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AN EXTRAPOLATION THEOREM IN NON-EUCLIDEAN GEOMETRIES AND ITS APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We prove a generalization of an extrapolation theorem in the fashion of García-Cuerva and Rubio de Francia towards $\mathcal{R}$-boundedness on weighted Lebesgue spaces over locally compact abelian groups. This result can be applied to show maximal L^p regularity for differential operators that correspond to parabolic evolution equations subject to more general spatial geometries, for example the partially periodic Stokes operator. As a main tool, we generalize the classical Muckenhoupt theorem on maximal operators to locally compact abelian groups.

1. Introduction

In the setup of $\mathbb{R}^n$ the concept of Muckenhoupt weights has been studied extensively throughout the last four decades or so, with many remarkable results in the fields of harmonic analysis, weighted inequalities and partial differential equations (cf. [2], [9], [11], [12], [13], [14], [19], [20], [25]). For $q \in (1, \infty)$, a nonnegative weight function $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be in the Muckenhoupt class $A_q(\mathbb{R}^n)$ if

$$\mathcal{A}_q(\omega) := \sup_{r>0} \sup_{y\in\mathbb{R}^n} \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} \omega \, \mathrm{d}x \right) \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} \omega^{-\frac{q'}{q}} \, \mathrm{d}x \right)^{\frac{q}{q'}} < \infty,$$

where $B_r(x)$ denotes the open ball of radius r around the center x, and where q' is the Hölder conjugate of q. The weight ω is said to be in $A_1(\mathbb{R}^n)$ if there is a constant $c > 0$ such that $\mathcal{M}_{\mathbb{R}^n}\omega(x) \le c\omega(x)$ for almost all $x \in \mathbb{R}^n$. Here, $\mathcal{M}_{\mathbb{R}^n}$ denotes the usual (centered) Hardy-Littlewood maximal operator on $\mathbb{R}^n$. These classes of weights have been introduced by Muckenhoupt, who considered such weights for bounded intervals and products of intervals [21].

Muckenhoupt weights are known to possess several interesting properties. In particular, the maximal operator is bounded on weighted L^q-spaces for $q \in (1, \infty)$, see Theorem V.3.1 in [25]. This result was used by García-Cuerva and Rubio de Francia to show their Extrapolation Theorem in Section IV.5 of [13], which states that if a family of operators is uniformly bounded in $L^q_\omega(\mathbb{R}^n)$ for one $q \in [1, \infty)$ but all $\omega \in A_q(\mathbb{R}^n)$, then it is already

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bounded in $L^p_\nu(\mathbb{R}^n)$ for *all* $p \in (1,\infty)$ and all $\nu \in A_p(\mathbb{R}^n)$. Strengthening this result towards $\mathcal{R}$-boundedness of families of operators as defined in Section 4, Fröhlich [12] proved maximal L^p-regularity of the Stokes operator on weighted spaces $L^q_\omega(\Omega)$, where Ω is the whole space, the half space or a bounded domain of class $C^{1,1}$. For details about maximal L^p-regularity, see e.g. [6], [17].

In this paper, we wish to generalize the theory of Muckenhoupt weights and extrapolation towards locally compact abelian groups G. We apply the abstract methods obtained here in [23], [24] in order to obtain maximal regularity of the partially periodic Stokes operator and to treat a spatially periodic nonlinear model describing the dynamics of nematic liquid crystal flows. Let us briefly discuss how the results of the present paper affect the Stokes equations in $\mathbb{R}^n$ exhibiting a periodic behaviour in some of the space dimensions. For simplicity, assume that there is exactly one direction of periodicity, say in x_n-direction. That is, we consider for a fixed time $T > 0$ the set of equations

$$
\left\{
\begin{aligned}
\partial_t u - \Delta u + \nabla \mathfrak{p} &= f && \text{in } (0,T)\times\mathbb{R}^n,\\
\operatorname{div} u &= 0 && \text{in } (0,T)\times\mathbb{R}^n,\\
u(0) &= u_0 && \text{in } \mathbb{R}^n,\\
u(t,x',x_n+2\pi) &= u(t,x',x_n), &&\\
\lim_{|x'|\to\infty} u(t,x',x_n) &= 0, &&
\end{aligned}
\right.
\tag{1.1}
$$

with periodic external force $f(t,x',x_n) = f(t,x',x_n+2\pi)$. We analyze this problem in an L^q-setting, whence the decay condition in the last line of (1.1) will always be fulfilled, at least in the sense of summability. Taking the work of Kyed [18] on the time-periodic Stokes equations as an inspiration and introducing the locally compact abelian group $G := \mathbb{R}^{n-1}\times\mathbb{R}/2\pi\mathbb{Z}$, we can use the Fourier transform on G to define the Helmholtz projection $\mathbb{P} : L^q(G) \to L^q_\sigma(G)$, $1 < q < \infty$, via

$$
\mathcal{F}_G[\mathbb{P}f](\eta) := \left(I - \frac{\eta\otimes\eta}{|\eta|^2}\right)\mathcal{F}_G[f](\eta), \qquad 0 \neq \eta \in \hat{G} := \mathbb{R}^{n-1}\times\mathbb{Z},
$$

and equivalently reformulate (1.1) as an abstract Cauchy problem on the Banach space $L^q_\sigma(G)$ via

$$
\left\{
\begin{aligned}
\partial_t u + Au &= f && \text{in } (0,T),\\
u(0) &= u_0, &&
\end{aligned}
\right.
\tag{1.2}
$$

where the partially periodic Stokes operator A is defined as usual, *i.e.*, $A := -\mathbb{P}\Delta$. We wish to establish maximal L^p-regularity of the partially periodic Stokes operator, which is equivalent to the $\mathcal{R}$-boundedness of the family of resolvent operators $\{\lambda(\lambda+A)^{-1} | \lambda \in \Sigma_{\vartheta+\frac{\pi}{2}}\}$ with $\Sigma_{\vartheta+\frac{\pi}{2}} := \{\lambda\in\mathbb{C} : |\arg\lambda| < \vartheta + \frac{\pi}{2}, \lambda \neq 0\}$ for some $\vartheta \in (0,\pi/2)$. Thus, the question is how to obtain the corresponding $\mathcal{R}$-bounds. In [24] we show that at least uniform bounds can be obtained via Fourier analysis on the group G. These uniform bounds hold true not only in $L^q(G)$, but in fact in all Muckenhoupt weighted $L^q_\omega(G)$-spaces as introduced in Section 3 below. Therefore, the aim of this paper is to establish an extrapolation theorem which ensures that $\mathcal{R}$-boundedness follows already from uniform boundedness in all weighted spaces. Theorem 2 gives exactly this.

Note that on locally compact abelian groups one can define a nontrivial, translation invariant, regular measure μ, called *Haar measure* [1], [4], [15], [27], with $\mu(K) < \infty$ for all compact $K \subset G$. Furthermore, such a measure is unique up to multiplication with a constant. However, we often deal with the measure $\mathrm{d}\mu_\omega := \omega \, \mathrm{d}\mu$, which is not translation invariant anymore. Therefore, if not stated otherwise, we shall drop the translation-invariance condition on μ. For $q \in [1,\infty]$ one can thus introduce the space $L^q(G)$ of q-integrable functions $f : G \to \mathbb{R}$, which turns into a Banach space if equipped with the usual norm

$$\|f\|_q := \left(\int_G |f|^q \, \mathrm{d}\mu\right)^{\frac{1}{q}}, \qquad q \in [1,\infty),$$
$$\|f\|_\infty := \mu\text{-}\operatorname*{ess\,sup}_G |f|.$$

Further we introduce the notion $L^{q,\infty}(G)$ for the weak $L^q(G)$-space, as introduced e.g. in [26]. Note that the space of continuous functions with compact support $C_0(G)$ is dense in $L^q(G)$ for all $q \in [1,\infty)$, see Appendix E.8 of [22] for details.

