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ON A NONLINEAR SYSTEM OF PDE'S ARISING IN FREE CONVECTION

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Abstract. By employing a fixed point argument, we prove existence and uniqueness of the weak solution to a nonlinear system of PDE's arising in free convection. An adapted weak formulation is involved to let the solution as weak as possible and avoid any additional smoothness hypotheses.

1. Introduction

In this paper, we consider a model problem introduced in [1], and derived from a coupled system of partial differential equations arising in the study of free convection about a vertical flat plate embedded in a porous medium. In [1], some existence result has been obtained for small data, by using the inverse function theorem. In [4], under more satisfying hypotheses, the existence of a solution is obtained by an iterative method. The regularity of this solution is also studied.

Here, we come back to the weak formulation of this problem and, under reasonable hypotheses on the data, we prove by constructing a suitable contraction mapping, that there is one and only one weak solution.

Details about the physical background can be found, for example, in [5], [6], [7], [8], [9] and [10]. In these papers, the authors assume that convection takes place in a thin layer around the plate. This allows to make boundary-layer approximations, and to get similarity solutions by solving an ordinary differential equation of the type

$$f''' + ff'' + \mathbf{g}(f') = 0$$

on the half line $[0, +\infty)$, with the boundary conditions $f(0) = a$, $f'(0) = b$ (or $f''(0) = c$) and $f'(t) \to \ell$ as $t \to +\infty$, where ℓ is a root of the function $\mathbf{g}$. This boundary value problem has been widely studied, most of the time for some particular form of $\mathbf{g}$, but also in the general case. The well known Blasius problem is for $\mathbf{g} = 0$. For an overview of mathematical results about these problems, we refer to [2] and [3] and the reference therein.

Let us state now the problem we are interesting in. Let Ω be a bounded domain of $\mathbb{R}^2$ with sufficiently smooth boundary Γ. Let Γ_1 and Γ_2 be two parts of Γ, such that $meas\,(\Gamma_1) \neq 0$ and

$$\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \Gamma, \quad \Gamma_1 \cap \Gamma_2 = \emptyset.$$

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In Ω, we consider the boundary value system defined by

$$-\Delta\Psi + K.\nabla H = F \tag{1.1}$$

$$-\lambda\Delta H + \nabla H.\left(\nabla\Psi\right)^{\perp} + \nabla\Theta.\left(\nabla\Psi\right)^{\perp} = 0 \tag{1.2}$$

with mixed boundary conditions for Ψ

$$\Psi = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \frac{\partial\Psi}{\partial\nu} = 0 \text{ on } \Gamma_2, \tag{1.3}$$

and for H

$$H = 0 \text{ on } \Gamma, \tag{1.4}$$

where $\overrightarrow{\nu}$ is the unit outward normal vector on Γ and $(\nabla\Psi)^{\perp} = (\partial_y\Psi, -\partial_x\Psi)$. The unknown functions are the stream function Ψ and the temperature H. The functions F, Θ and K are given, and we suppose that

$$F \in L^2(\Omega)$$

and that the function Θ belongs to $H^2(\Omega)$ and satisfies

$$\Delta\Theta = 0 \text{ in } \Omega. \tag{1.5}$$

Let us notice that $H^2(\Omega) \subset C^0(\overline{\Omega})$. For the coefficients $K = (k_1, k_2)$, it is assumed that

$$K \in L^\infty(\Omega) \times L^\infty(\Omega). \tag{1.6}$$

See [1] for details about the derivation of the problem (1.1)-(1.4). In the following, we will denote by $(\cdot,\cdot)$ the $L^2(\Omega)$-scalar product, and by $\|\cdot\|$ (resp. $|\cdot|_2$, $|\cdot|_\infty$ and $|\cdot|_{\infty,\Gamma}$) the norm of $H^1(\Omega)$ (resp. $L^2(\Omega)$, $L^\infty(\Omega)$ and $L^\infty(\Gamma)$). We also denote by $H_0^1(\Omega,\Gamma_1)$ the subset of functions of $H^1(\Omega)$ that vanish on Γ_1.

