} \dfrac{dx_1}{ds} &= -\sin\theta, \\ \dfrac{dr}{ds} &= \cos\theta, \\ \dfrac{d\theta}{ds} &= W(v) - c\cos\theta - (n-2)\dfrac{\sin\theta}{r}, \\ -c\dfrac{\partial v}{\partial x_1} &= g(\chi_{\Omega(0)}, v). \end{cases} \tag{2.2}$$

We remark that x_1, r, θ, v and c are unknown and that the interface will be determined by the first three equations for any c if v is given on it. We recall that the interface where $V > 0$ (resp. $V < 0$, $V = 0$) is denoted by a front (resp. a back, a tip). We denote the front and the back solutions of the first three equations by

$$(x_{1,+}, r_+, \theta_+), \quad (x_{1,-}, r_-, \theta_-)$$

respectively for a given function v and a constant c. Note that the graph $x_1 = w_+(|x'|)$ (resp. $x_1 = w_-(|x'|)$) corresponds to the front (resp. the back). We will specify $(x_{1,+}, r_+, \theta_+)$, $(x_{1,-}, r_-, \theta_-)$ later.

Remark 1. When $n = 2$, the last term of the equation for θ in (2.2) vanishes. The equation of $d\theta/ds$ may have a singularity at $r = 0$ when $n \geq 3$. This causes the new difficulty to show the existence of the front and the back.

3. Existence and uniqueness of the front

We may assume that $v = 0$ on

$$\Omega^+(t) = \{(x_1, r\omega) \in \mathbb{R}^n \mid x_1 \geq w_+(r) + ct,\ \omega \in S^{n-2}\}.$$

Especially $v = 0$ on the front. Thus (2.2) becomes

$$\begin{cases} \dfrac{dx_{1,+}}{ds} &= -\sin\theta_+, \\ \dfrac{dr_+}{ds} &= \cos\theta_+, \\ \dfrac{d\theta_+}{ds} &= a - c\cos\theta_+ - (n-2)\dfrac{\sin\theta_+}{r}. \end{cases} \tag{3.1}$$

Note that $w_+(r_+(s)) = x_{1,+}(s)$ as long as the solution of (3.1) exists. Due to the independence of the equations for r and θ on x_1, it turns out that the solution with some shift along the x_1-axis also satisfies (3.1). Thus, we can put the initial value as follows:

$$x_{1,+}(0) = 0, \quad r_+(0) = 0, \quad \theta_+(0) = 0. \tag{3.2}$$

To show the existence of the solution of the initial value problem (3.1)-(3.2) of the front, we first consider the condition for the $d\theta_+(0)/ds$. Using the l'Hospital rule, we have

$$\begin{aligned} \frac{d\theta_+}{ds}(0) &= a - c\cos\theta(0) - (n-2)\cos\theta(0)\frac{d\theta_+}{ds}(0) \\ &= a - c - (n-2)\frac{d\theta_+}{ds}(0), \end{aligned}$$

which implies

$$\frac{d\theta_+}{ds}(0) = \frac{a-c}{n-1} > 0.$$

The next lemma guarantees the existence and uniqueness of the local solution of (3.1)-(3.2).

Lemma 1. *For each $c \in (0, a)$, there exists a unique solution $(x_{1,+}, r_+, \theta_+)(s)$ of the system* (3.1) *with* (3.2) *such that*

$$\begin{aligned} x_{1,+}(s) &= -\frac{a-c}{2(n-1)s^2} + O(s^3), \\ r_+(s) &= s + O(s^2), \\ \theta_+(s) &= \frac{a-c}{n-1}s + O(s^2) \end{aligned}$$

for $0 \leq s \leq s_0$ where s_0 is sufficiently small.

This lemma can be shown by the similar argument to Guo-Ninomiya-Tsai [9, Lemma 3.1]. So, we omit the proof.

When $\theta_s > 0$, s can be written as the function of θ. Set $R(\theta) := r_+(s(\theta))$, then $R(\theta)$ satisfies the following equations:

$$\begin{cases} \dfrac{dR}{d\theta} = \dfrac{R\cos\theta}{(a - c\cos\theta)R - (n-2)\sin\theta}, \\ R(0) = 0, \quad \dfrac{dR}{d\theta}(0) = \dfrac{n-1}{a-c}. \end{cases} \tag{3.3}$$

By Lemma 1, the local solution $R(\theta)$ exists and is positive for $\theta \in (0, \theta(s_0)]$.

Lemma 2. *The following inequality holds:*

$$R(\theta) \geq \frac{(n-1)\sin\theta}{a - c\cos\theta},$$

as long as the solution of (3.3) *exists.*

Proof. A simple calculation leads us to

$$\frac{d}{d\theta}\left(R - \frac{(n-1)\sin\theta}{a-c\cos\theta}\right) = \frac{(n-2)\cos\theta}{(a-c\cos\theta)R-(n-2)\sin\theta}\left(\frac{(n-1)\sin\theta}{a-c\cos\theta} - R\right) + \frac{(n-1)c\sin^2\theta}{(a-c\cos\theta)^2}.$$

This lemma immediately follows from the Grownwall inequality. □

Thus the solution of (3.3) can be extended for any $\theta \in [0, \pi/2]$. Since $dx_{1,+}/ds$ is monotone increasing in $\theta \in (0, \pi/2)$, the solution of the initial value problem (3.1)-(3.2) also exists uniquely by Lemmas 1 and 2.

Now we denote the size of the front by $r_0(c) := R(\pi/2)$.

Proposition 1. *The function $r_0(c)$ is strictly increasing in $c \in (0, a)$. Furthermore, we have $r_0(c) \to +\infty$ as $c \nearrow a$.*

Proof. By (3.3) we have

$$\left(\frac{\partial R}{\partial c}\right)_\theta = A(\theta)\Big\{R^2\cos^2\theta - (n-2)\frac{\partial R}{\partial c}\sin\theta\cos\theta\Big\},$$

where

$$A(\theta) := \frac{1}{\Big\{(a-c\cos\theta)R-(n-2)\sin\theta\Big\}^2}.$$

Using the Grownwall inequality, we see that $(\partial R/\partial c)(\theta) \geq 0$. Then we compute that

$$\frac{\partial R}{\partial c}(\theta) = \int_0^\theta \left[A(\tilde\theta)R^2(\tilde\theta)\cos^2\tilde\theta\exp\left(\int_0^{\tilde\theta}(n-2)A(\hat\theta)\sin\hat\theta\cos\hat\theta d\hat\theta\right)\right]d\tilde\theta \times\exp\left(-\int_0^\theta (n-2)A(\tilde\theta)\sin\tilde\theta\cos\tilde\theta d\tilde\theta\right).$$

If $(\partial R/\partial c)(\pi/2) = 0$, from the last equality, we know that $R(\theta) = 0$ for all $\theta \in [0, \pi/2]$ and get a contradiction. Therefore, $(\partial R/\partial c)(\pi/2) > 0$ and we get the conclusion.

For $\theta \in (0, \pi/2)$, it easy to see that

$$\frac{dR}{d\theta} = \frac{R\cos\theta}{(a-c\cos\theta)R-(n-2)\sin\theta} \geq \frac{\cos\theta}{a-c\cos\theta}.$$

Integrating both sides from 0 to $\pi/2$, we obtain that

$$r_0(c) \geq -\frac{\pi}{2c} + \frac{2a}{c\sqrt{a^2-c^2}}\tan^{-1}\left(\frac{a+c}{\sqrt{a^2-c^2}}\right).$$

Since the right hand side of the last inequality tends to infinity as c tends to a, we get the conclusion. □

Proposition 2. *When $c = 0$, the front is a semi-circle. Namely,*

$$x_{1,+}(s) = \frac{1}{a}(\cos(as) - 1), \quad r_+(s) = \frac{1}{a}\sin(as), \quad \theta_+(s) = as.$$

Furthermore, we have $r_0(0) = (n-1)/a$.