As we wish to carry over as many concepts known from classical harmonic analysis as possible to the general setting, we will have to assume that the group G is furnished with something that resembles the concept of balls and that the measure μ enjoys a doubling property with respect to these balls. We therefore make the following assumption.

Assumption 1. Suppose that G is a locally compact abelian group equipped with a nontrivial and regular measure μ, such that $\mu(K) < \infty$ for all compact $K \subset G$. Furthermore, assume that there is a local base of $0 \in G$ consisting of relatively compact measurable neighbourhoods U_k, $k \in \mathbb{Z}$, such that

(i) $\bigcup_{k\in\mathbb{Z}} U_k = G$,

(ii) $U_k \subset U_m$, if $k \le m$,

(iii) there exist a positive constant A and a mapping $\theta : \mathbb{Z} \to \mathbb{Z}$ such that for all $k \in \mathbb{Z}$ and all $x \in G$

$$k < \theta(k),$$
$$U_k - U_k \subset U_{\theta(k)},$$
$$\mu(x + U_{\theta(k)}) \le A\mu(x + U_k).$$

Observe that necessarily $A \ge 1$ because $U_k \subset U_{\theta(k)}$.

Remark 1. From now on, we will always assume that the locally compact abelian group G admits a family of sets $(U_k)_{k\in\mathbb{Z}}$ satisfying Assumption 1. We will call any set of the form $x+U_k$, $x \in G$, $k \in \mathbb{Z}$ a *base set*. It is instructive to think of such base sets as an equivalent of balls in the $\mathbb{R}^n$ with center in x and radius 2^k. Observe that by the following considerations we can assume without loss of generality the base sets to be symmetric and the function θ to be non-decreasing.

(i) Replace θ by $\tilde{\theta}$ defined via

$$\tilde{\theta}(k) := \min\{l \in \mathbb{Z} : l > k \text{ with } U_k - U_k \subset U_l\}.$$

The thusly defined function is non-decreasing and satisfies $\tilde{\theta}(k) \le \theta(k)$ for all $k \in \mathbb{Z}$. Therefore, for all $x \in G$ and $k \in \mathbb{Z}$,

$$\mu(U_{\tilde{\theta}(k)}) \le \mu(U_{\theta(k)}) \le A\mu(U_k),$$

and hence we may assume that θ is non-decreasing.

(ii) We call a set $U \subset G$ *symmetric* if $U = -U$. Since G is abelian, the set $V := U - U$ is symmetric for any $U \subset G$. Replacing the base sets U_k by the symmetric sets $V_k := U_k - U_k$ and replacing the doubling constant A by A^2, we may assume that all of our base sets are symmetric. Indeed, the V_k still form a local base of $0 \in G$ consisting of relatively compact neighbourhoods, see e.g. Appendix B.4 of [22]. The inclusion $V_k \subset V_m$ for $k \leq m$ is obvious and the union of the $V_k (\supset U_k)$ covers the whole group.

Concerning condition (iii) of Assumption 1, we see

$$V_k - V_k \subset U_{\theta(k)} - U_{\theta(k)} = V_{\theta(k)}.$$

Moreover, the doubling property will be fulfilled with constant A^2, since for all $x \in G$ and all $k \in \mathbb{Z}$

$$\mu(x + V_{\theta(k)}) \leq \mu(x + U_{\theta^2(k)}) \leq A^2 \mu(x + U_k) \leq A^2 \mu(x + V_k).$$

Thus, from now on we will assume the base sets U_k to be symmetric and we will write $U_k - U_k = U_k + U_k =: 2U_k$.

Remark 2. Among the most prominent groups satisfying Assumption 1 are the groups $\mathbb{R}$, $\mathbb{Z}$, the torus $\mathbb{T}$ and finite products of these groups.

(i) In the case of the real numbers $\mathbb{R}$ equipped with the Lebesgue measure, define $U_k := (-2^{k-1}, 2^{k-1})$, $A = 2$ and $\theta(k) = k + 1$.
(ii) For integers, an analogous construction to (i) corresponding to the counting measure satisfies Assumption 1. Namely, choose $U_k := (-2^{k-1}, 2^{k-1}) \cap \mathbb{Z}$, $A = 3$ and $\theta(k) = k + 1$.
(iii) If one chooses the arc length as a measure on the torus, possible choices are $U_k := \{z \in \mathbb{C} : |\arg z| < 2^k\}$, $A = 2$ and $\theta = k + 1$.

Moreover, if G is a locally compact abelian group with an increasing sequence $(U_k)_{k \in \mathbb{Z}}$ of compact open subgroups, such that

$$\bigcup_{k \in \mathbb{Z}} U_k = G, \qquad \bigcap_{k \in \mathbb{Z}} U_k = \{0\},$$

then Assumption 1 is fulfilled if and only if $A := \sup_{k \in \mathbb{Z}} |U_k : U_{k-1}| < \infty$, and one may take $\theta(k) = k + 1$ in that case. See Examples 2.1.3 in [10] for details.

Let us define the (centered) *maximal operator* on G. Suppose $f \in L^1_{\mathrm{loc}}(G)$ and define the sublinear operator

$$\mathcal{M}_G f(x) := \sup_{k \in \mathbb{Z}} \frac{1}{\mu(x + U_k)} \int_{x + U_k} |f| \, \mathrm{d}\mu. \tag{1.3}$$

Note that $\mathcal{M}_G f$ is obviously lower semi-continuous and therefore measurable.

Our two main theorems can be viewed as direct generalizations of their equivalents in the classical $\mathbb{R}^n$-setup. For the definition of $A_q(G)$-consistency see Section 3.

Theorem 1. *Let G be a locally compact abelian group satisfying Assumption 1 and assume $q \in (1, \infty)$ and $\omega \in A_q(G)$. Then $\mathcal{M}_G$ is bounded in $L^q_\omega(G)$ with an $A_q(G)$-consistent bound.*

Theorem 2. *Let G be a locally compact abelian group satisfying Assumption 1. Suppose that $r, q \in (1, \infty)$, $\omega \in A_q(G)$ and that $\Omega \subset G$ is measurable. Moreover, assume that $\mathcal{T}$ is a* *family of linear operators such that for all* $v \in A_r(G)$ *there is an* $A_r(G)$*-consistent constant* $c_r = c_r(v) > 0$ *with*

$$\|Tf\|_{L^r_v(\Omega)} \leq c_r \|f\|_{L^r_v(\Omega)}$$

for all $f \in L^r_v(\Omega)$ *and all* $T \in \mathcal{T}$*. Then every* $T \in \mathcal{T}$ *can be extended to* $L^q_\omega(\Omega)$ *and* $\mathcal{T}$ *is* $\mathcal{R}$*-bounded in* $\mathcal{L}(L^q_\omega(\Omega))$ *in the sense of Definition 4 with an* $A_q(G)$*-consistent* $\mathcal{R}$*-bound* c_q*.*

This paper is organized as follows. In Section 2 we provide further properties of the group G subject to Assumption 1 and the maximal operator $\mathcal{M}_G$. In the case of a translation-invariant measure μ, most of the results in this section are known and can be found in Chapter 2 of [10]. Section 3 is devoted to establishing Theorem 1. Finally, in Section 4, we provide basic information about $\mathcal{R}$-boundedness and prove Theorem 2.