2. Weak formulation

In order to define a variational formulation of the previous problem, let us assume that Ψ and H are classical solutions of (1.1) and (1.2) in Ω, such that the boundary conditions (1.3) and (1.4) hold. Multiplying (1.1) and (1.2) by test functions $u \in H_0^1(\Omega,\Gamma_1)$ and $v \in H_0^1(\Omega)$, and integrating on Ω, we get

$$\int_\Omega \nabla\Psi.\nabla u\, dx + \int_\Omega u\, K.\nabla H dx = \int_\Omega F u\, dx$$

and

$$\lambda\int_\Omega \nabla H.\nabla v\, dx + \int_\Omega v\,\nabla H.\left(\nabla\Psi\right)^{\perp} dx + \int_\Omega v\,\nabla\Theta.\left(\nabla\Psi\right)^{\perp} dx = 0. \tag{2.1}$$

Since the system is formulated in the above form we are more able to think about solutions $\Psi \in H_0^1(\Omega,\Gamma_1)$ and $H \in H_0^1(\Omega)$. If this is the case, the third integral in the latter equality is still well defined (this is due to the fact that $\Theta \in H^2(\Omega)$), whereas, a priori, it is not anymore the case for the second one.

Let us clarify this point. To this end, for $u, v, w \in H^1(\Omega)$ such that $u\nabla v.(\nabla w)^{\perp} \in L^1(\Omega)$, we set

$$a(u,v,w) = \int_\Omega u\nabla v.\left(\nabla w\right)^{\perp} dx = \left(u\nabla v, (\nabla w)^{\perp}\right)$$

and let us show the following results.

Lemma 1. *Let $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$ such that one of them vanishes on the boundary of Ω. For $w \in H^1(\Omega)$ we have*

$$a(u,v,w) = -a(v,u,w).$$

In particular, for every $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and every $w \in H^1(\Omega)$ we have : $a(u,u,w) = 0$.

Proof. For $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$ and $w \in H^1(\Omega)$ the quantities $a(u,v,w)$ and $a(v,u,w)$ are well defined. Since, moreover, $uv \in H_0^1(\Omega)$, we have

$$\begin{aligned} a(u,v,w) + a(v,u,w) &= \left(u\nabla v + v\nabla u, (\nabla w)^\perp\right) = \left(\nabla(uv), (\nabla w)^\perp\right) \\ &= -\left(\operatorname{div}\left((\nabla w)^\perp\right), uv\right)_{H^{-1}(\Omega), H_0^1(\Omega)} = 0, \end{aligned}$$

because $\operatorname{div}\left((\nabla w)^\perp\right) = 0$. □

Lemma 2. *Let $u, v \in H_0^1(\Omega)$. For $w \in H^2(\Omega)$ we have*

$$a(u,v,w) = -a(v,u,w).$$

In particular, for every $u \in H_0^1(\Omega)$ and every $w \in H^2(\Omega)$ we have : $a(u,u,w) = 0$.

Proof. First, because $H^1(\Omega) \hookrightarrow L^4(\Omega)$, the quantities $a(u,v,w)$ and $a(v,u,w)$ are well defined for all $u, v \in H_0^1(\Omega)$ and $w \in H^2(\Omega)$. On the other hand, by Lemma 1, for all $\varphi, \psi \in \mathcal{D}(\Omega)$ and all $w \in H^2(\Omega)$, we have

$$a(\varphi, \psi, w) = -a(\psi, \varphi, w),$$

and the conclusion then follows from the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$. □

Taking into account Lemma 1, we can replace the second integral in (2.1) by

$$-\int_\Omega H\nabla v.\,(\nabla\Psi)^\perp\,dx$$

which is well defined, if $H \in L^\infty(\Omega)$. Having that in mind, we state the following (equivalent) definition.