Proof. When $n \geq 3$, we can rewrite the first equality in (3.3) as follows:

$$(aR^{n-2} - (n-2)R^{n-3}\sin\theta)dR - R^{n-2}\cos\theta d\theta = 0$$

Since

$$\frac{\partial}{\partial\theta}(aR^{n-2} - (n-2)R^{n-3}\sin\theta) = -(n-2)R^{n-3}\cos\theta = \frac{\partial}{\partial R}(-R^{n-2}\cos\theta)$$

and $R(0) = 0$, it is easy to show that

$$R(\theta) = \frac{(n-1)\sin\theta}{a}.$$

This implies that $r_+(s) = (n-1)\sin\theta_+(s)/a$. Then we get the conclusion by substituting this into (3.1)-(3.2). ☐

Remark 2. We know that $r_0(c) > (n-1)/a$ for every $c \in (0, a)$ by Propositions 1 and 2.

Recall that the solution of the front with some shift along the x_1-axis is also a solution. Thus we have shown the existence of the front satisfying (3.1) with

$$r_+(0) = 0 \text{ when } \theta_+(0) = 0, \tag{3.4}$$

$$x_{1,+}(s_0) = 0 \text{ when } \theta_+(s_0) = \frac{\pi}{2}. \tag{3.5}$$

Hereafter we always assume the above property for the front.

4. Existence and uniqueness of the back

Recall that $w_\pm$ is defined in (2.1). Since s can be represented as a function of r, we define $(w_+(r), \varphi_+(r)) = (x_{1,+}(s(r)), \theta_+(s(r)))$. Then, the second equation of (1.8) becomes

$$-cv_{x_1} = g(\chi_{\Omega(0)}, v) = \begin{cases} g(1, v), & \text{if } w_-(r) \leq x_1 \leq w_+(r),\ |r| \leq r_0(c), \\ g(0, v), & \text{otherwise.} \end{cases}$$

Due to the symmetry assumption, v depends only on $x_1 - ct$ and $r = |x'|$. Hence we use $v(x_1 - ct, r)$ instead of $v(x_1 - ct, x')$. Since $v = 0$ on the front, we get

$$w_+(r) - x_1 = \int_0^{v(x_1,r)} \frac{c}{g(1,\xi)} d\xi. \tag{4.1}$$

We introduce two functions G_0 and G_1 defined by

$$G_1^{-1}(v) := \int_0^v \frac{d\xi}{g(1,\xi)}, \qquad G_0^{-1}(v) := \int_1^v \frac{d\xi}{g(0,\xi)}$$

respectively. If g is given by (1.6), then

$$\begin{aligned} G_0^{-1}(v) &= \frac{g_3}{g_2}(1-v) - \frac{g_4}{g_2}\log v, \\ G_1^{-1}(v) &= \frac{g_3 v}{g_1 g_3 - g_2} - \frac{g_4 g_2}{(g_2 - g_1 g_3)^2}\log\frac{g_4 g_1 + (g_1 g_3 - g_2)v}{g_4 g_1} \end{aligned}$$

as in [5]. From the definition of G_1 and (4.1), we obtain

$$v(x_1, r) = G_1\left(\frac{w_+(r) - x_1}{c}\right)$$

for $w_-(r) \le x_1 \le w_+(r),\ 0 \le r \le r_0$.

Similar calculations induce

$$v(x_1, r) = \begin{cases} 0, & \text{if } x_1 \ge w_+(r) \text{ or } r \ge r_0, \\ G_1\left(\dfrac{w_+(r) - x_1}{c}\right), & \text{if } w_-(r) \le x_1 \le w_+(r),\ 0 \le r \le r_0, \\ G_0\left(G_0^{-1}(v_-(r)) + \dfrac{w_-(r) - x_1}{c}\right), & \text{if } x_1 \le w_-(r),\ 0 \le r \le r_0, \end{cases} \tag{4.2}$$

where

$$v_-(r) := G_1\left(\frac{w_+(r) - w_-(r)}{c}\right).$$

Using $w_+(r)$ and $v_-(r)$, we can derive the equation of the back as follows:

$$\begin{cases} \dfrac{dx_{1,-}}{ds} = -\sin\theta_-, \\ \dfrac{dr_-}{ds} = \cos\theta_-, \\ \dfrac{d\theta_-}{ds} = W\left(G_1\left(\dfrac{w_+(r_-) - x_{1,-}}{c}\right)\right) - c\cos\theta_- - (n-2)\dfrac{\sin\theta_-}{r_-} \end{cases} \tag{4.3}$$

with the initial condition

$$(x_{1,-}(0), r_-(0), \theta_-(0)) = \left(0, r_0(c), \frac{\pi}{2}\right). \tag{4.4}$$

The local existence and uniqueness of solutions of (4.3) and (4.4) will be guaranteed by Lemma 3. We denote the solution of (4.3) with (4.4) by $(x_{1,-}(s), r_-(s), \theta_-(s)) = (x_{1,-}, r_-, \theta_-)(s; c, b)$. By the assumption that the traveling spot is symmetric with respect to the x_1-axis, we look for the positive constants $b^* = b^*(c)$ and $s^* = s^*(c)$ such that the solution should satisfy the following conditions:

$$r_-(s^*; b^*) = 0, \quad \theta_-(s^*; b^*) = \pi. \tag{4.5}$$

Note that the third equation of the system (4.3) has the singularity when $r_-(s^*; b^*) = 0$.

4.1. **Basic properties.** To show the existence and uniqueness of a solution of (4.3)–(4.5), we prepare several properties of the local solution of (4.3) and (4.4).

Lemma 3. *For each* $c \in (0, a)$ *and each* $b > 0$*, there exists a unique solution* $(x_{1,-}, r_-, \theta_-)(s)$ *of the system* (4.3) *with* (4.4) *such that*

$$\begin{aligned} x_{1,-}(s) &= -s + O(s^2), \\ r_-(s) &= r_0(c) - \frac{1}{2}\left(a - \frac{n-2}{r_0(c)}\right) s^2 + O(s^3), \\ \theta_-(s) &= \frac{\pi}{2} + \left(a - \frac{n-2}{r_0(c)}\right) s + O(s^2) \end{aligned}$$

for $0 \le s \ll 1$.

Since the proof is similar to [9, Lemma 3.1], we omit it.

For the simplicity of notation, we ignore the subscript minus sign from now on. By the similar argument as in [5, Lemma 4.2], the next lemma can be shown.

Lemma 4. *For* $b > 0$*, the following statements hold.*

(i) *If there is a* $s_* > 0$ *such that* $\theta(s_*) = 3\pi/2$ *and* $\pi/2 < \theta(s) < 3\pi/2$ *for* $0 < s < s_*$*, then* $\theta(s) > 3\pi/2$ *for* $s > s_*$ *with* $s - s_*$ *small.*

(ii) *If there is a* $s_* > 0$ *such that* $\theta(s_*) = \pi/2$ *and* $\pi/2 < \theta(s) < 3\pi/2$ *for* $0 < s < s_*$*, then* $\theta(s) < \pi/2$ *for* $s > s_*$ *with* $s - s_*$ *small.*

Proof. If the assumption of (i) holds, then we have $d\theta/ds(s_*) \ge 0$. When $d\theta/ds(s_*) > 0$, it is done. Otherwise, if $d\theta/ds(s_*) = 0$, we have

$$\begin{aligned} \frac{d^2\theta}{ds^2}(s_*) &= \left\{ -bG_1'\left(\frac{w_+ - x_1}{c}\right)\frac{w_{+,r}r_s - x_{1,s}}{c} + c\frac{d\theta}{ds}\sin\theta - (n-2)\frac{\cos\theta \cdot \frac{d\theta}{ds} r - \frac{dr}{ds}\sin\theta}{r^2} \right\}\Bigg|_{s=s_*} \\ &= \frac{b}{c}\, G_1'\left(\frac{w_+(r(s_*)) - x_1(s_*)}{c}\right) > 0 \end{aligned}$$

and get the conclusion of (i).

