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EXISTENCE OF STANDING WAVES SOLUTION FOR A NONLINEAR SCHRÖDINGER EQUATION IN $\mathbb{R}^N$

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Abstract. In this paper, we investigate the existence of a positive solution for the following class of elliptic equation

$$-\epsilon^2\Delta u + V(x)u = f(u) \ \text{ in } \mathbb{R}^N,$$

where $\epsilon > 0$ is a positive parameter, f has a subcritical growth and V is a positive potential verifying some conditions.

1. Introduction

In recent years, many authors have considered the existence of solution for the following class of elliptic equation

$$\begin{cases} -\epsilon^2\Delta u + V(x)u = f(u) & \text{in } \mathbb{R}^N, \\ u > 0 \ \text{ in } \ \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$(P)_\epsilon$}$$

where $\epsilon > 0$ is a positive parameter, $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions with V being a nonnegative function and f having a subcritical or critical growth. The existence and concentration of positive solutions for general semilinear elliptic equations $(P)_\epsilon$ for the case $N \geq 2$ have been extensively studied, see for example, Ackermann and Szulkin [2], Alves, do Ó and Souto [4], Bartsch, Pankov and Wang [5], do Ó and Souto [9], del Pino and Felmer [6, 8], del Pino, Felmer and Miyagaki [8], Floer and Weinstein [10], Oh [11], Rabinowitz [12], Wang [13] and their references.

The knowledge of the solutions of $(P)_\epsilon$ has a great importance for studying the existence of *standing wave solutions* for, the nonlinear Schrödinger equation

$$i\epsilon\frac{\partial\Psi}{\partial t} = -\epsilon^2\Delta\Psi + W(z)\Psi - f(\Psi) \ \text{ for all } \ z \in \mathbb{R}^N, \tag{NLS}$$

which are solutions of the form $\Psi(x,t) = \exp(-iEt/\epsilon)u(x)$, where u is a solution of $(P)_\epsilon$. The equation (NLS) is one of the main objects of the quantum physics, because it appears in problems involving nonlinear optics, plasma physics and condensed matter physics, see [10] and [11] for more details about these topics.

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In a seminal paper, Rabinowitz [12] introduced the following condition on V

$$0 < \inf_{z \in \mathbb{R}^N} V(z) < \liminf_{|z| \to +\infty} V(z). \tag{V_8}$$

Later Wang [13] showed that these solutions concentrate at global minimum points of V as ϵ tends to 0.

In [6], del Pino and Felmer established the existence of positive solutions which concentrate around local minimum of V, by introducing a penalization method. More precisely, they assumed that there is an open and bounded set $\mathcal{O}$ compactly contained in $\mathbb{R}^N$ such that

$$0 < \gamma \leq V_0 = \inf_{z \in \mathcal{O}} V(z) < \min_{z \in \partial \mathcal{O}} V(z). \tag{V_1}$$

Motivated by this result, Alves, do Ó and Souto [4] and do Ó and Souto [9] studied the same type of problem with f having critical growth for $N \geq 3$ and exponential critical growth for $N = 2$ respectively.

In [7], del Pino, Felmer and Miyagaki considered the case where potential V has a geometry like saddle, essentially they assumed the following conditions on V: First of all, they fixed two subspaces $X, Y \subset \mathbb{R}^N$ such that

$$\mathbb{R}^N = X \oplus Y.$$

By supposing that V is bounded, they fixed $c_0, c_1 > 0$ satisfying

$$c_0 = \inf_{z \in \mathbb{R}^N} V(z) > 0$$

and

$$c_1 = \sup_{x \in X} V(x).$$

Furthermore, they also supposed that $V \in C^2(\mathbb{R}^N)$ and it verifies the following geometric conditions:

(V_1)

$$c_0 = \inf_{R>0} \sup_{x \in \partial B_R(0) \cap X} V(x) < \inf_{y \in Y} V(y).$$

(V_2) The functions V, $\frac{\partial V}{\partial x_i}$ and $\frac{\partial^2 V}{\partial x_i \partial x_j}$ are bounded in $\mathbb{R}^N$ for all $i, j \in \{1, ..., N\}$.