Definition 1. We will say that a couple (Ψ, H) is a WEAK SOLUTION of the problem (1.1)-(1.4), if $\Psi \in H_0^1(\Omega, \Gamma_1)$ and $H \in L^\infty(\Omega) \cap H_0^1(\Omega)$, and if the integral identities

$$\begin{aligned} (\nabla\Psi, \nabla u) + (K.\nabla H, u) &= (F, u) \\ \lambda(\nabla H, \nabla v) - a(H, v, \Psi) + a(v, \Theta, \Psi) &= 0 \end{aligned}$$

hold for any $u \in H_0^1(\Omega, \Gamma_1)$ and for any $v \in H_0^1(\Omega)$.

3. Existence and uniqueness of a weak solution

3.1. **A priory estimates.** We will need the following lemma.

Lemma 3. *Let $\Psi \in H^2(\Omega)$. If $H \in H_0^1(\Omega)$ satisfies*

$$\forall v \in H_0^1(\Omega), \quad \lambda(\nabla H, \nabla v) - a(H, v, \Psi) = -a(v, \Theta, \Psi), \tag{3.1}$$

then

$$\inf_\Gamma \Theta \le H + \Theta \le \sup_\Gamma \Theta \tag{3.2}$$

and

$$\inf_{\Gamma}\Theta - \sup_{\Omega}\Theta \le H \le \sup_{\Gamma}\Theta - \inf_{\Omega}\Theta. \tag{3.3}$$

In particular, we have $H \in L^{\infty}(\Omega)$ and $|H|_{\infty}$ is bounded independently of Ψ.

Proof. The ingredients of the proof are in [1, Proposition 3.2] ; for convenience, and because the hypotheses are slightly different, we write it here. Let us set $l = \sup_{\Gamma}\Theta$ and $H^{+} = \sup\{H + \Theta - l\,;0\}$. Since $H^{+} \in H_0^1(\Omega)$, then (1.5), (3.1) and Lemma 2 imply

$$\begin{aligned}\lambda(\nabla H^{+}, \nabla H^{+}) &= \lambda(\nabla H, \nabla H^{+}) + \lambda(\nabla\Theta, \nabla H^{+}) = \lambda(\nabla H, \nabla H^{+})\\ &= a(H, H^{+}, \Psi) - a(H^{+}, \Theta, \Psi)\\ &= a(H, H^{+}, \Psi) + a(\Theta, H^{+}, \Psi)\\ &= a(H + \Theta, H^{+}, \Psi) = a(H^{+}, H^{+}, \Psi) = 0.\end{aligned}$$

It follows that $|\nabla H^{+}|_2 = 0$ and hence $H^{+} = 0$. This gives the second inequality of (3.2). To obtain the other one, we set $l' = \inf_{\Gamma}\Theta$ and $H^{-} = \inf\{H + \Theta - l'\,;0\}$ and proceed in the same way. The inequalities (3.3) follow immediately from (3.2). ☐

3.2. **A contraction.** Let $\mathbf{W} = H_0^1(\Omega, \Gamma_1) \times H_0^1(\Omega)$. On the Hilbert space $\mathbf{W}$ we define the norm $\| \, \|_{\mathbf{W}}$ by

$$\|(\Psi, H)\|_{\mathbf{W}} = \kappa|\nabla\Psi|_2 + |\nabla H|_2$$

where $\kappa > 0$ is a constant that we will choose later.

Let $\mathbf{D} = \mathcal{D}(\Omega, \Gamma_1) \times \mathcal{D}(\Omega)$ where $\mathcal{D}(\Omega, \Gamma_1)$ is the subset of $\mathcal{D}(\bar{\Omega})$ whose elements are supported far away from Γ_1 and let $\mathbf{F} : \mathbf{D} \to \mathbf{W}$ be the application defined in the following way. If $(\Psi, H) \in \mathbf{D}$, then $\mathbf{F}(\Psi, H) = (\tilde{\Psi}, \tilde{H})$ where $\tilde{\Psi}$ and $\tilde{H}$ are the unique solutions of the linear problems

$$\forall u \in H_0^1(\Omega, \Gamma_1), \quad (\nabla\tilde{\Psi}, \nabla u) = -(K.\nabla H, u) + (F, u), \tag{3.4}$$

$$\forall v \in H_0^1(\Omega), \quad \lambda(\nabla\tilde{H}, \nabla v) - a(\tilde{H}, v, \Psi) = -a(v, \Theta, \Psi). \tag{3.5}$$