2. Harmonic Analysis on Locally Compact Abelian Groups

We first provide some basic properties that follow directly from Assumption 1.

Proposition 1. *Suppose Assumption 1 is satisfied.*

(i) *For every* $x \in G$ *and* $k \in \mathbb{Z}$ *it holds* $\mu(x + U_k) > 0$*.*

(ii) *The interiors of the base sets* U_k *cover* G*, i.e.,* $\bigcup_{k\in\mathbb{Z}} \mathring{U}_k = G$*. In particular, for every compact* $K \subset G$ *there is* $k \in \mathbb{Z}$ *such that* $K \subset U_k$*.*

Proof. (i) By Assumption 1 (i) and the regularity of measure μ, we easily deduce $\mu(x + U_k) \to \mu(G)$ as $k \to \infty$. Since μ is nontrivial, we have $\mu(G) > 0$ and hence there exists $K \in \mathbb{Z}$ with $\mu(x + U_K) > 0$. Then for $k \in \mathbb{Z}$, Assumption 1 (iii) gives $k < \theta(k)$, which shows that for all $k \in \mathbb{Z}$ there exists $N \in \mathbb{N}$ with $\theta^N(k) \geq K$. Hence

$$0 < \mu(x + U_K) \leq \mu(x + U_{\theta^N(k)}) \leq A^N \mu(x + U_k),$$

proving the assertion.

(ii) It suffices to show that for every $k \in \mathbb{Z}$ we have $U_k \subset \mathring{U}_{\theta(k)}$ and then use property (i) of Assumption 1. So fix $k \in \mathbb{Z}$ and choose an open neighbourhood O of $0 \in G$ such that $O \subset U_k$. Then we have

$$U_k \subset O' := \bigcup_{x \in U_k} (x + O) \subset 2U_k.$$

Observe that O' is open, since it is the union of the open sets $x + O$. It follows $U_k \subset O' \subset U_{\theta(k)}$ and by definition of the interior even $U_k \subset O' \subset \mathring{U}_{\theta(k)}$, which is what we wanted to show.

For the assertion about the compact set K we note that $\{\mathring{U}_k\}_{k\in\mathbb{Z}}$ is an open cover of K and we thus find a finite subcover by compactness. But since the base sets U_k are nested, so are their interiors, and so the finite subcover consists only of the largest element. Hence there is $k \in \mathbb{Z}$ with $K \subset \mathring{U}_k \subset U_k$. □

One can define the uncentered maximal operator M_G in an analogous way, if one takes the supremum in (1.3) not only over all $k \in \mathbb{Z}$, but also over all centers $y \in G$ such that $x \in y + U_k$. By a similar reasoning as for the centered maximal operator, $M_G f$ is measurable. In fact, the uncentered maximal operator is comparable to the centered maximal operator.

Lemma 1. *Let $f \in L^1_{loc}(G)$. Then*

$$\mathcal{M}_G f \le M_G f \le A^2 \mathcal{M}_G f. \tag{2.1}$$

Moreover, for all $x \in G$ it holds

$$M^\theta_G f(x) := \sup_{k\in\mathbb{Z}} \sup_{y+U_k \ni x} \frac{1}{\mu(y+U_{\theta^2(k)})} \int_{y+U_k} |f|\, d\mu \le \mathcal{M}_G f(x).$$

Proof. The first inequality of (2.1) is obvious. For the second inequality, let $x, y \in G$ and $k \in \mathbb{Z}$ be such that $x \in y + U_k$. Hence, we obtain $x + U_k \subset y + U_{\theta(k)}$, and the doubling property yields

$$\mu(x+U_{\theta(k)}) \le A\mu(x+U_k) \le A\mu(y+U_{\theta(k)}) \le A^2\mu(y+U_k).$$

On the other hand $y + U_k \subset x + U_{\theta(k)}$, and thus

$$\begin{aligned} \frac{1}{\mu(y+U_k)} \int_{y+U_k} |f|\,\mathrm{d}\mu &\le \frac{1}{\mu(y+U_k)} \int_{x+U_{\theta(k)}} |f|\,\mathrm{d}\mu \\ &\le \frac{A^2}{\mu(x+U_{\theta(k)})} \int_{x+U_{\theta(k)}} |f|\,\mathrm{d}\mu. \end{aligned}$$

Taking the supremum first on the right-hand side and then on the left-hand side yields (2.1). The second assertion follows analogously if one observes that $x + U_{\theta(k)} \subset y + U_{\theta^2(k)}$. □

As the measure μ possesses the doubling property, we expect the weak estimate

$$\mu(\{x \in G : \mathcal{M}_G f(x) > t\}) \le \frac{A}{t}\|f\|_1, \qquad t > 0, \tag{2.2}$$

and even the stronger form

$$\mu(\{x \in G : \mathcal{M}_G f(x) > t\}) \le \frac{2A}{t} \int_{\{|f|>t/2\}} |f|\,\mathrm{d}\mu, \qquad t > 0. \tag{2.3}$$

In order to show this, we need the following covering lemma due to Edwards and Gaudry [10] to apply the known technique from the $\mathbb{R}^n$-setting.

Lemma 2. *Let E be a subset of G and $k : E \to \mathbb{Z}$ a mapping bounded from above such that for every $k_0 \in \mathbb{Z}$ the set $\{x \in E : k(x) \ge k_0\}$ is relatively compact in G. Then there is a sequence $(x_n) \subset E$, finite or infinite, such that*

(i) *the sequence $(k_n) := (k(x_n))$ is decreasing,*
(ii) *the sets $x_n + U_{k_n}$ are pairwise disjoint and*
(iii) *$E \subset \bigcup (x_n + 2U_{k_n})$.*

Proof. The lemma has been proven in Lemma 2.2.1 of [10] in the case of an translation invariant measure μ. The proof in the more general case considered here needs some modifications.

If there is a finite sequence $x_1, \dots, x_m$ of points of E such that (ii) and (iii) are satisfied, then one can always achieve (i) by relabelling and there is nothing further to prove. Hence, assume that there is no such finite sequence.