By the similar argument as the above, the conclusion of (ii) holds. □

Next, we consider the following open region:

$$D := \mathbb{R} \times (0, r_0(c)) \times \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

For any $c \in (0, a)$ and $b > 0$, we define exit-lengths $\underline{s}$, $\overline{s}$, $S(c, b)$ and an exit-point $(x_e, r_e, \theta_e)(c, b)$ as follows:

Definition 1. (exit-lengths and exit-points)

(i) If there is a positive number $\underline{s}$ such that the orbit stays in D for $0 < s < \underline{s}$, $r(\underline{s}) > 0$, $\theta(\tau) < \pi/2$ for $0 < \tau - \underline{s} \ll 1$, then $S(c, b) = \underline{s}$ and $(x_e, r_e, \theta_e)(c, b) = (x(\underline{s}), r(\underline{s}), \pi/2)$;

(ii) if there is a positive number $\overline{s}$ such that the orbit stays in D for $0 < s < \overline{s}$, $r(\overline{s}) > 0$, $\theta(\tau) > 3\pi/2$ for $0 < \tau - \overline{s} \ll 1$, then $S(c, b) = \overline{s}$ and $(x_e, r_e, \theta_e)(c, b) = (x(\overline{s}), r(\overline{s}), 3\pi/2)$;

(iii) if there is a positive number $\hat{s}$ such that the orbit stays in R for $0 < s < \hat{s}$ and $r(\hat{s}) = 0$, then $S(c, b) = \hat{s}$ and $(x_e, r_e, \theta_e)(c, b) = (x(\hat{s}), 0, \theta(\hat{s}))$.

From Lemma 4, the definitions of exit-length and exit-point are well-defined.

When $s = 0$, we have $d\theta/ds = a - (n-2)/r_0 > 0$ by $r_0 > (n-1)/a$. Then we can define the following.

Definition 2. For the solution (x_1, r, θ) of the system (4.3) with (4.4) on $[0, S(c,b))$, we denote by $\tilde{s}(c,b)$ the first positive s such that $d\theta/ds = 0$ holds. Then we also define $\tilde{r} := r(\tilde{s}; c, b)$, $\tilde{z} := r_0(c) - \tilde{r}$ and $\tilde{\theta} := \theta(\tilde{s}; c, b)$.

Set $z := r_0(c) - r$. Since r is strictly decreasing for $s \in (0, S)$, we can define $X(z) := x_1(s(z))$ and $\Theta(z) := \theta(s(z))$. Also, we set

$$z_e := r_0(c) - r_e, \quad X_+(z) := w_+(r_0(c) - z).$$

Then (X, Θ) satisfies the following differential equations

$$\begin{cases} \dfrac{dX}{dz} = \dfrac{\sin\Theta}{\cos\Theta}, \\ \dfrac{d\Theta}{dz} = -\dfrac{a - c\cos\Theta - bG_1(\dfrac{X_+(z) - X(z)}{c}) - (n-2)\dfrac{\sin\Theta}{r_0(c) - z}}{\cos\Theta} \end{cases} \tag{4.6}$$

with

$$X(0) = 0, \quad \Theta(0) = \frac{\pi}{2}. \tag{4.7}$$

The following lemmas can be shown by using the comparison principle with respect to the parameters b and c (see also [9, Lemma 3.3]).

Lemma 5. *Fix a $c > 0$. For $0 < b_1 < b_2$, the solutions $(X_i(z), \Theta_i(z)) := (X(z; c, b_i), \Theta(z; c, b_i))$ of (4.6) and (4.7) defined on $[0, z(S(c, b_i))]$, $(i = 1, 2)$ satisfy*

$$X_1(z) \geq X_2(z), \quad \Theta_1(z) \geq \Theta_2(z)$$

on $(0, \min\{z(S(c, b_1)), z(S(c, b_2), r_0(c))\})$ as long as $X_+(z) > X_1(z)$.

Lemma 6. *Fix a $b > 0$. For $0 < c_1 < c_2$, the solutions $(X_i(z), \Theta_i(z)) := (X(z; c_i, b), \Theta(z; c_i, b))$ of (4.6) and (4.7) defined on $[0, z(S(c_i, b))]$, $(i = 1, 2)$ satisfy*

$$X_2(z) \geq X_1(z), \quad \Theta_2(z) \geq \Theta_1(z)$$

on $(0, \min\{z(S(c_1, b)), z(S(c_2, b), r_0(c))\})$ as long as $X_+(z) > X_2(z)$.

Next, we borrow an idea from [6, Lemma 3.4-3.5] to give some properties between the front and the back. Now we construct a special solution to (4.3).

Proposition 3. *Given $b > 0$. Set*

$$(\hat{x}(s), \hat{r}(s), \hat{\theta}(s)) := \left(x_{1,+}(-s) - c\,G_1^{-1}\left(\frac{2a}{b}\right), r_+(-s), \theta_+(-s) + \pi \right), \quad s \geq 0.$$

Then $(\hat{x}, \hat{y}, \hat{\theta})$ becomes a solution of the system (4.3) defined for $s \in [0, s_+]$ where $r_+(-s_+) = 0$ and so

$$(\hat{X}, \hat{\Theta}) := \left(X_+(z) - c\,G_1^{-1}\left(\frac{2a}{b}\right), \Theta_+(z) + \pi \right)$$

is also a solution of (4.6).

Proof. Using $2a = bG_1(G_1^{-1}(2a/b)) = bG_1((x_{1,+}(-s) - \hat{x}(s))/c)$, we can easily check that $(\hat{x}, \hat{r}, \hat{\theta})$ is the solution of (4.3). □

The next lemma shows the relation between $\Theta_+(z)$ and $\Theta(z)$.

Lemma 7. *Fix a $c \in (0, a)$. For each $b > 0$, the solution $(x(s; c, b), r(s; c, b), \theta(s; c, b))$ of* (4.3) *and* (4.4) *satisfies one of the following properties :*

(i) *$\pi/2 < \Theta(z) < \Theta_+(z) + \pi$ for any $z \in (0, z_e)$.*

(ii) *There exists a $z_p = z_p(b) \in (0, z_e)$ such that $\pi/2 < \Theta(z) < \Theta_+(z) + \pi$ for any $z \in (0, z_p)$ and $\Theta(z_p) = \Theta_+(z_p) + \pi$.*

Moreover, in the case of (ii), *we have*

$$X_+(z) - X(z) < c\, G_1^{-1}\left(\frac{2a}{b}\right) \quad \text{for } \ 0 < z \le z_e,$$

$$\Theta(z) > \Theta_+(z) + \pi \quad \text{for } \ z \in (z_p, z_e).$$

By using the similar argument as the proof in [5, Lemma 3.3], the conclusion of Lemma 7 can be shown.