Let us notice that the coercivity on $H_0^1(\Omega)$ of the bilinear form

$$(\tilde{H}, v) \mapsto \lambda(\nabla\tilde{H}, \nabla v) - a(\tilde{H}, v, \Psi)$$

follows from Lemma 2. If now $(\Psi_1, H_1) \in \mathbf{D}$ and $(\Psi_2, H_2) \in \mathbf{D}$, then we deduce from (3.4) and (3.5) that, for all $u \in H_0^1(\Omega, \Gamma_1)$ and for all $v \in H_0^1(\Omega)$, we have

$$(\nabla(\tilde{\Psi}_1 - \tilde{\Psi}_2), \nabla u) = -(K.\nabla(H_1 - H_2), u), \tag{3.6}$$

$$\lambda(\nabla(\tilde{H}_1 - \tilde{H}_2), \nabla v) - a(\tilde{H}_1, v, \Psi_1) + a(\tilde{H}_2, v, \Psi_2) = -a(v, \Theta, \Psi_1 - \Psi_2). \tag{3.7}$$

where $(\tilde{\Psi}_i, \tilde{H}_i) = \mathbf{F}(\Psi_i, H_i)$, $i = 1, 2$. Let us choose $u = \tilde{\Psi}_1 - \tilde{\Psi}_2$. On one hand, thanks to (1.6), we get from (3.6)

$$|\nabla(\tilde{\Psi}_1 - \tilde{\Psi}_2)|_2^2 = -(K.\nabla(H_1 - H_2), \tilde{\Psi}_1 - \tilde{\Psi}_2) \le |K|_{\infty}|\nabla(H_1 - H_2)|_2|\tilde{\Psi}_1 - \tilde{\Psi}_2|_2,$$

and hence

$$|\nabla(\tilde{\Psi}_1 - \tilde{\Psi}_2)|_2 \le C|K|_{\infty}|\nabla(H_1 - H_2)|_2 \tag{3.8}$$

where C is the the Poincaré constant of Ω. On the other hand, using Lemma 1, we can rewrite (3.7), taking $v = \tilde{H}_1 - \tilde{H}_2$, as

$$\begin{aligned}
\lambda|\nabla(\tilde{H}_1 - \tilde{H}_2)|_2^2 &= a(\tilde{H}_1, \tilde{H}_1 - \tilde{H}_2, \Psi_1) - a(\tilde{H}_2, \tilde{H}_1 - \tilde{H}_2, \Psi_2) - a(\tilde{H}_1 - \tilde{H}_2, \Theta, \Psi_1 - \Psi_2) \\
&= a(\tilde{H}_2, \tilde{H}_1, \Psi_1) - a(\tilde{H}_2, \tilde{H}_1, \Psi_2) + a(\Theta, \tilde{H}_1 - \tilde{H}_2, \Psi_1 - \Psi_2) \\
&= a(\tilde{H}_2, \tilde{H}_1, \Psi_1 - \Psi_2) + a(\Theta, \tilde{H}_1 - \tilde{H}_2, \Psi_1 - \Psi_2) \\
&= a(\tilde{H}_2, \tilde{H}_1 - \tilde{H}_2, \Psi_1 - \Psi_2) + a(\Theta, \tilde{H}_1 - \tilde{H}_2, \Psi_1 - \Psi_2) \\
&= a(\tilde{H}_2 + \Theta, \tilde{H}_1 - \tilde{H}_2, \Psi_1 - \Psi_2) \\
&\le |\tilde{H}_2 + \Theta|_\infty |\nabla(\tilde{H}_1 - \tilde{H}_2)|_2 |\nabla(\Psi_1 - \Psi_2)|_2
\end{aligned}$$

and thanks to Lemma 3 we arrive to

$$|\nabla(\tilde{H}_1 - \tilde{H}_2)|_2 \le \frac{1}{\lambda}|\Theta|_{\infty,\Gamma}|\nabla(\Psi_1 - \Psi_2)|_2. \tag{3.9}$$