Then, arguing exactly as in [10], we find a sequence of points $(x_n)_{n\in\mathbb{N}} \subset E$ such that (i) and (ii) are satisfied and such that $k_n = \max\{k(x) : x \in A_{n-1}\}$, where

$$A_n := E \setminus \left(\bigcup_{1\le l\le n} x_l + 2U_{k_l} \right).$$

It remains to proof (iii), *i.e*, that the intersection over all A_n is empty. Were this not the case, there would exist a point $x \in E$ belonging to every A_n, yielding $k_n \ge k(x)$ for all $n \in \mathbb{N}$. Therefore, by assumption, the set $M := \{x_n : n \in \mathbb{N}\}$ is relatively compact in G. Since $U_{k_n} \subset U_{k_1}$ and U_{k_1} is relatively compact, it follows that

$$F := \bigcup_{n\in\mathbb{N}} (x_n + U_{k_n}) \subset M + U_{k_1}$$

is relatively compact and so $\mu(F) \le \mu(\overline{F}) < \infty$. On the other hand, the compact set $\overline{M}$ is contained in a base set U_K, $K \in \mathbb{Z}$, by Proposition 1 (ii). Hence $x_n \in U_K$ for all $n \in \mathbb{N}$. Furthermore, by the monotonicity of θ, we find $N \in \mathbb{N}$ with $\theta^N(k(x)) \ge K$, and so $0 \in x_n + U_{\theta^N(k(x))}$. This shows

$$U_K \subset \big(x_n + U_{\theta^N(k(x))}\big) + U_K \subset x_n + 2U_{\theta^N(k(x))} \subset x_n + U_{\theta^{N+1}(k(x))}.$$

Since the $x_n + U_{k_n}$ are disjoint, this finally yields

$$\begin{aligned}\mu(F) &= \sum_{n\in\mathbb{N}} \mu(x_n + U_{k_n}) \ge \sum_{n\in\mathbb{N}} \mu(x_n + U_{k(x)}) \\ &\ge A^{-(N+1)} \sum_{n\in\mathbb{N}} \mu(x_n + U_{\theta^{N+1}(k(x))}) \ge A^{-(N+1)} \sum_{n\in\mathbb{N}} \mu(U_K) = \infty,\end{aligned}$$

since $\mu(U_K) > 0$ by Proposition 1 (i). This contradicts the finiteness of $\mu(F)$. Hence $\bigcap_{n\in\mathbb{N}} A_n$ is empty, finishing the proof. □

Theorem 3. *Let $q \in (1, \infty]$. Then the maximal operator $\mathcal{M}_G$ is bounded in $L^q(G)$. Furthermore, $\mathcal{M}_G$ is weakly bounded in $L^1(G)$,* i.e., *estimate* (2.2) *(and even* (2.3)*) holds true.*

Proof. Since $\mathcal{M}_G f$ is lower semi-continuous for $f \in L^1_{\mathrm{loc}}(G)$ and obviously $\mathcal{M}_G f(x) \le \|f\|_\infty$ almost everywhere, the maximal operator extends to a bounded operator in $L^\infty(G)$.

Let us now establish (2.2). Assume that $t > 0$ is such that $\mu(G) > \frac{A}{t}\|f\|_1$, since otherwise the assertion is trivial. As we want to apply Lemma 2, consider the set

$$E_t := \{x \in G : \mathcal{M}_G f(x) > t\}.$$

If E_t is empty, there is nothing to prove. Otherwise, choose a compact subset $E_t' \subset E_t$ and define a function $k : E_t' \to \mathbb{Z}$ via

$$k(x) := \max\left\{ k \in \mathbb{Z} : \frac{1}{\mu(x + U_k)} \int_{x+U_k} |f| \,\mathrm{d}\mu > t \right\}.$$

This mapping is certainly well-defined. Indeed, if there was no maximal $k \in \mathbb{Z}$, then we would find a sequence $(k_n) \subset \mathbb{Z}$ with $k_n \to \infty$ as $n \to \infty$ such that for all $n \in \mathbb{N}$ it holds

$$\frac{A}{t}\|f\|_1 \ge \frac{1}{t}\|f\|_1 \ge \frac{1}{t} \int_{x+U_{k_n}} |f| \,\mathrm{d}\mu \ge \mu(x + U_{k_n}) \to \mu(G), \qquad \text{as } n \to \infty,$$

contradicting our assumption.

We have to show that the mapping k is bounded from above. Assume again otherwise. Then we find $(x_n)_{n\in\mathbb{N}} \subset E_t'$ such that $k_n := k(x_n) \to \infty$ as $n \to \infty$. Since E_t' is compact, there is a $K \in \mathbb{Z}$ with $\bigcup_{n\in\mathbb{N}}\{x_n\} \subset E_t' \subset U_K$ by Proposition 1 ii. Taking sufficiently large $n \in \mathbb{N}$, we obtain $k_n \geq K$. Therefore $0 \in x_n + U_{k_n}$ and consequently $U_{k_n} \subset x_n + U_{\theta(k_n)}$. Hence, we see

$$\mu(U_{k_n}) \leq \mu(x_n + U_{\theta(k_n)}) \leq A\mu(x_n + U_{k_n}).$$

But then

$$\frac{A}{t}\|f\|_1 \geq \frac{A}{t}\int_{x_n+U_{k_n}} |f| \,\mathrm{d}\mu \geq A\mu(x_n + U_{k_n}) \geq \mu(U_{k_n}) \to \mu(G), \text{ as } n \to \infty,$$

yielding again a contradiction.

Since for every $k_0 \in \mathbb{Z}$ the set $\{x \in E_t' : k(x) \geq k_0\}$ is a subset of the compact E_t' and therefore relatively compact in G, we can invoke Lemma 2 to obtain a finite or infinite sequence of points x_n such that $E_t' \subset \bigcup (x_n + 2U_{k_n})$, but the sets $x_n + U_{k_n}$ are pairwise disjoint and it holds $\mu(x_n + U_{k_n}) < \frac{1}{t}\int_{x_n+U_{k_n}} |f| \,\mathrm{d}\mu$. Assume the obtained sequence to be infinite, the finite case being even easier. This yields

$$\begin{aligned}\mu(E_t') &\leq \sum_{n=1}^{\infty} \mu(x_n + 2U_{k_n}) \leq A\sum_{n=1}^{\infty} \mu(x_n + U_{k_n}) \\ &\leq \frac{A}{t}\sum_{n=1}^{\infty}\int_{x_n+U_{k_n}} |f| \,\mathrm{d}\mu \leq \frac{A}{t}\|f\|_1.\end{aligned}$$

Observe that this estimate is independent of the compact subset $E_t' \subset E_t$. Since E_t is open by the lower semi-continuity of the maximal operator and since the measure μ is inner regular, we may take the supremum over all compact subsets of E_t to obtain (2.2). Therefore $\mathcal{M}_G$ is continuous from $L^1(G)$ to $L^{1,\infty}(G)$.

Inequality (2.3) can be verified by a standard argument using (2.2), see e.g. Chapter I.3.1 of [25].

Since $\mathcal{M}_G$ is weakly bounded in $L^1(G)$ and bounded in $L^\infty(G)$, it is also bounded in $L^q(G)$ for $q \in (1,\infty)$ by the Marcinkiewicz interpolation theorem, see Appendix A in [10]. □

3. Muckenhoupt Weights

Assume that G is a locally compact abelian group with a measure μ satisfying Assumption 1.

Definition 1. Given a weight function $\omega \in L^1_{\mathrm{loc}}(G)$, we denote by μ_ω the measure defined via $\mu_\omega(E) := \int_E \omega \,\mathrm{d}\mu$ and by $L^q_\omega(G)$ the space of all measurable functions such that the q-norm with respect to the measure μ_ω is finite. Furthermore we denote by $\mathcal{M}_{G,\omega}$ the maximal operator defined as in (1.3) with respect to the measure μ_ω.