(V_3) V satisfies the Palais-Smale condition, that is, if $(x_n) \subset \mathbb{R}^N$ is a sequence such that $(V(x_n))$ is bounded and $\nabla V(x_n) \to 0$, then (x_n) possesses a convergent subsequence in $\mathbb{R}^N$.

Using the above conditions on V, and supposing that

$$c_1 < 2^{\frac{2(p-1)}{N+2-p(N-2)}} c_0,$$

del Pino, Felmer and Miyagaki showed the existence of positive solutions for the following problem

$$-\epsilon^2 \Delta u + V(z) u = |u|^{p-2} u \ \text{ in } \ \mathbb{R}^N,$$

where $p \in (2, 2^*)$ if $N \geq 3$ and $p \in (2, +\infty)$ if $N = 1, 2$, for $\epsilon > 0$ small enough. The main tool used was the variational method, more precisely, the authors found critical points of the functional

$$E_\epsilon(u) = \int_{\mathbb{R}^N} (\epsilon^2|\nabla u|^2 + |u|^2)\,dx$$

on the manifold

$$\mathcal{M} = \left\{ u \in H^1(\mathbb{R}^N) \cap P \,:\, \int_{\mathbb{R}^N} |u|^p\,dx = 1 \right\},$$

where P denotes the cone of nonnegative functions of $H^1(\mathbb{R}^N)$. Recently, in [3], Alves has studied the same type of problem with f having an exponential critical growth and $N = 2$.

In [8], del Pino and Felmer have considered the following assumptions on V:

(V_4) V is of class C^1 and there exists an $\alpha > 0$ such that

$$V(z) \geq \alpha, \quad \forall z \in \mathbb{R}^N.$$

Locally, it was fixed an open and bounded set $D \subset \mathbb{R}^N$ and subsets $B_0, B \subset D$ with B connected. Using these sets, we denoted by Γ the class of all continuous functions $\phi : B \to D$ with the property that $\phi(y) = y$ for all $y \in B_0$. Define the min-max value c as

$$c = \inf_{\phi\in\Gamma} \sup_{y\in B} V(\phi(y)), \tag{1.1}$$

and assume additionally

(V_5)

$$\sup_{y\in B_0} V(y) < c.$$

(V_6) For all $\phi \in \Gamma$, $\phi(B) \cap \{y \in D \,:\, V(y) \geq c\} \neq \emptyset$.

(V_7) For all $y \in \partial D$ such that $V(y) = c$, one has $\partial_\nu V(y) \neq 0$, where ∂_ν denotes the tangential derivative.

Motivated by the above papers, in the present article we show the existence of solution for $(P)_\epsilon$, by considering two new class of potential V, namely :

Class 1: The potential V verifies the Palais-Smale condition.

(A_0) There exists a $V_0 > 0$ such that $V(x) \geq V_0, \quad \forall x \in \mathbb{R}^N$.

(A_1) $V \in C^2(\mathbb{R}^N)$ and V, $\frac{\partial V}{\partial x_i}$ and $\frac{\partial^2 V}{\partial x_i \partial x_j}$ are bounded in $\mathbb{R}^N$ for all $i, j \in \{1, ..., N\}$.

(A_2) V verifies the Palais-Smale condition, that is, if $(x_n) \subset \mathbb{R}^N$ is a sequence such that $(V(x_n))$ is bounded and $\nabla V(x_n) \to 0$, then (x_n) possesses a convergent subsequence in $\mathbb{R}^N$.

Class 2: The potential V does not have critical point on the boundary of some bounded domain.

In this class of potential, we suppose that V verifies $(A_0) - (A_1)$ and the following additional condition:

(A_3) There is a bounded domain $\Lambda \subset \mathbb{R}^N$, such that $\nabla V(x) \neq 0$ for all $x \in \partial\Lambda$.

Related to the function f, we assume that

(f_1) $\displaystyle\limsup_{s\to 0^+} \frac{f(s)}{s} = 0.$

(f_2) There exists $p \in (2, 2^*)$, such that

$$\limsup_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 0;$$

(f_3) There exists $\theta > 2$ such that

$$0 < \theta F(s) \le s f(s) \qquad \forall s > 0;$$

where $F(s) = \int_0^s f(t)\,dt$.