Lemma 8. *For any $c \in (0, a)$, $\Theta(z_e(c, 0)) \in (\pi, 3\pi/2]$.*

Proof. If $z_e(c, 0) < r_0(0) < r_0(c)$, we have $\Theta(z_e(c, 0)) = 3\pi/2$ or $\pi/2$ by Definition 1. The comparison principle implies that

$$\Theta(z_e(c, 0)) = \Theta(z_e(c, 0); c, 0) > \Theta(z_e(c, 0); 0, 0) > \frac{\pi}{2}.$$

Hence, we know that $\Theta(z_e(c, 0)) = 3\pi/2$.

For the case $z_e(c, 0) \ge r_0(0)$, by the comparison principle, we have

$$\Theta(r_0(0); c, 0) > \Theta(r_0(0); 0, 0) = \pi.$$

Hence, there exists a $z_1 \in (0, r_0(0))$ such that $\Theta(z_1; c, 0) = \pi$. Now we claim that $\Theta(z; c, 0) > \pi$ for any $z \in (z_1, z_e(c, 0)]$. If not, there exists a $z_2 \in (z_1, z_e(c, 0)]$ such that $\Theta(z_2; c, 0) = \pi$ and $d\Theta(z_2; c, 0)/dz \le 0$. But we also have

$$\frac{d\Theta(z_2; c, 0)}{dz} = a + c > 0$$

and get a contradiction. Therefore, we get the conclusion. □

From the next two lemmas, we know that $\Theta(z_e) = \pi/2$ for b sufficiently large. Although the proofs of the next two lemmas are similar to the proofs in [5, Lemma 5.2-5.3], for the reader's convenience, we provide the details here.

Lemma 9. *For each $\gamma \in (\pi/2, \pi]$, set*

$$\tilde{b} := \tilde{b}(\gamma, c) = \frac{a + c}{G_1(X_+(R(\gamma; c, 0))/c)}.$$

where $s_1 \in [0, \hat{s})$ satisfies $\theta(s_1; c, b) = \gamma$. If $b > \tilde{b}$, then

$$\frac{\pi}{2} < \tilde{\theta}(c, b) \le \gamma.$$

Moreover,

$$\lim_{b \to +\infty} \tilde{\theta}(c, b) = \frac{\pi}{2}, \quad \lim_{b \to +\infty} \tilde{r}(c, b) = r_0(c). \tag{4.8}$$

Proof. Fix a $\gamma \in (\pi/2, \pi]$. Since $R(\gamma; c, 0)$ belongs to the interval $(0, r_0(c))$ by Lemma 8, $\tilde{b}$ is well defined. By the contradiction argument, we assume that there exist a $b > \tilde{b}$ and the smallest $s_1 > 0$ such that $\theta(s_1; c, b) = \gamma$. Then we know that $x_1(s_1) < 0$ and $\theta'(s_1; c, b) > 0$. By the comparison principle, we also have $r(s_1; c, b) > r(s_1; c, 0)$. From these results and the fact G_1 is increasing, we obtain that

$$\begin{aligned}\theta'(s_1) &= a - c\cos\gamma - bG_1((X_+(r(s_1; c, b)) - x_1(s_1; c, b))/c) - \frac{(n-2)\sin\gamma}{r(s_1; c, b)}\\ &< a + c - bG_1(X_+(r(s_1; c, b))/c)\\ &< a + c - bG_1(X_+(R(\gamma; c, 0))/c)\\ &< a + c - \tilde{b}G_1(X_+(R(\gamma; c, 0))/c) = 0.\end{aligned}$$

Hence, this contradicts the definition of s_1. So we have $\pi/2 < \tilde{\theta}(c, b) \le \gamma$ for all $b > \tilde{b}$.

Since

$$\lim_{\gamma \to \pi/2} R(\gamma; c, 0) = r_0(c),$$

we have

$$\lim_{\gamma \to \pi/2} \tilde{b} = +\infty.$$

Therefore, (4.8) holds. □

Lemma 10. *For each $c \in (0, a)$, there is a positive constant $\underline{b} := \underline{b}(c)$ such that $S(c, b) = \underline{s}$ for all $b \ge \underline{b}$.*

Proof. By Lemma 9, for $\gamma = 3\pi/4$, we can take b_1 so large that

$$\tilde{\theta}(c, b) \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right), \quad \tilde{r}(c, b) \in \left(\frac{7}{8} r_0(c), r_0(c)\right)$$

for $b > b_1$. From this, we have

$$0 < \tilde{z} < \frac{r_0(c)}{8}, \quad \Theta'(z) > 0 \quad \text{for} \quad z \in (0, \tilde{z}).$$

When $\Theta'(z) = 0$ and $\Theta \in (\pi/2, \pi)$, we obtain that

$$\frac{d^2\Theta}{dz^2} = \frac{bG_1'\Big(\dfrac{X_+(z) - X(z)}{c}\Big)\dfrac{X_+'(z) - \tan\Theta}{c} + \dfrac{(n-2)\sin\Theta}{(r_0(c) - z)^2}}{\cos\Theta} < 0$$

holds. This also implies that $\Theta(z)$ is decreasing on $(\tilde{z}, z_e)$. So $\pi/2 < \Theta(z) < 3\pi/4$ holds for $z \in (\tilde{z}, z_e)$.

If we assume that $z_e \ge r_0(c)$, $\Theta(z)$ can be defined on $[0, r_0(c)]$ with

$$\Theta(z) \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right) \quad \text{for} \quad z \in [0, r_0(c)].$$

Hence, for $z \in [0, r_0(c)]$, we obtain $X(z) < 0$ by $X(0) = 0$ and

$$\frac{dX}{dz}(z) = \frac{\sin\Theta}{\cos\Theta} < 0.$$

Since G_1 is increasing, we know that $G_1((X_+(z)-X(z))/c) \geq G_1(X_+(z)/c)$. Thus, for $\Theta \in (\pi/2, 3\pi/4)$,

$$\begin{aligned} \frac{d\Theta}{dz} &= -\frac{a - c\cos\Theta - bG_1\Big(\dfrac{X_+(z)-X(z)}{c}\Big) - \dfrac{(n-2)\sin\Theta}{r_0(c)-z}}{\cos\Theta} \\ &< \sqrt{2}\left(a - c\cos\Theta - bG_1\Big(\frac{X_+(z)}{c}\Big)\right) \\ &< \sqrt{2}(a + c - bG_1\Big(\frac{X_+(z)}{c}\Big)) \\ &< \sqrt{2}\left(a + c - b\min_{z\in[r_0(c)/4, r_0(c)/2]} G_1\Big(\frac{X_+(z)}{c}\Big)\right) \end{aligned}$$

holds. For any $b \geq \underline{b}$ where

$$\underline{b} := \max\left\{ b_1, \frac{a + c + \dfrac{2\pi}{\sqrt{2}r_0(c)}}{\min_{z\in[r_0(c)/4, r_0(c)/2]} G_1\Big(\dfrac{X_+(z)}{c}\Big)} \right\},$$

we have

$$\frac{d\Theta}{dz} < -\frac{2\pi}{r_0(c)}.$$

Integrating from $r_0(c)/4$ to $r_0(c)/2$ on both sides yields

$$\Theta\left(\frac{r_0(c)}{2}\right) - \Theta\left(\frac{r_0(c)}{4}\right) < -\frac{2\pi}{r_0(c)}\left(\frac{r_0(c)}{2} - \frac{r_0(c)}{4}\right) = -\frac{\pi}{2}.$$

It contradicts $|\Theta(r_0(c)/2) - \Theta(r_0(c)/4)| < \pi/4$ by $\Theta(z) \in [\pi/2, 3\pi/4)$. From this, we show that $S(c, b) = \underline{s}$ for b sufficiently large. □

4.2. **Existence and uniqueness of a solution of** (4.3)–(4.5)**.** Now we define the following set:

$$B := \left\{ b > 0 \;\middle|\; \begin{array}{l} \text{There exists a constant } z_p = z_p(b) \in (0, z_e) \text{ such that} \\ \Theta(z_p) = \Theta_+(z_p) + \pi \text{ and} \\ \pi/2 < \Theta(z) < \Theta_+(z) + \pi \text{ for any } z \in (0, z_p) \end{array} \right\}.$$

Then we have the following property of B.