Definition 2. Let $q \in (1,\infty)$. A function $0 \leq \omega \in L^1_{loc}(G)$ is called an $A_q(G)$-*weight* if

$$\mathcal{A}_q(\omega) := \sup_{U\subset G}\left(\frac{1}{\mu(U)}\int_U \omega \,\mathrm{d}\mu\right)\left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{q'}{q}} \,\mathrm{d}\mu\right)^{\frac{q}{q'}} < \infty, \tag{3.1}$$

where the supremum runs over all base sets $U \in G$. In that case, $\mathcal{A}_q(\omega)$ is called the $A_q(G)$-constant of ω. We say that ω belongs to the Muckenhoupt class $A_q(G)$ or even $\omega \in A_q(G)$ if it is an $A_q(G)$-weight.

Furthermore, we call a locally integrable, nonnegative function ω an $A_1(G)$-weight if there exists a constant $c \geq 0$ such that

$$\mathcal{M}_G\omega(x) \leq c\omega(x), \qquad \text{a.a. } x \in G. \tag{3.2}$$

The infimum over all these constants is called the $A_1(G)$-constant of ω and is denoted by $\mathcal{A}_1(\omega)$.

We call a constant $c = c(\omega) > 0$ that depends on $A_q(G)$-weights $A_q(G)$-*consistent*, if for each $d > 0$ we have

$$\sup\{c(\omega) : \omega \text{ is an } A_q(G)\text{-weight with } \mathcal{A}_q(\omega) < d\} < \infty.$$

Let us note some important observations on basic properties of the Muckenhoupt classes.

Proposition 2. (i) *Let $\omega \in A_q(G)$ for $q \in (1,\infty)$. Then the following hold true.*

(a) *$\omega \in A_p(G)$ for $p \in (q,\infty)$ and $\mathcal{A}_p(\omega)$ is $A_q(G)$-consistent. Here, even the end-point case $q = 1$ is allowed.*

(b) *$\omega^{-\frac{q'}{q}} \in A_{q'}(G)$, where q' is the Hölder conjugate of q. Moreover, $\mathcal{A}_{q'}(\omega^{-\frac{q'}{q}})$ is $A_q(G)$-consistent.*

(ii) *Let $0 \leq \omega \in L^1_{loc}(G)$ and let $q \in [1,\infty)$. Then $\omega \in A_q(G)$ if and only if there is an $A_q(G)$-consistent constant $c > 0$ such that for every nonnegative measurable function $f : G \to \mathbb{R}$ and every base set $U \subset G$ it holds*

$$\left(\frac{1}{\mu(U)}\int_U f\,d\mu\right)^q \leq \frac{c}{\mu_\omega(U)}\int_U f^q\omega\,d\mu. \tag{3.3}$$

(iii) (a) *Let $r, q \in [1,\infty)$ with $q < r$ and let $\omega_0 \in A_q(G)$, $\omega_1 \in A_1(G)$. Then it holds $\omega_0 \cdot \omega_1^{q-r} \in A_r(G)$ and*

$$\mathcal{A}_r(\omega_0 \cdot \omega_1^{q-r}) \leq \mathcal{A}_q(\omega_0)\mathcal{A}_1(\omega_1)^{r-q}.$$

(b) *Let $r, q \in (1,\infty)$ with $r < q$ and let $\omega_0 \in A_q(G)$, $\omega_1 \in A_1(G)$. Then it holds $(\omega_0^{r-1} \cdot \omega_1^{q-r})^{1/(q-1)} \in A_r(G)$ and*

$$\mathcal{A}_r((\omega_0^{r-1} \cdot \omega_1^{q-r})^{1/(q-1)}) \leq \mathcal{A}_q(\omega_0)^{\frac{r-1}{q-1}}\mathcal{A}_1(\omega_1)^{\frac{q-r}{q-1}}.$$

(iv) *Let $q \in [1,\infty)$ and $\omega \in A_q(G)$. Then*

(a) *the measure μ_ω is regular and has the doubling property, i.e.,*

$$\mu_\omega(x + U_{\theta(k)}) \leq c_\omega\mu_\omega(x + U_k)$$

for all $x \in G$ and $k \in \mathbb{Z}$, where $c_\omega > 0$ is an $A_q(G)$-consistent constant,

(b) *slightly more general, for any base set U and any measurable subset $S \subset U$ we have*

$$\left(\frac{\mu(S)}{\mu(U)}\right)^q \leq c\frac{\mu_\omega(S)}{\mu_\omega(U)}, \tag{3.4}$$

where $c > 0$ is the bound appearing in (3.3),

(c) *it holds $L^\infty(G) = L^\infty_\omega(G)$ with equal norms,*

(d) *$\mathcal{M}_{G,\omega}$ is bounded in $L^p_\omega(G)$ for all $1 < p \leq \infty$ and weakly bounded in $L^1_\omega(G)$ with an $A_q(G)$-consistent bound.*

Proof. Parts (i) and (ii) are analogous to [13]. The respective consistencies are apparent from the proof given there. Part (iii) is analogous to Lemma 2.1 of [9]. Thus, we concentrate on part (iv).

Regularity follows by Lebesgue's Theorem on Dominated Convergence. To verify the doubling property, simply use (3.3) with $U = x + U_{\theta(k)}$ and $f = \chi_{x+U_k}$. Since μ has the doubling property with doubling constant A, we obtain (iva) with $c_\omega = cA^q$.

For (ivb), we argue analogously, using (3.3) with $f = \chi_S$.

To show (ivc), recall that the norm on $L^\infty_\omega(G)$ can be represented via

$$\|f\|_{L^\infty_\omega(G)} = \sup\{r \in \mathbb{R} : \mu_\omega(\{x \in G : f(x) > r\}) > 0\},$$

and a similar expression for the norm on $L^\infty(G)$, if we replace the measure μ_ω by the measure μ. Since μ_ω is absolutely continuous with respect to μ, clearly $\|f\|_{L^\infty_\omega(G)} \le \|f\|_{L^\infty(G)}$. Moreover, $\omega > 0$ almost everywhere, excepting the trivial case $\omega = 0$. Indeed, if $\omega = 0$ on a set S such that $\mu(S) > 0$, we get in virtue of (3.4) that $\mu_\omega(U) = 0$ for every base set U containing S. If S is not contained in any base set, then consider the set $\tilde{S} := S \cap U$ for some base set U large enough such that $\mu(\tilde{S}) > 0$, which certainly exists, since otherwise

$$\mu(S) = \mu\left(\bigcup_{k\in\mathbb{Z}} (S \cap U_k)\right) \le \sum_{k\in\mathbb{Z}} \mu(S \cap U_k) = 0.$$

Hence, $\omega = 0$ almost everywhere on every base set containing $\tilde{S}$ and thus on the whole group G. This shows that for every nontrivial Muckenhoupt weight ω we have $\omega > 0$ almost everywhere. Thus, μ is absolutely continuous with respect to μ_ω. Consequently $\|f\|_{L^\infty(G)} = \|f\|_{L^\infty_\omega(G)}$.

The boundedness of the maximal operators follows by Theorem 3. Marcinkiewicz' interpolation theorem yields together with part (iva) the $A_q(G)$-consistency of the bound. □

The Muckenhoupt weights can be characterized as those weight functions such that the maximal operator is weakly bounded in the weighted function space $L^q_\omega(G)$. In fact, Theorem 1 states that for $q \in (1, \infty)$ the maximal operator is bounded in $L^q_\omega(G)$ even in the strong sense. However, we first focus on the weak boundedness, which is true also for $q = 1$.