Now, the estimates (3.8) and (3.9) give

$$\begin{aligned}
\|\mathbf{F}(\Psi_1, H_1) - \mathbf{F}(\Psi_2, H_2)\|_{\mathbf{W}} &= \kappa|\nabla(\tilde{\Psi}_1 - \tilde{\Psi}_2)|_2 + |\nabla\tilde{H}_1 - \tilde{H}_2|_2 \\
&\le \kappa C|K|_\infty|\nabla(H_1 - H_2)|_2 + \tfrac{1}{\lambda}|\Theta|_{\infty,\Gamma}|\nabla(\Psi_1 - \Psi_2)|_2 \\
&\le \max\left\{\kappa C|K|_\infty \ ; \ \tfrac{1}{\kappa\lambda}|\Theta|_{\infty,\Gamma}\right\}\|(\Psi_1, H_1) - (\Psi_2, H_2)\|_{\mathbf{W}}.
\end{aligned}$$

In order to have the best constant, we choose $\kappa = \sqrt{\frac{|\Theta|_{\infty,\Gamma}}{\lambda C|K|_\infty}}$ and we obtain

$$\|\mathbf{F}(\Psi_1, H_1) - \mathbf{F}(\Psi_2, H_2)\|_{\mathbf{W}} \le \sqrt{\tfrac{C|K|_\infty|\Theta|_{\infty,\Gamma}}{\lambda}}\,\|(\Psi_1, H_1) - (\Psi_2, H_2)\|_{\mathbf{W}}.$$

It follows that $\mathbf{F} : \mathbf{D} \to \mathbf{W}$ is Lipschitz continuous, with the Lipschitz constant

$$\beta = \sqrt{\tfrac{C|K|_\infty|\Theta|_{\infty,\Gamma}}{\lambda}}.$$

Since $\mathbf{D}$ is dense in the Banach space $\mathbf{W}$, the map $\mathbf{F}$ can be extended to $\overline{\mathbf{F}} : \mathbf{W} \to \mathbf{W}$ which is still Lipschitz continuous, with the same Lipschitz constant β.

If now $\beta < 1$, then $\overline{\mathbf{F}}$ is a *contraction* and hence has a unique fixed point, say (Ψ_*, H_*). Let us show that (Ψ_*, H_*) is then the unique weak solution of the problem (1.1)-(1.4). By density, there exists a sequence $(\Psi_n, H_n) \in \mathbf{D}$ such that $(\Psi_n, H_n) \to (\Psi_*, H_*)$ in $\mathbf{W}$. In other words, we have

$$\Psi_n \to \Psi_* \quad \text{and} \quad H_n \to H_* \ \text{ in } \ H^1(\Omega) \ \text{ as } \ n \to +\infty. \tag{3.10}$$

If we set $(\tilde{\Psi}_n, \tilde{H}_n) = \mathbf{F}(\Psi_n, H_n)$, then

$$\forall u \in H_0^1(\Omega, \Gamma_1), \quad (\nabla\tilde{\Psi}_n, \nabla u) = -(K.\nabla H_n, u) + (F, u), \tag{3.11}$$

$$\forall v \in H_0^1(\Omega), \quad \lambda(\nabla\tilde{H}_n, \nabla v) - a(\tilde{H}_n, v, \Psi_n) = -a(v, \Theta, \Psi_n). \tag{3.12}$$

Since $(\tilde{\Psi}_n, \tilde{H}_n) \to \overline{\mathbf{F}}(\Psi_*, H_*) = (\Psi_*, H_*)$ in $\mathbf{W}$, by taking the limits as $n \to +\infty$ in (3.11), we obtain

$$\forall u \in H_0^1(\Omega, \Gamma_1), \quad (\nabla\Psi_*, \nabla u) = -(K.\nabla H_*, u) + (F, u). \tag{3.13}$$