Lemma 11. *The set B is nonempty, bounded and open.*

Since the proof of this lemma can be done with an argument similar to [6, Proposition 3.7], we do not repeat it here.

Lemma 12. *For any sufficiently small $\delta > 0$, there exists a b_δ such that $\Theta(r_0(c) - \delta; b_\delta) = \pi$*

Proof. By Lemma 11, for any sufficiently small $\delta > 0$, there exist a $\underline{b}_\delta$ and a $z_{p,\delta} := z_p(\underline{b}_\delta)$ such that $\Theta(z_{p,\delta}; \underline{b}_\delta) = \Theta_+(z_{p,\delta}) + \pi > \pi$. Also, by Lemma 9 and 10, there exists a $\overline{b}_\delta$ such that $\Theta(r_0(c) - \delta; \overline{b}_\delta) = \pi/2$. By the continuous dependence on b at $r_0(c) - \delta$, there exists a $b_\delta \in (\underline{b}_\delta, \overline{b}_\delta)$ such that $\Theta(r_0(c) - \delta; b_\delta) = \pi$. □

Taking the monotone decreasing sequence $\{\delta_j\}$ such that $\delta_j \to 0$ as $j \to \infty$, we define b^* by the limit of b_{δ_j} as $j \to \infty$, i.e.

$$b^* := \lim_{j\to\infty} b_{\delta_j}. \tag{4.9}$$

Lemma 13. *For small $\eta_0 > 0$, take a positive integer J satisfying*

$$\delta_J < \frac{(n-2)\sin\eta_0}{2(a+c)}, \quad \Theta_+(r_0(c) - \delta_J) < \eta_0. \tag{4.10}$$

Fix an integer $j \geq J$. For any $\eta > \eta_0$, if there exist a $z_\eta \in (0, r_0(c))$ and a $s_\eta \in (0, S(c,b))$ such that

$$z(s_\eta) = z_\eta, \qquad |\Theta(z_\eta; b_{\delta_j}) - \pi| = \eta,$$

then

$$|\Theta(z; b_{\delta_j}) - \pi| \geq \eta$$

for any $z \geq z_\eta$. Moreover, $X(z; b_{\delta_j})$ is also bounded.

Proof. First, we show that $X(z)$ is bounded if $|\Theta(z; b_{\delta_j}) - \pi| < \eta$ for any $z \in [r_0(c) - \delta_j, z_e)$. Indeed, since

$$-\tan(\pi+\eta) = \tan(\pi-\eta) \leq \frac{dX}{dz}(z; b_{\delta_j}) = \tan\Theta(z; b_{\delta_j}) \leq \tan(\pi+\eta),$$

the integration of both sides from $z = r_0(c) - \delta_j$ to z implies that

$$|X(z; b_{\delta_j}) - X(r_0(c) - \delta_j; b_{\delta_j})| \leq \tan(\pi+\eta)(z - (r_0(c) - \delta_j)) \leq \delta_j \tan(\pi+\eta).$$

Then we know that $X(z)$ is bounded for $z \in [r_0(c) - \delta_j, z_e)$ with $|\Theta(z; b_{\delta_j}) - \pi| < \eta$.

Next, we claim that $\Theta(z; b_{\delta_j}) > \pi + \eta$ for $z > z_\eta$ and $X(z; b_\delta)$ is bounded, if $\Theta(z_\eta; b_{\delta_j}) = \pi + \eta$. From the last inequality of (4.10), there exists $z_p(b_{\delta_j}) \in (r_0(c) - \delta_j, z_\eta)$ such that

$$\Theta(z_p(b_{\delta_j}); b_{\delta_j}) = \Theta_+(z_p(b_{\delta_j})) + \pi.$$

By Lemma 7,

$$X_+(z) - X(z; b_{\delta_j}) < c\, G_1^{-1}\left(\frac{2a}{b_{\delta_j}}\right)$$

for $z_p(b_{\delta_j}) \leq z < z_e$. From this, we obtain that

$$\begin{aligned}
\frac{d\Theta}{dz}(z; c, b_{\delta_j}) &= -\frac{1}{\cos\Theta}\left(a - b_{\delta_j} G_1\Big(\frac{X_+(z) - X(z)}{c}\Big) - c\cos\Theta - \frac{(n-2)\sin\Theta}{r_0(c) - z}\right) \\
&\geq -\frac{1}{\cos\Theta}\left(a - b_{\delta_j}\frac{2a}{b_{\delta_j}} - c - \frac{(n-2)\sin(\pi+\eta)}{\delta_j}\right) \\
&= -\frac{1}{\cos\Theta}\left(\frac{(n-2)\sin\eta}{\delta_j} - (a+c)\right) \\
&\geq -\frac{1}{\cos\Theta}\left(\frac{(n-2)\sin\eta_0}{\delta_J} - (a+c)\right) \\
&= -\frac{1}{\cos\Theta}(a+c) > 0
\end{aligned}$$

as long as $\Theta(z; b_{\delta_j}) \in (\pi+\eta, 3\pi/2)$. Thus, it follows from $\Theta(z_\eta; b_{\delta_j}) = \pi+\eta$ that $\Theta(z; b_{\delta_j}) > \pi + \eta$ for $z \in (z_\eta, z_e)$. Similarly, we have

$$\theta(s; b_{\delta_j}) - \theta(s_\eta; b_{\delta_j}) \geq (s - s_\eta)(a + c).$$

By

$$\theta(s; b_{\delta_j}) - \theta(s_\eta; b_{\delta_j}) \leq \frac{3\pi}{2} - (\pi + \eta) = \frac{\pi}{2} - \eta,$$

we derive that

$$s - s_\eta \leq \frac{\pi/2 - \eta}{a + c}.$$

Since $|dx/ds| = |-\sin\theta| \leq 1$, by integrating from $s = s_\eta$ to s, we obtain that

$$\begin{aligned} |x(s; b_{\delta_j})| &\leq |x(s_\eta; b_{\delta_j})| + |s - s_\eta| \\ &\leq |x(s_\eta; b_{\delta_j})| + \left|\frac{\pi/2 - \eta}{a + c}\right|. \end{aligned}$$

Hence, $x(s; b_{\delta_j})$ is bounded.

By a similar argument to the case $\theta(s_\eta; b_{\delta_j}) = \pi + \eta$, if $\theta(s_\eta; b_{\delta_j}) = \pi - \eta$, we obtain that $\Theta(z; b_{\delta_j}) < \pi - \eta$ for $z > z_\eta$ and $X(z; b_{\delta_j})$ is bounded. □

Lemma 14.

$$\lim_{\delta \to +0} \Theta(r_0(c) - \delta; b^*) = \pi$$

holds.

Proof. We show this lemma by a contradiction argument. Assume that there exist a $\eta > 0$ and a $z_{2\eta}$ such that

$$|\Theta(z_{2\eta}; b^*) - \pi| = 2\eta \tag{4.11}$$

Here $\hat{\delta}$ is defined by $z_{2\eta} = r_0(c) - \hat{\delta}$.