Theorem 4. *Let $0 \le \omega \in L^1_{loc}(G)$ and let $q \in [1, \infty)$. Then $\omega \in A_q(G)$ if and only if $\mathcal{M}_G$ is bounded from $L^q_\omega(G)$ to $L^{q,\infty}_\omega(G)$ with an $A_q(G)$-consistent bound.*

Proof. Assume $\omega \in A_q(G)$. We can apply Proposition 2 (iv) to obtain that $\mathcal{M}_{G,\omega}$ is weakly bounded in $L^1_\omega(G)$ with an $A_q(G)$-consistent bound. Therefore, the "only if" implication follows by the arguments given in Chapter V.2.2 of [25].

Conversely, assume that $\mathcal{M}_G$ is bounded from L^q_ω to $L^{q,\infty}_\omega$. Let $f \ge 0$ be measurable and let $U \subset G$ be a base set. If

$$(f_U) := \frac{1}{\mu(U)} \int_U f \, \mathrm{d}\mu = 0,$$

there is nothing left to prove. Hence, assume $(f_U) > 0$ and observe that for every $x \in U$ it holds $(f_U) \le M_G f(x) \le A^2 \mathcal{M}_G f(x)$. Fixing $0 < t < (f_U)$, we obtain

$$\begin{aligned} U &= \{x \in U : \mathcal{M}_G f(x) \ge (f_U)/A^2\} \\ &\subset \{x \in U : \mathcal{M}_G f(x) > t/A^2\} \subset \{x \in G : \mathcal{M}_G f(x) > t/A^2\}, \end{aligned}$$

and by the weak boundedness of the maximal operator we obtain

$$\mu_\omega(U) \leq \frac{cA^{2q}}{t^q} \int_U |f|^q \, \mathrm{d}\mu_\omega.$$

Letting $t \to (f_U)$, we finally see

$$(f_U)^q \mu_\omega(U) \leq cA^{2q} \int_U |f|^q \, \mathrm{d}\mu_\omega,$$

and in virtue of Proposition 2 (ii) we obtain $\omega \in A_q(G)$. □

If we want to strengthen Theorem 4 towards strong boundedness, we will necessarily have to exclude the case $q = 1$: There are counterexamples even for the group $G = \mathbb{R}^n$. Take for example $\omega = 1$. It is easy to see that applying the maximal operator to a nontrivial nonnegative integrable function never yields an integrable function.

However, if $1 < q = p$, then we do obtain such a strong estimate. In the classical setting $G = \mathbb{R}^n$, this is called the Muckenhoupt theorem. It is usually proven via the so-called reverse Hölder inequality (cf. [14], [25]), which in turn shows for $q \in (1, \infty)$ that $\omega \in A_q(G)$ implies $\omega \in A_p(G)$ for some smaller $p < q$. Then the Marcinkiewicz interpolation theorem may be applied to show the assertion. Unfortunately, the proof of the reverse Hölder inequality heavily relies on the existence of dyadic cubes. In our situation, we lack of such a concept. However, Jawerth [16] found a different approach avoiding the reverse Hölder inequality and hence suitable to adapt to our situation. Later, Lerner [19] significantly simplified the argument.

Proof of Theorem 1. Let $f \in L^q_\omega(G)$ and assume that $U := x + U_k$, $x \in G$, $k \in \mathbb{Z}$ is a base set. Define

$$\mathcal{A}_{q,U}(\omega) := \left(\frac{1}{\mu(U)} \int_U \omega \, \mathrm{d}\mu \right) \left(\frac{1}{\mu(U)} \int_{x+U_{\theta^2(k)}} \omega^{-\frac{q'}{q}} \, \mathrm{d}\mu \right)^{\frac{q}{q'}}.$$

Note that $\mathcal{A}_{q,U}(\omega)$ can be estimated by $A^{2q}\mathcal{A}_q(\omega)$. We write $\nu := \omega^{-q'/q}$ and observe that $\nu \in A_{q'}(G)$ by Proposition 2 (ii). We calculate

$$\begin{aligned}
\frac{1}{\mu(U)} \int_U |f| \, \mathrm{d}\mu &= \mathcal{A}_{q,U}(\omega)^{\frac{q'}{q}} \left(\frac{\mu(U)}{\mu_\omega(U)} \left(\frac{1}{\mu_\nu(x + U_{\theta^2(k)})} \int_U |f|\nu^{-1} \, \mathrm{d}\mu_\nu \right)^{\frac{q}{q'}} \right)^{\frac{q'}{q}} \\
&\leq A^{2q'} \mathcal{A}_q(\omega)^{\frac{q'}{q}} \left(\frac{\mu(U)}{\mu_\omega(U)} \left(\inf_{y \in U} M^\theta_{G,\nu}(f\nu^{-1})(y) \right)^{\frac{q}{q'}} \right)^{\frac{q'}{q}} \\
&\leq A^{2q'} \mathcal{A}_q(\omega)^{\frac{q'}{q}} \left(\frac{1}{\mu_\omega(U)} \int_U M^\theta_{G,\nu}(f\nu^{-1})^{\frac{q}{q'}} \, \mathrm{d}\mu \right)^{\frac{q'}{q}} \\
&\leq A^{2q'} \mathcal{A}_q(\omega)^{\frac{q'}{q}} \left(\frac{1}{\mu_\omega(U)} \int_U \mathcal{M}_{G,\nu}(f\nu^{-1})^{\frac{q}{q'}} \, \mathrm{d}\mu \right)^{\frac{q'}{q}},
\end{aligned}$$

where the last estimate has been proven in Lemma 1. Therefore we deduce

$$\mathcal{M}_G f(x) \leq A^{2q'} \mathcal{A}_q(\omega)^{\frac{q'}{q}} \left(\mathcal{M}_{G,\omega}(\mathcal{M}_{G,\nu}(f\nu^{-1})^{\frac{q}{q'}} \omega^{-1})(x) \right)^{\frac{q'}{q}}.$$

Consequently,

$$\begin{aligned}\|\mathcal{M}_G f\|_{L^q_\omega(G)} &\le A^{2q'}\mathcal{A}_q(\omega)^{\frac{q'}{q}}\|\mathcal{M}_{G,\omega}(\mathcal{M}_{G,\nu}(f\nu^{-1})^{\frac{q}{q'}}\omega^{-1})\|^{\frac{q'}{q}}_{L^{q'}_\omega(G)}\\ &\le A^{2q'}\mathcal{A}_q(\omega)^{\frac{q'}{q}}\|\mathcal{M}_{G,\omega}\|^{\frac{q'}{q}}_{L^{q'}_\omega(G)}\|\mathcal{M}_{G,\nu}(f\nu^{-1})\|_{L^q_\nu(G)}\\ &\le A^{2q'}\mathcal{A}_q(\omega)^{\frac{q'}{q}}\|\mathcal{M}_{G,\omega}\|^{\frac{q'}{q}}_{L^{q'}_\omega(G)}\|\mathcal{M}_{G,\nu}\|_{L^q_\nu(G)}\|f\|_{L^q_\omega(G)}.\end{aligned}\tag{3.5}$$

Here, $\|\mathcal{M}_{G,\omega}\|_{L^{q'}_\omega(G)}$ is the operator norm which is bounded by an $A_q(G)$-consistent bound $C(\omega)$ by Proposition 2 (iv). Similarly, $\|\mathcal{M}_{G,\nu}\|_{L^q_\nu(G)}$ is bounded by an $A_q(G)$-consistent bound $\tilde{C}(\omega)$ by Proposition 2 (ii) and (iv). Hence, the assertion follows. □

Remark 3. As remarked in [19], in the classical case $G = \mathbb{R}^n$ the bounds of $\mathcal{M}_{\mathbb{R}^n,\omega}$ and $\mathcal{M}_{\mathbb{R}^n,\nu}$ are uniform in $\omega \in A_q(\mathbb{R}^n)$ by the Besicovič covering theorem. Therefore, the bound of the maximal operator in (3.5) reduces to $cA_q(\omega)^{q'/q}$ with $c = c(n,q) > 0$ in this case. Buckley [3] showed that this estimate is sharp, *i.e.*, the exponent q'/q is the best possible. It would be interesting to investigate if one can achieve a sharp result also in the more general setup considered here.