As $n \to +\infty$, we also have $(\nabla \tilde{H}_n, \nabla v) \to (\nabla H_*, \nabla v)$ and $a(v, \Theta, \Psi_n) \to a(v, \Theta, \Psi_*)$, for all $v \in H_0^1(\Omega)$. It follows that $a(\tilde{H}_n, v, \Psi_n)$ has a finite limit as $n \to +\infty$. To compute this limit, let us extract from $(\tilde{H}_n)$ a subsequence $(\tilde{H}_{n_k})$ such that

$$\tilde{H}_{n_k}(x) \to H_*(x) \text{ a.e. in } \Omega \text{ as } k \to +\infty.$$

From Lemma 3, the sequence $(\tilde{H}_n)$ is bounded in $L^\infty(\Omega)$ by some constant $c = c(\Theta)$, and thus $H_* \in L^\infty(\Omega)$ and we have $|H_*|_\infty \leq c$. Therefore, on one hand, from (3.10), we have

$$|a(\tilde{H}_n, v, \Psi_n - \Psi_*)| \leq c\|v\|\|\Psi_n - \Psi_*\| \to 0 \text{ as } n \to +\infty,$$

and, on the other hand, from the Lebesgue theorem, it holds

$$a(\tilde{H}_{n_k}, v, \Psi_*) \to a(\tilde{H}_*, v, \Psi_*) \text{ as } k \to +\infty,$$

It follows that

$$a(\tilde{H}_n, v, \Psi_n) \to a(\tilde{H}_*, v, \Psi_*) \text{ as } n \to +\infty,$$

and (3.12) gives

$$\forall v \in H_0^1(\Omega), \quad \lambda(\nabla H_*, \nabla v) - a(H_*, v, \Psi_*) = -a(v, \Theta, \Psi_*). \tag{3.14}$$

From (3.13), (3.14) and the fact that $H_* \in L^\infty(\Omega)$, we obtain that (Ψ_*, H_*) is a weak solution of the problem (1.1)-(1.4).

The uniqueness follows from the fact that any weak solution of (1.1)-(1.4) is a fixed point of $\overline{\mathbf{F}}$. In fact, let (Ψ, H) be a weak solution of the problem (1.1)-(1.4), and $(\Psi_n, H_n) \in \mathbf{D}$ be a sequence such that

$$\Psi_n \to \Psi \text{ and } H_n \to H \text{ in } H^1(\Omega) \text{ as } n \to +\infty.$$

Let us set $(\tilde{\Psi}, \tilde{H}) = \overline{\mathbf{F}}(\Psi, H)$ and $(\tilde{\Psi}_n, \tilde{H}_n) = \mathbf{F}(\Psi_n, H_n)$. Arguing as above, we can take the limits as $n \to +\infty$ in (3.11)-(3.12), and we obtain

$$\forall u \in H_0^1(\Omega, \Gamma_1), \quad (\nabla \tilde{\Psi}, \nabla u) = -(K.\nabla H, u) + (F, u),$$

$$\forall v \in H_0^1(\Omega), \quad \lambda(\nabla \tilde{H}, \nabla v) - a(\tilde{H}, v, \Psi) = -a(v, \Theta, \Psi).$$

This immediatly gives that $\tilde{\Psi} = \Psi$ and $\tilde{H} = H$.

To summarize, we have proved the following result.