Since $\Theta(z; b)$ is continuous with respect to b at $z = z_{2\eta}$, for any $\varepsilon \in (0, \eta)$, there exists a large integer j such that

$$|\Theta(z_{2\eta}; b_{\delta_j}) - \Theta(z_{2\eta}; b^*)| < \varepsilon \tag{4.12}$$

holds. By (4.11) and (4.12), we get

$$|\Theta(z_{2\eta}; b_{\delta_j}) - \pi| > 2\eta - \varepsilon > \eta$$

From Lemma 13, for $z \geq z_{2\eta}$, we know that

$$|\Theta(z; b_{\delta_j}) - \pi| > \eta$$

By the fact $\Theta(r_0(c) - \delta_j; b_{\delta_j}) = \pi$, we have

$$r_0(c) - \delta_j \leq r_0(c) - \hat{\delta}$$

which implies that $\delta_j \geq \hat{\delta} > 0$. Then we get a contradiction by $\delta_j \to 0$ as $j \to \infty$. Therefore, the proof of this lemma has been completed. □

Proof of Theorem 1. The existence of the front has been shown in Section 3. For the back, it is shown that there exists a b^* given by (4.9) such that the corresponding solution of (4.3) satisfies the conditions (4.4) and (4.5). These results show the existence of Ω for any $c \in (0, a)$. To show that $\partial\Omega$ is C^2, we only need to treat it at the tip. Since

$$\begin{aligned} &x_{1,+}(s_0) = 0 = x_{1,-}(0), \quad r_+(s_0) = r_0(c) = r_-(0) \\ &\frac{dx_{1,+}}{ds}(s_0) = -1 = \frac{dx_{1,-}}{ds}(0), \quad \frac{dr_+}{ds}(s_0) = 0 = \frac{dr_-}{ds}(0), \\ &\frac{d^2x_{1,+}}{ds^2}(s_0) = 0 = \frac{d^2x_{1,-}}{ds^2}(0), \quad \frac{d^2r_+}{ds^2}(s_0) = -a + \frac{n-2}{r_0(c)} = \frac{d^2r_-}{ds^2}(0), \end{aligned}$$

we see that $\partial\Omega$ is C^2. For the uniqueness of b^*, by the fact $X_+(z) - X(z; b^*) > 0$ for all $z \in (0, r_0(c)]$, it follows from Lemma 5

From the formula of $v(x_1, r)$ in (4.2), it is easy to see that $v \in C^1(\mathbb{R}^n \setminus \partial\Omega) \cap C^0(\mathbb{R}^n)$ and $v(x_1, x') = 0$ if $|x'| \geq r_0$ or $x_1 \geq w_+(r)$. To show (1.9), we only need to consider the case x_1 tends to $-\infty$. When $x_1 \to -\infty$, we have $G_0^{-1}(v) \to \infty$. This implies that v tends to 0 as $x_1 \to -\infty$. Therefore, we get the conclusion. □

5. Properties of traveling spots

In this section we consider the profiles of traveling spots. In order to study them, we prepare the following lemmas.

Lemma 15. *The back of a traveling spot never intersects the front except for the tip* $(X_+(0), r_0(c))$.

Proof. Applying Proposition 7 to the back of traveling spots, we have

$$\frac{\pi}{2} < \Theta(z) < \Theta_+(z) + \pi \tag{5.1}$$

for any $0 < z < r_0(c)$. Then we derive that $(X_+ - X)(z)$ is increasing in $z \in (0, r_0(c))$ by $\Theta_+ \in (0, \pi/2)$, $\Theta \in (\pi/2, 3\pi/2)$ and

$$\frac{d(X_+ - X)(z)}{dz} = \frac{\sin(\Theta_+ - \Theta)}{\cos\Theta_+ \cos\Theta} > 0.$$

Hence we conclude to the non-existence of self-intersection points of traveling spots. □

Lemma 16. *For the back of a traveling spot,* Θ *can attain* π *at most once in* $z \in (0, r_0(c))$. *Moreover, if* $\Theta(z_q) = \pi$, *then*

$$\frac{d\Theta}{dz}(z_q) \geq 0. \tag{5.2}$$

Proof. First we note that $\Theta(r_0(c)) = \pi$. If there are more than two roots of $\Theta(z) = \pi$, we can assume that there are z_1 and z_2 $(0 < z_1 < z_2 \leq r_0(c))$ such that $\Theta(z_1) = \Theta(z_2) = \pi$ and $\Theta(z) < \pi$ for any $z \in (z_1, z_2)$. Then there is a $z_3 \in (z_1, z_2)$ such that

$$d\Theta/dz(z_3) = 0, \quad d^2\Theta/dz^2(z_3) \geq 0, \quad \frac{\pi}{2} < \Theta(z_3) < \pi.$$

On the other hand, we compute that

$$\begin{aligned}\frac{d^2\Theta}{dz^2} &= -\frac{\sin\Theta}{\cos^2\Theta}\frac{d\Theta}{dz}\left\{a - c\cos\Theta - bG_1\left(\frac{X_+ - X}{c}\right) - (n-2)\frac{\sin\Theta}{r_0(c)-z}\right\} \\ &- \frac{1}{\cos\Theta}\left\{c\sin\Theta\frac{d\Theta}{dz} - bG_1'\left(\frac{X_+ - X}{c}\right)\frac{1}{c}\frac{d(X_+ - X)(z)}{dz}\right. \\ &\left. -(n-2)\left[\frac{\sin\Theta}{(r_0(c)-z)^2} + \frac{\cos\Theta}{r_0(c)-z}\frac{d\Theta}{dz}\right]\right\}\end{aligned}$$

Substituting $z = z_3$ to the above equality, we have

$$\frac{d^2\Theta}{dz^2}(z_3) = \left\{\frac{b}{c\cos\Theta(z_3)}G_1'\left(\frac{X_+ - X}{c}\right)\frac{d(X_+ - X)(z_3)}{dz} + \frac{n-2}{\cos\Theta(z_3)}\left[\frac{\sin\Theta(z_3)}{(r_0(c)-z_3)^2}\right]\right\} < 0$$

since $\Theta_+(z_3) \in (0, \pi/2)$, $\Theta(z_3) \in (\pi/2, \pi)$ and the functions G_1 and $X_+ - X$ are increasing. This contradicts the definition of z_3. Thus the first statement has been shown.

If there is a z_q such that $\Theta(z_q) = \pi$ and $d\Theta/dz(z_q) < 0$, then there is a $z_0(> z_q)$ close to z_q such that $\Theta(z_q) > \pi$. This implies that there is another root of $\Theta(z) = \pi$ which is less than z_q, namely, there are at least two roots. This contradicts the first statement. □

Lemma 17. *As for the back of a traveling spot, $\Theta(z)$ is increasing while $\Theta(z) \in (\pi/2, \pi)$.*

Proof. We show this lemma by contradiction. Assume that there is a $z_1 \in (0, r_0(c))$ such that

$$\frac{d\Theta}{dz}(z_1) < 0, \quad \frac{\pi}{2} < \Theta(z_1) < \pi.$$

According to Lemma 16, $\Theta(z) \in (\pi/2, \pi)$ for any $z \in (0, z_1)$. Since $\Theta(r_0(c)) = \pi$, there is a $z_2 \in (z_1, r_0(c))$ such that

$$\frac{d\Theta}{dz}(z_2) = 0, \quad \frac{d^2\Theta}{dz^2}(z_2) \geq 0, \quad \frac{\pi}{2} < \Theta(z_2) < \pi.$$

As in the proof of Lemma 16, we have

$$\frac{d^2\Theta}{dz^2}(z_2) = \left\{\frac{b}{c\cos\Theta(z_2)}\left[G_1'\left(\frac{X_+ - X}{c}\right)\frac{d(X_+ - X)(z_2)}{dz}\right] + \frac{n-2}{\cos\Theta(z_2)}\left[\frac{\sin\Theta(z_2)}{(r_0(c)-z_2)^2}\right]\right\} < 0.$$

This contradicts the definition of z_2. Hence the proof is complete. □

5.1. **Convexity.** In this study, we classify the traveling spot into two types as in [9]:

(i) $\theta'(s) > 0$ on $[0, s^*)$ (Convex type),
(ii) $\theta'(s)$ changes its sign in $[0, s^*)$ (Non-convex type).