The statement of our main result is the following

Theorem 1. *Suppose that V belongs to the Class 1 or 2 and f satisfies $(f_1) - (f_3)$. Then, the problem $(P)_\epsilon$ has a positive solution for $\epsilon > 0$ small enough.*

In the proof of the Theorem 1, we will use variational methods, more precisely, the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [1] combined with some arguments developed by del Pino and Felmer [6], for more details see Section 2.

The paper is organized as follows. In the next section, inspired by [6], we study the existence of a solution for a class of auxiliary problems. In Section 3, we prove the main theorem supposing that V belongs to Class 1, while in Section 4, we consider the case which V belongs to Class 2. Finally, in Section 5, we make some final considerations for elliptic problems with f having critical growth for $N \ge 3$ and exponential critical growth for $N = 2$.

2. del Pino and Felmer's approach

In this section, following an idea found in del Pino and Felmer [6], we will study the existence of a solution for a special class of elliptic problems associated with $(P)_\epsilon$.

Since we intend to prove the existence of positive solutions, hereafter we consider

$$f(s) = 0, \quad \forall s \le 0.$$

Using the change of variable $v(x) = u(\epsilon x)$, it is possible to prove that $(P)_\epsilon$ is equivalent to the following problem

$$\begin{cases} -\Delta u + V(\epsilon x)u = f(u) \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \ \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{$(P)'_\epsilon$}$$

Therefore, in what follows, we prove the existence of a positive solution for $(P)'_\epsilon$. For this purpose, we start observing that from (A_0), we can work in $H^1(\mathbb{R}^N)$ with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\epsilon x)|u|^2) dx \right)^{\frac{1}{2}},$$

which is equivalent to the usual norm.

The Euler-Lagrange functional associated with $(P)'_\epsilon$ is given by

$$I_\epsilon(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u) dx, \ \ \forall u \in H^1(\mathbb{R}^N).$$

From the conditions on f, the functional $I_\epsilon \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and its Gâteaux derivative is

$$I'_\epsilon(u)v = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(\epsilon x)uv) dx - \int_{\mathbb{R}^N} f(u) v dx, \ \ \forall u, v \in H^1(\mathbb{R}^N).$$

It is easy to check that the critical points of I_ϵ are weak solutions of $(P)'_\epsilon$.

In the sequel, let us denote by $I_\infty : H^1(\mathbb{R}^N) \to \mathbb{R}$ the functional

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty |u|^2)dx - \int_{\mathbb{R}^N} F(u)dx,$$

where

$$V_\infty = \max_{x\in\mathbb{R}^N} V(x).$$

Furthermore, let us denote by c_∞ the mountain pass level associated with I_∞, that is,

$$c_\infty = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\infty(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } I_\infty(\gamma(1)) < 0\}. \tag{2.1}$$

Here, we would like to point out that c_∞ depends only on V_∞, θ and f.

2.1. **An auxiliary problem.** Given a bounded domain $\Omega \subset \mathbb{R}^N$, we fix the numbers $k = \frac{2\theta}{\theta-2} > 2$ and $a > 0$ be the value at which

$$\frac{f(a)}{a} = \frac{V_0}{k},$$

where $V_0 > 0$ is given in (A_0). Using these numbers, we set the functions

$$\tilde{f}(s) = \begin{cases} 0, & s \le 0, \\ f(s), & 0 \le s \le a, \\ \frac{V_0}{k}s, & s \ge a \end{cases}$$

and

$$g(x,s) = \chi(x)f(s) + (1-\chi)\tilde{f}(s), \quad \forall (x,s) \in \mathbb{R}^N \times \mathbb{R},$$

where χ denotes the characteristic function associated with Ω, that is,

$$\chi(x) = \begin{cases} 1, & x \in \Omega \\ 0, & x \in \Omega^c. \end{cases}$$

Using the above functions, we will study the existence of a positive solution for the following problem

$$(AP)_\epsilon \qquad \begin{cases} -\Delta u + V(\epsilon x)u = g_\epsilon(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where

$$g_\epsilon(x,s) = g(\epsilon x, s), \quad \forall (x,s) \in \mathbb{R}^N \times \mathbb{R}.$$

The above problem is strongly related to $(P)'_\epsilon$, because if u is a solution of $(AP)_\epsilon$ verifying

$$u(x) < a, \quad \forall x \in \mathbb{R}^N \setminus \Omega_\epsilon,$$

where $\Omega_\epsilon = \Omega/\epsilon$, then u will be a solution for $(P)'_\epsilon$.