4. $\mathcal{R}$-boundedness and Extrapolation Theorem

This section is devoted to establishing an extrapolation theorem generalizing the classical extrapolation theorem due to García-Cuerva and Rubio de Francia towards $\mathcal{R}$-boundedness. Therefore, we first provide some basic facts about $\mathcal{R}$-bounded families of operators.

Definition 3. We call the sequence of functions $(r_j)_{j\in\mathbb{N}}$ defined via

$$\begin{aligned} r_j &: [0,1] \to \{-1,1\},\\ r_j(t) &:= \operatorname{sgn}[\sin(2^{j-1}\pi t)],\end{aligned}$$

the sequence of *Rademacher functions.*

Remark 4. Note that the Rademacher functions are symmetric, independent and $\{-1,1\}$-valued random variables on the probability space $([0,1],\mathcal{B},\lambda)$, where $\mathcal{B}$ is the Borel algebra on $[0,1]$ and λ is the corresponding Borel measure. In fact, all arguments used in this section can be be transferred from Rademacher functions to symmetric, independent, $\{-1,1\}$-valued random variables on $[0,1]$ without any changes.

Definition 4. Let $\mathcal{X}$ be a Banach space. A subset $\mathcal{T} \subset \mathcal{L}(\mathcal{X})$ is called $\mathcal{R}$-bounded, if there exists a constant $c > 0$ such that

$$\int_0^1 \Big\|\sum_{j=1}^n r_j(t)T_jx_j\Big\|_{\mathcal{X}}\,\mathrm{d}t \le c\int_0^1\Big\|\sum_{j=1}^n r_j(t)x_j\Big\|_{\mathcal{X}}\,\mathrm{d}t\tag{4.1}$$

for all $T_1,\dots,T_N \in \mathcal{T}$, $x_1,\dots,x_n \in \mathcal{X}$ and $n \in \mathbb{N}$. Here, $(r_j)_{j\in\mathbb{N}}$ is the sequence of Rademacher functions.

The smallest constant $c > 0$ such that (4.1) holds is called $\mathcal{R}$-*bound of* $\mathcal{T}$ and is denoted by $\mathcal{R}_1(\mathcal{T})$.

For $1 \le p < \infty$, we can replace the condition (4.1) in Definition 4 by

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t) T_j x_j \Big\|_{\mathcal{X}}^p \,\mathrm{d}t \le \mathcal{R}_p(\mathcal{T}) \int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}}^p \,\mathrm{d}t, \tag{4.2}$$

due to the following lemma, which is known as *Kahane's inequality.*

Lemma 3. *Let $(r_j)_{j\in\mathbb{N}}$ be the sequence of Rademacher functions. Then there is a constant $k_p > 0$ such that for every Banach space $\mathcal{X}$ and for all $x_1, \dots x_n \in \mathcal{X}$*

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}} \mathrm{d}t \le \Big(\int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}}^p \mathrm{d}t \Big)^{\frac{1}{p}} \le k_p \int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}} \mathrm{d}t.$$

Hence, (4.1) *holds with a bound $\mathcal{R}_1(\mathcal{T}) := k_p \mathcal{R}_p(\mathcal{T})^{\frac{1}{p}}$ if* (4.2) *holds with a bound $\mathcal{R}_p(\mathcal{T})$, and* (4.2) *holds with a bound $\mathcal{R}_p(\mathcal{T}) := (k_p \mathcal{R}_1(\mathcal{T}))^p$ if* (4.1) *holds with a bound $\mathcal{R}_1(\mathcal{T})$.*

Proof. See Theorem 11.1 in [7]. ☐

In the particular case that $\mathcal{X}$ is an $L^q(X, \mu_X)$-space, where (X, μ_X) is a measure space, we can give a characterization of $\mathcal{R}$-boundedness that is much easier to handle. It relies on the following *Khinchin's inequality.*

Lemma 4. *Let $0 < q < \infty$ and $(r_j)_{j\in\mathbb{N}}$ be the sequence of Rademacher functions. Then there is a constant $c_q > 0$ such that*

$$c_q^{-1} \Big(\sum_{j=1}^n |a_j|^2 \Big)^{\frac{1}{2}} \le \Big(\int_0^1 \Big| \sum_{j=1}^n r_j(t) a_j \Big|^q dt \Big)^{\frac{1}{q}} \le c_q \Big(\sum_{j=1}^n |a_j|^2 \Big)^{\frac{1}{2}}, \tag{4.3}$$

for all $a_1, \dots, a_n \in \mathbb{C}$ and all $n \in \mathbb{N}$.

Proof. See Theorem 1.10 in [7]. ☐

Proposition 3. *Let $(X, \mathcal{A}, \mu_X)$ be a measure space, $q \in (1, \infty)$ and write $\mathcal{X} := L^q(X, \mu_X)$. Then $\mathcal{T} \subset \mathcal{L}(\mathcal{X})$ is $\mathcal{R}$-bounded if and only if there is a constant $c > 0$ such that*

$$\Big\| \Big(\sum_{j=1}^n |T_j f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathcal{X}} \le c \cdot \Big\| \Big(\sum_{j=1}^n |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathcal{X}}, \tag{4.4}$$

for all $T_1, \dots, T_n \in \mathcal{T}$, $f_1, \dots, f_n \in \mathcal{X}$ and $n \in \mathbb{N}$.

Proof. See e.g. Lemma 4.2 of [12]. ☐

Remark 5. If in the situation of Proposition 3 the constant c appearing in (4.4) is $A_q(G)$-consistent, then also the $\mathcal{R}$-bound of $\mathcal{T}$ is $A_q(G)$-consistent. Indeed, from the proof of Lemma 4.2 in [12] it is apparent that $\mathcal{R}_q(\mathcal{T}) = c_q^2 c$ is $A_q(G)$-consistent; here, c_q is the constant from Khinchin's inequality (4.3) which is independent of ω. But since $\mathcal{R}_1(\mathcal{T}) = k_q \mathcal{R}_q(\mathcal{T})^{1/q}$ by Lemma 3, and since k_q is independent of the underlying Banach space and therefore in particular $A_q(G)$-consistent, we see that $\mathcal{R}_1(\mathcal{T})$ is $A_q(G)$-consistent.