First, for $c = 0$, we have $b^*(0) = 0$, $s^* = \pi/2a$ and the corresponding solution

$$(x_{1,-}, r_-, \theta_-)(s; 0, 0) = \Big(-\frac{1}{a}\sin as, \frac{1}{a}(\cos as), \pi/2 + as\Big)$$

for $s \in [0, s^*]$. Since we have the uniqueness of $b^*(c)$ for each $c \in (0, a)$ and the comparison principle Lemma 5, by the similar argument as in [9, Lemma 6.1-6.3], we have the following two lemmas.

Lemma 18. *$b^*(c)$ is continuous in $c \in (0, a)$. Moreover, if $0 < c_1 < c_2 < a$ and $\theta'(s; c_i, b^*(c_i)) > 0$ for all $s \in [0, s^*(c_i))$, $i = 1, 2$, then $b^*(c_1) < b^*(c_2)$.*

Lemma 19. $\lim_{c \to 0^+} b^*(c) = 0$.

By the last two lemmas, the continuity of $\theta'(s;c,b^*(c))$ with respect to s, c, $b^*(c)$ and the fact $\theta'(s;0,0)=a>0$, we have $\theta'(s;c,b^*(c))>0$ for all $s\in[0,s^*(c)]$ for the sufficiently small $c>0$. This completes the proof of Theorem 2 (i). Moreover, we have

$$\lim_{c\to 0}(x_{1,+},r_+,\theta_+)(s;c) \quad = (x_{1,+},r_+,\theta_+)(s;0)$$

and

$$\lim_{c\to 0}(x_{1,-},r_-,\theta_-)(s;c,b^*(c)) \quad = (x_{1,-},r_-,\theta_-)(s;0,0).$$

Therefore, Theorem 2 (iii) holds and the region Ω of the traveling spot converges to a disk with the radius $1/a$ as c tends to 0.

Now we will show that the traveling spot is non-convex when c is close to a. From the results shown in the above, there exists a $c_0\in(0,a)$ such that the traveling spot is convex for all $c\in(0,c_0]$. On the other hand, we set $W_+(\theta):=w_+(R(\theta))$ where $w_+(r)=x_{1,+}(s(r))$. Then $W_+(\theta)$ satisfies the following system.

$$\begin{cases} \dfrac{dW_+(\theta)}{d\theta} = \dfrac{-\sin\theta}{a-c\cos\theta-(n-2)\dfrac{\sin\theta}{R(\theta)}} \\ \\ W_+(\pi/2)=0. \end{cases}$$

For $\theta\in[0,\pi/2]$, we have

$$\frac{dW_+(\theta)}{d\theta} \le \frac{-\sin\theta}{a-c\cos\theta} \tag{5.3}$$

Integrating both sides of (5.3) from 0 to $\pi/2$, we have

$$W_+(0;c)\ge \frac{1}{c}\log\frac{a}{a-c}.$$

Then it is easy to see that $W_+(0;c)$ tends to ∞ when c tends to a. Since $b^*(c_0)$ is bounded, there exists a $\underline{c}>c_0$ such that

$$W_+(0;c)>aG_1^{-1}\left(\frac{2a}{b^*(c_0)}\right)$$

for all $c\in[\underline{c},a)$. Now we claim that the traveling spot is of type non-convex for all $c\in[\underline{c},a)$. If not, there exists a $\hat{c}\in[\underline{c},a)$ such that $\theta'(s;\hat{c},b^*(\hat{c}))>0$, $\theta_-(s;\hat{c},b^*(\hat{c}))\in(\pi/2,\pi)$ and $x_{1,-}(s;\hat{c},b^*(\hat{c}))<0$ for all $s\in[0,s^*(\hat{c}))$. By Lemma 18, we also have $b^*(\hat{c})>b^*(c_0)$. Since G_1 is increasing, we derive that

$$a-\hat{c}\cos\theta_- - b^*(\hat{c})G_1\left(\frac{w_+(r_-)-x_{1,-}}{c}\right)-(n-2)\frac{\sin\theta_-}{r_-} < 2a-b^*(c_0)G_1\left(\frac{w_+(r_-)}{a}\right)$$

Then we know that $\theta'(s;\hat{c},b^*(\hat{c}))<0$ near $s=s^*(\hat{c})$ and get a contradiction. Therefore, we proved that the traveling spot is non-convex when c is close to a. The proof of Theorem 2 (ii) has been done.

5.2. **Convergence to a planar wave.** Now, it remains to show Theorem 2 (iv) holds. When $c = a$, it is easy to see that $(x_{1,+}, r_+, \theta_+)(s) = (0, s, 0)$ is the planar wave solution of (3.1)-(3.2) for any $s \geq 0$. So we have $\varphi_+(r) = 0$ for any $r \geq 0$. By Lemma 18, we define $b^*(a) := \lim_{c \nearrow a} b^*(c)$. For convenience, we denote $b^*(a)$ by b^*. For $b = b^*$ and $c = a$, we also derive that that $(X, \Theta)(z) = (X_+(z) - aG_1^{-1}(2a/b^*), \pi)$ is the planar solution of (4.6) for all $z \geq 0$ by

$$\frac{dX}{dz} = 0 = \frac{\sin \pi}{\cos \pi},$$

$$\frac{d\Theta}{dz} = 0 = -\frac{a - a\cos\pi - b^* G_1\left(\dfrac{X_+(z) - (X_+(z) - aG_1^{-1}(2a/b^*))}{a}\right) - (n-2)\dfrac{\sin\pi}{r_0(c) - z}}{\cos\pi}.$$

To show that the front converges to the planar wave as $c \nearrow a$, we claim that for any $L > 0$, $\varphi_+(r) = \Theta(r_0(c) - r)$ converges to 0 uniformly in $|r| < L$ as $c \nearrow a$. Lemma 2 implies that

$$r > k(\sin\varphi_+(r); c)$$

where

$$k(s; c) := \frac{s}{a - c\sqrt{1 - s^2}}.$$

Since the function k is monotone increasing in $s \in (0, \sqrt{1 - (c/a)^2})$, we can define the inverse function k^{-1} on $(0, 1/\sqrt{a^2 - c^2})$. Then, we have

$$\sin\varphi_+(r) < k^{-1}(r; c) < \sqrt{1 - \frac{c^2}{a^2}}$$

for $0 < r < 1/\sqrt{a^2 - c^2}$. When $c \nearrow a$, $k^{-1}(r; c)$ tends to 0 for any $r > 0$. Hence we get the conclusion. This means that the front converges to the planar wave locally uniformly as $c \nearrow a$, i.e.,

$$\lim_{c \nearrow a} X_+(z) = 0. \tag{5.4}$$

Next we consider the back. Now we want to show that

$$X_+(z) - X(z) - a\,G_1^{-1}\left(\frac{2a}{b^*}\right) \to 0 \tag{5.5}$$

as $c \nearrow a$. Since

$$-\frac{d\theta}{dr}(0) = a - b^* v(X(r_0(c)), 0) + c - (n-2)\lim_{r \to 0} \frac{\sin\theta}{r},$$

we have

$$-\frac{d\theta}{dr}(0) = \frac{a - b^* v(X(r_0(c)), 0) + c}{n - 1}.$$

As seen in the previous subsection, if c is sufficiently close to a, $\theta'(s)$ becomes negative near $r = 0$, namely,

$$-\frac{d\theta}{dr}(0) = \frac{a - b^* v(X(r_0(c)), 0) + c}{n - 1} < 0,$$

which yields

$$v(X(r_0(c)), 0) = G_1\left(\frac{X_+(r_0(c)) - X(r_0(c))}{c}\right) > \frac{a + c}{b^*}.$$