Theorem 1. *Let C be the Poincaré constant of Ω. If we have*

$$C\,|K|_\infty\,|\Theta|_{\infty,\Gamma} < \lambda, \tag{3.15}$$

then problem (1.1)-(1.4) *has one and only one weak solution.*

Remark 1. The existence of the solution is still ensured even if the contraction is lost. That is to say if $divK \in L^\infty(\Omega)$ and (3.15) is relaxed as

$$C\,|K|_\infty\,|\Theta|_{\infty,\Gamma} \leq \lambda, \tag{3.16}$$

the problem (1.1)-(1.4) still has at least one weak solution. Indeed, choose a sequence of real numbers $\lambda_n > \lambda$ which ensure (3.15) if λ_n is replaced by λ. Of course thanks to the above theorem there exist $\Psi_n \in H_0^1(\Omega, \Gamma_1)$ and $H_n \in L^\infty(\Omega) \cap H_0^1(\Omega)$ solutions to problem (1.1)-(1.4) when λ_n is replaced by λ i.e.

$$\begin{aligned} (\nabla \Psi_n, \nabla u) + (K.\nabla H_n, u) &= (F, u), \\ \lambda_n (\nabla H_n, \nabla v) - a(H_n, v, \Psi_n) + a(v, \Theta, \Psi_n) &= 0, \end{aligned}$$

for any $u \in H_0^1(\Omega, \Gamma_1)$ and any $v \in H_0^1(\Omega)$. Arguing as in [4, Lemma 3.2], we may show that the sequences Ψ_n and H_n are bounded in $H_0^1(\Omega, \Gamma_1)$ and $L^\infty(\Omega) \cap H_0^1(\Omega)$ respectively. Then there exist $\Psi \in H_0^1(\Omega, \Gamma_1)$ and $H \in L^\infty(\Omega) \cap H_0^1(\Omega)$ such that -up to a subsequence- we have

$$\begin{aligned} \Psi_n &\rightharpoonup \Psi, \ H_n \rightharpoonup H \ \text{ in } H^1(\Omega), \\ \Psi_n &\to \Psi, \ H_n \to H \ \text{ in } L^2(\Omega), \end{aligned}$$

as $n \to +\infty$. These allow us to pass to the limit in the above system when $n \to +\infty$ and end up with an existence result if the hypothesis (3.15) is relaxed to (3.16).

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References

[1] S. Akesbi, B. Brighi and J.-D. Hoernel : *Steady free convection in a bounded and saturated porous medium.* Proceedings of the 2004 Swiss-Japanese Seminar "Recent advances on Elliptic and Parabolic Issues" World Scientific Publishing Co. Pte. Ltd. (2006) 1–17.

[2] B. Brighi, *The equation $f''' + ff'' + \mathbf{g}(f') = 0$ and the associated boundary value problems.* Results. Math. **61** (2012), no. 3-4, 355–391.

[3] B. Brighi, A. Fruchard and T. Sari, *On the Blasius problem*, Adv. Differential Equations, **13** (2008), no. 5-6, 509–600.

[4] B. Brighi and S. Guesmia, *Existence of Solutions and Iterative Approximations for Nonlinear Systems Arising in Free Convection.* Anal. Appl. **7** (2009) no. 3, 225–241.

[5] M. A. Chaudhary, J.H. Merkin and I. Pop, *Similarity solutions in free convection boundary-layer flows adjacent to vertical permeable surfaces in porous media: I prescribed surface temperature*, European J. Mech. B Fluids **14** (1995), no. 2, 217–237.

[6] M. A. Chaudhary, J.H. Merkin and I. Pop, Similarity solutions in free convection boundary-layer flows adjacent to vertical permeable surfaces in porous media: II prescribed surface heat flux, Heat and Mass Transfer **30** (1995), 341–347.

[7] P. Cheng and W. J. Minkowycz, *Free-convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dike*, J. Geophys. Res. **82** (1977), no. 14, 2040–2044.

[8] E. I. Ene and D. Poliševski, *Thermal flow in porous media*, D. Reidel Publishing Company, Dordrecht, 1987.

[9] J. H. Merkin and G. Zhang, *On the similarity solutions for free convection in a saturated porous medium adjacent to impermeable horizontal surfaces*, Wärme und Stoffübertr. **25** (1990), 179–184.

[10] R. A. Wooding, *Convection in a saturated porous medium at large Rayleigh number or Peclet number*, J. Fluid. Mech. **15** (1963), 527–544.

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