Proposition 3 suggests that a vector-valued extrapolation theorem is sufficient to pass from uniform to $\mathcal{R}$-bounds. In fact, in the classical situation $G = \mathbb{R}^n$, such a vector-valued theorem has been proven by García-Cuerva and Rubio de Francia already in their book [13]. Since then, their original arguments have been improved and simplified several times, see for example [5] and [9]. We will state here a more abstract version following the latter approach.

Proposition 4. *Let G be a locally compact abelian group satisfying Assumption 1 and let $\Omega \subset G$ be measurable. Moreover, let $r \in [1,\infty)$ and assume that there is*

$$\mathcal{F} \subset \{(f,g) : f,g : \Omega \to \mathbb{R} \textit{ are nonnegative, measurable functions}\},$$

such that for every $\upsilon \in A_r(G)$,

$$\|g\|_{L^r_\upsilon(\Omega)} \le \tilde{c}\|f\|_{L^r_\upsilon(\Omega)}, \qquad (f,g) \in \mathcal{F}, \tag{4.5}$$

with an $A_r(G)$-consistent constant $\tilde{c} = \tilde{c}(\upsilon) > 0$. Then for every $q \in (1,\infty)$ and every $\omega \in A_q(G)$,

$$\|g\|_{L^q_\omega(\Omega)} \le c\|f\|_{L^q_\omega(\Omega)}, \qquad (f,g) \in \mathcal{F}, \tag{4.6}$$

with an $A_q(G)$-consistent constant $c = c(q,\omega) > 0$.

If the constant $\tilde{c}(\omega)$ appearing in (4.5) *is of the form $N(A_r(\omega))$, where N is an increasing function, then one obtains for the constant in* (4.6)

$$c(q,\omega) = \begin{cases} N(A_q(\omega)(2\|\mathcal{M}_G\|_{L^q_\omega(G)})^{r-q}), & \textit{if } q < r, \\ N(A_q(\omega)^{\frac{r-1}{q-1}}(2\|\mathcal{M}_G\|_{L^{q'}_\nu(G)})^{\frac{q-r}{q-1}}), & \textit{if } q > r. \end{cases} \tag{4.7}$$

where $\nu := \omega^{-q'/q}$.

Proof. See Theorem 3.1 of [9] for a proof in the classical case $\mathbb{R}^n$. A quick inspection shows that the arguments given there only use elementary calculations, Hölder's inequality, the theorem of Hahn-Banach, the factorization properties stated in Proposition 2 (iii) and, as the main ingredient, the boundedness of the maximal operator in L^q_ω and $L^{q'}_\nu$. Hence, the assertion carries over to our setting. □

Remark 6. The bound obtained in (4.7) is of particular interest in regard of Remark 3, since it provides sharp bounds for the extrapolation theorem in the classical case $G = \mathbb{R}^n$ due to the sharp dependence of the maximal operator on the Muckenhoupt weight, as pointed out by Dragičević, Grafakos, Pereyra and Petermichl [8].

Remark 7. Proposition 4 contains an extrapolation theorem on locally compact abelian groups in the style of García-Cuerva and Rubio de Francia.

(i) Choose

$$\mathcal{F}_{cl} := \{(|f|,|Tf|) : f : \Omega \to \mathbb{R} \text{ continuous with compact support}\}.$$

If $T : L^r_\upsilon(\Omega) \to L^r_\upsilon(\Omega)$ is bounded with an $A_r(G)$-consistent bound, then we always have

$$\|g\|_{L^r_\upsilon(\Omega)} = \|Tf\|_{L^r_\upsilon(\Omega)} \le c\|f\|_{L^r_\upsilon(\Omega)}, \qquad (f,g) \in \mathcal{F}_{cl},$$

and thus Proposition 4 gives us

$$\|Tf\|_{L^q_\omega(\Omega)} = \|g\|_{L^r_\omega(\Omega)} \le c\|f\|_{L^q_\omega(\Omega)}, \qquad (f,g) \in \mathcal{F}_{cl},$$

with an $A_q(G)$-consistent constant $c = c(q,\omega) > 0$. By density, this yields the $A_q(G)$-consistent boundedness of T in $L^q_\omega(\Omega)$.

(ii) We also get a vector-valued version of Proposition 4, *i.e.*, under the assumption of the theorem we have for all $p, q \in (1, \infty)$ and for all $\omega \in A_q(G)$

$$\left\| \left(\sum_{j=1}^{n} g_j^p \right)^{1/p} \right\|_{L^q_\omega(\Omega)} \leq c \left\| \left(\sum_{j=1}^{n} f_j^p \right)^{1/p} \right\|_{L^q_\omega(\Omega)},$$

for all finite sequences $\{(f_j, g_j)\}_{j=1}^n \subset \mathcal{F}$, where $c = c(q, p, \omega) > 0$ is $A_q(G)$-consistent. To see this, consider

$$\mathcal{F}_p := \left\{ (F, G) = \left(\left(\sum_{j=1}^{n} f_j^p \right)^{1/p}, \left(\sum_{j=1}^{n} g_j^p \right)^{1/p} \right) : \{(f_j, g_j)\}_{j=1}^n \subset \mathcal{F} \right\},$$

and observe that Proposition 4 applied with q replaced by p gives for all $\nu \in A_p(G)$ and $(F, G) \in \mathcal{F}_p$

$$\|G\|^p_{L^p_\nu(\Omega)} = \sum_{j=1}^{n} \int_\Omega g_j^p \, \mathrm{d}\mu_\nu \leq c \sum_{j=1}^{n} \int_\Omega f_j^p \, \mathrm{d}\mu_\nu \leq c \|F\|^p_{L^p_\nu(\Omega)},$$

with an $A_p(G)$-consistent constant $c = c(p, \nu) > 0$. Thus, taking the pth-root, we obtain $\|G\|_{L^p_\nu(G)} \leq c\|F\|_{L^p_\nu(G)}$ for all $(F, G) \in \mathcal{F}_p$. If we apply now Proposition 4 again, but this time with exponents $r = p$, $q = q$ and $\mathcal{F} = \mathcal{F}_p$, we obtain

$$\left\| \left(\sum_{j=1}^{n} g_j^p \right)^{1/p} \right\|_{L^q_\omega(\Omega)} = \|G\|_{L^q_\omega(\Omega)} \leq c\|F\|_{L^q_\omega(\Omega)} = c \left\| \left(\sum_{j=1}^{n} f_j^p \right)^{1/p} \right\|_{L^q_\omega(\Omega)},$$

with an $A_q(G)$-consistent constant $c = c(q, p, \omega) > 0$.

We can finally give the proof of our main theorem.

Proof of Theorem 2. We will choose

$$\mathcal{F} := \{(|f|, |Tf|) : f : \Omega \to \mathbb{R} \text{ continuous with compact support}, \ T \in \mathcal{T}\}.$$

Then using the vector-valued extrapolation estimate in Remark 7 (ii) with $p = 2$, we obtain

$$\left\| \left(\sum_{j=1}^{n} |T_j f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q_\omega(\Omega)} \leq c \left\| \left(\sum_{j=1}^{n} |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q_\omega(\Omega)},$$

for all $T_1, \dots T_n \in \mathcal{T}$, $f_1, \dots, f_n$ and all $n \in \mathbb{N}$. Hence, Proposition 3 yields the $\mathcal{R}$-boundedness of $\mathcal{T}$ and Remark 5 shows that the $\mathcal{R}$-bound is $A_q(G)$-consistent. ☐

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