Associated with $(AP)_\epsilon$, we have the energy functional $J_\epsilon : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\epsilon x)|u|^2)\,dx - \int_{\mathbb{R}^N} G_\epsilon(x,u)\,dx,$$

where

$$G_\epsilon(x,s) = \int_0^s g_\epsilon(x,t)\,dt, \quad \forall (x,s) \in \mathbb{R}^N \times \mathbb{R}.$$

From the conditions on f, and hence on g, $J_\epsilon \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and one has

$$J_\epsilon'(u)v = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(\epsilon x)uv)dx - \int_{\mathbb{R}^N} g_\epsilon(x,u)vdx, \; \forall u, v \in H^1(\mathbb{R}^N).$$

Thus, critical points of J_ϵ correspond to weak solutions of $(AP)_\epsilon$.

Repeating the same arguments found in [6], it is easy to see that J_ϵ verifies the hypotheses of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [1] for all $\epsilon > 0$. Therefore, there exists a $u_\epsilon \in H^1(\mathbb{R}^N)$ such that

$$J_\epsilon(u_\epsilon) = c_\epsilon > 0 \quad \text{and} \quad J_\epsilon'(u_\epsilon) = 0,$$

where

$$c_\epsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\epsilon(\gamma(t))$$

with

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } J_\epsilon(\gamma(1)) < 0\}.$$

Observing that

$$J_\epsilon(u) \le I_\infty(u) \quad \forall u \in H^1(\mathbb{R}^N),$$

we have

$$c_\epsilon \le c_\infty, \quad \forall \epsilon > 0. \tag{2.2}$$

The lemma below establishes an important estimate from above for the H^1-norm of the family (u_ϵ).

Lemma 1. *For all $\epsilon > 0$, the solution u_ϵ of $(AP)_\epsilon$ satisfies the estimate*

$$\|u_\epsilon\|^2 \le 2kc_\infty.$$

Proof: Using the fact that u_ϵ is a critical point of J_ϵ, we must have

$$c_\epsilon = J_\epsilon(u_\epsilon) = J_\epsilon(u_\epsilon) - \frac{1}{\theta}J_\epsilon'(u_\epsilon)u_\epsilon \ge \frac{(\theta-2)}{4\theta}\|u_\epsilon\|^2 \ge \frac{1}{2k}\|u_\epsilon\|^2.$$

Now, the result follows by combining the above inequality with (2.2). □

Here, we would like to point out that in Lemma 1, the norm $\|u_\epsilon\|$ is bounded from above, by a constant that depends only on V_∞, θ and f, then the constant does not depend on $\epsilon > 0$.

3. Proof of Theorem 1: The Class 1

In this section, we will prove the Theorem 1, by supposing that V belongs to Class 1. For this purpose we will use the results obtained in Section 2 fixing

$$\Omega = B_{R_\epsilon}(0),$$

where $R_\epsilon = \frac{1}{\epsilon}$. Consequently, we know that there is a solution $u_\epsilon \in H^1(\mathbb{R}^N)$ for $(AP)_\epsilon$.

In what follows, our goal is to prove that there is $\epsilon_0 > 0$ such that

$$u_\epsilon(x) < a, \quad \forall x \in \mathbb{R}^N \setminus B_{\frac{R_\epsilon}{\epsilon}}(0) \quad \text{and} \quad \forall \epsilon \in (0, \epsilon_0).$$

Lemma 2. *The function u_ϵ verifies the following estimate*

$$\max_{x\in\partial B_{\frac{R_\epsilon}{\epsilon}}(0)} u_\epsilon(x) \to 0, \quad \epsilon \to 0.$$

Proof: Assume by contradiction that there exists an $\epsilon_n \to 0$ and $\gamma > 0$ such that

$$\max_{x\in\partial B_{\frac{R_{\epsilon_n}}{\epsilon_n}}(0)} u_n(x) \geq \gamma \quad \forall n \in \mathbb{N},$$

where $u_n = u_{\epsilon_n}$. From now on, we fix $x_n \in \partial B_{\frac{R_{\epsilon_n}}{\epsilon_n}}(0)$ satisfying

$$u_n(x_n) = \max_{x\in\partial B_{\frac{R_{\epsilon_n}}{\epsilon_n}}(0)} u_{\epsilon_n}(x).$$