This immediately implies that

$$c\,G_1^{-1}\left(\frac{a+c}{b^*}\right) \le X_+(r_0(c)) - X(r_0(c)) \le c\,G_1^{-1}\left(\frac{2a}{b^*}\right).$$

Then we have

$$\lim_{c\nearrow a}(X_+(r_0(c)) - X(r_0(c))) = a\,G_1^{-1}\left(\frac{2a}{b^*}\right).$$

Next we show (5.5) by a contradiction argument in the interval $[r_0(c)-L, r_0(c))$. For $c \nearrow a$, there are $\delta > 0$ and $z_c \in [r_0(c)-L, r_0(c))$ such that

$$X_+(z_c) - X(z_c) \le c\,G_1^{-1}\left(\frac{2a-\delta}{b^*}\right). \tag{5.6}$$

Recall that the traveling spot is non-convex if c is close to a. Lemma 16 implies that there is a unique root z_q such that $\Theta(z_q) = \pi$ and $\Theta(z) > \pi$ for any $z \in (z_q, r_0(c))$. Moreover, we have

$$X_+(z_q) - X(z_q) < c\,G_1^{-1}\left(\frac{a+c}{b^*}\right)$$

by

$$\frac{d\Theta}{dz}(z_q) = a + c - b^*\,G_1\left(\frac{X_+(z_q) - X(z_q)}{c}\right) > 0.$$

For any $z \in (z_q, r_0(c))$,

$$X(z) < X(r_0(c)).$$

It turns out from (5.4) that, for $z_q < z < r_0(c)$,

$$a\,G_1^{-1}\left(\frac{2a}{b^*}\right) \ge X_+(z) - X(z) \ge X_+(z) - X(r_0(c)) \to a\,G_1^{-1}\left(\frac{2a}{b^*}\right) \tag{5.7}$$

as $c \nearrow a$. Thus by the definition of z_c, $r_0(c) - L \le z_c \le z_q$. By Lemma 17, we have

$$\frac{\pi}{2} \le \Theta(z) < \Theta(z_c) < \pi$$

for $0 \le z < z_c$. Thus

$$X(z_c) < X(0) = X_+(0).$$

On the other hand, for the front, we have

$$\begin{aligned}\frac{dx_+}{d\theta_+} &= -\frac{\sin\theta_+}{a - c\cos\theta_+ - \dfrac{(n-2)\sin\theta_+}{r_0(c)-z}}\\ &\le -\frac{\sin\theta_+}{a - c\cos\theta_+}.\end{aligned}$$

This implies

$$X_+(0) - X_+(z_c) \le \left[-\frac{1}{c}\log\left(a - c\cos\theta\right)\right]_{\theta=\Theta_+(z_c)}^{\theta=\pi/2} = -\frac{1}{c}\log\frac{a}{a - c\cos\Theta_+(z_c)}.$$

Since $\Theta_+(z) \to 0$ for any $z \in [r_0(c)-L, r_0(c)]$ as $c \nearrow a$, we obtain that

$$X_+(z_c) - X_+(0) \to \infty$$

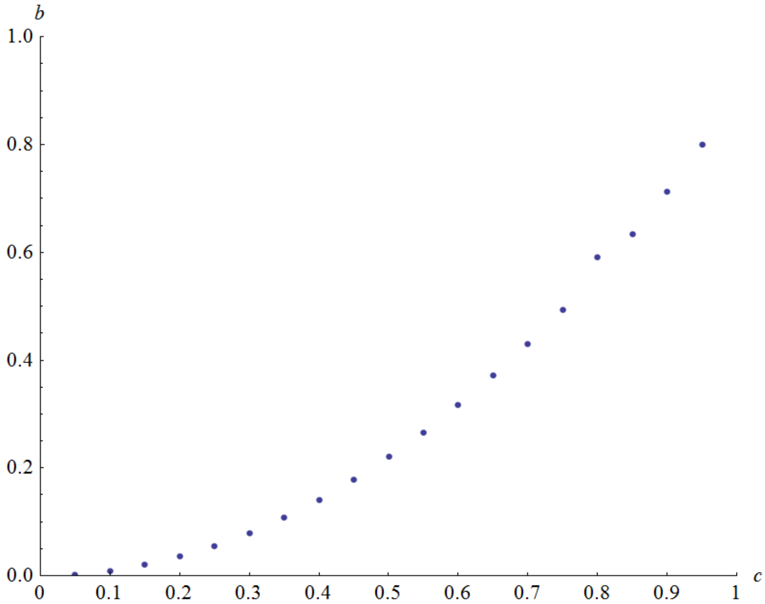


FIGURE 3. Relationship between c and b when $n = 3$, $g_1 = 1$, $g_2 = 44$, $g_3 = 100$, $g_4 = 1$ and $a = 1$.

as $c \nearrow a$. Therefore

$$X_+(z_c) - X(z_c) \geq X_+(z_c) - X_+(0) \to \infty.$$

This contradicts (5.6). Thus (5.5) has been shown. Now we give the proof of Theorem 2 (iv) as follows.

Proof of Theorem 2 (iv). From the results shown in the above, we show that for any $L > 0$, $\varphi_+(r) = \Theta(r_0(c) - r)$ converges to 0 uniformly in $|r| < L$ as $c \nearrow a$. This implies that the solution of the front $(x_{1,+}, r_+, \theta_+)(s; c)$ converges to the planar wave $(x_{1,+}, r_+, \theta_+)(s) = (0, s, 0)$. For the back, we have $X_+(z; c) - X(z; c)$ converges to $a\, G_1^{-1}(2a/b^*)$ as $c \nearrow a$. Therefore, we derive that the solution for the back $(X, \Theta)(z; c)$ converges to the planar wave $(X_+(z) - a\, G_1^{-1}(2a/b^*), \pi)$. □

6. CONCLUSION

Traveling spot is one of the patterns observed in n-dimensional excitable media ($n \geq 3$). To capture this pattern we have used interface equations derived from the singular limit problem of the FitzHugh-Nagumo type equations, introduced in [5]. The traveling spot mainly consists of the front and the back. The system for the front is a set of three-component ordinary differential equations (3.1), (3.4), and (3.5). The system for the back is coupled with that for the front and three-component ordinary differential equations (4.3), (4.4), and (4.5). For $n \geq 3$, these systems include the singularities when $r = 0$. This makes the problem more complicated than the case [5] where $n = 2$. The existence of the traveling spot is shown by proving the existence of the front and the back if the speed c is smaller than that of a planar traveling wave. More precisely, for any $c \in [0, a)$, there is a constant

b such that the traveling spot uniquely exists up to the shift where a is a parameter of the system corresponding to the speed of a planar traveling wave. Unfortunately, we cannot show the existence of c for a given constant b. However, by numerical computations, the relationship between c and b seems to be monotone, see Figure 3. In addition, we classify the traveling spot into two cases: convex and non-convex. Furthermore, we obtain that the traveling spot is close to a planar wave when the wave speed c is close to a.

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