Therefore,

$$u_n(x_n) \geq \gamma, \quad \forall n \in \mathbb{N}.$$

By Lemma 1, (u_n) is bounded in $H^1(\mathbb{R}^N)$. Thereby, setting $w_n = u_n(\cdot + x_n)$, we can guarantee that (w_n) is also bounded in $H^1(\mathbb{R}^N)$ and it satisfies

$$\begin{cases} -\Delta w_n + V(\epsilon_n x + \epsilon_n x_n) w_n = g(\epsilon_n x + \epsilon_n x_n, w_n), x \in \mathbb{R}^N, \\ \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

Using bootstrap arguments, it is possible to show that (w_n) converges uniformly on compact set to its weak limit $w \in H^1(\mathbb{R}^N)$. Then, $w \in C(\mathbb{R}^N)$ and $w(0) \geq \gamma$, which implies that $w \neq 0$. Moreover, by (A_1), there exists a subsequence of $(\epsilon_n x_n)$, still denoted by $(\epsilon_n x_n)$, such that

$$\alpha_1 = \lim_{n\to+\infty} V(\epsilon_n x_n),$$

for some $\alpha_1 > 0$. Since for each $\phi \in H^1(\mathbb{R}^N)$, the equality below holds

$$\int_{\mathbb{R}^N} \nabla w_n \nabla \phi \, dx + \int_{\mathbb{R}^N} V(\epsilon_n x + \epsilon_n x_n) w_n \phi \, dx - \int_{\mathbb{R}^N} g(\epsilon_n x + \epsilon_n x_n, w_n) \phi \, dx = o_n(1)\|\phi\|,$$

taking the limit of $n \to +\infty$, let us deduce that w is a nontrivial solution of the problem

$$\Delta u - \alpha_1 u + \tilde{g}(x, u) = 0, \quad x \in \mathbb{R}^N, \tag{3.1}$$

where

$$\tilde{g}(x, s) = \tilde{\chi}(x) f(s) + (1 - \tilde{\chi}(x)) \tilde{f}(s),$$

for some $\tilde{\chi} \in L^\infty(\mathbb{R}^N)$. Thus, by regularity theory, $w \in L^\infty(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$.

For each $j \in \mathbb{N}$, there is $\phi_j \in C_0^\infty(\mathbb{R}^N)$ such that

$$\|\phi_j - w\| \leq 1/j,$$

that is,

$$\|\phi_j - w\| = o_j(1).$$

Using $\frac{\partial \phi_j}{\partial x_i}$ as a test function, we get

$$\int_{\mathbb{R}^N} \nabla w_n \nabla \frac{\partial \phi_j}{\partial x_i} \, dx + \int_{\mathbb{R}^N} V(\epsilon_n x + \epsilon_n x_n) w_n \frac{\partial \phi_j}{\partial x_i} \, dx - \int_{\mathbb{R}^N} g(\epsilon_n x + \epsilon_n x_n, w_n) \frac{\partial \phi_j}{\partial x_i} \, dx = o_n(1).$$

Now, applying the Lebesgue's Theorem, we deduce that

$$\int_{\mathbb{R}^N} \nabla w_n \nabla \frac{\partial \phi_j}{\partial x_i} \, dx = \int_{\mathbb{R}^N} \nabla w \nabla \frac{\partial \phi_j}{\partial x_i} \, dx + o_n(1)$$

and

$$\int_{\mathbb{R}^N} g(\epsilon_n x + \epsilon_n x_n, w_n) \frac{\partial \phi_j}{\partial x_i}\, dx = \int_{\mathbb{R}^N} \tilde{g}(x, w) \frac{\partial \phi_j}{\partial x_i}\, dx + o_n(1).$$

From the above limit with (3.1), we find

$$\limsup_{n \to +\infty} \left| \int_{\mathbb{R}^N} (V(\epsilon_n x + \epsilon_n x_n) - V(\epsilon_n x_n)) w_n \frac{\partial \phi_j}{\partial x_i}\, dx \right| = 0.$$

As ϕ_j has compact support, the above limit gives

$$\limsup_{n \to +\infty} \left| \int_{\mathbb{R}^N} (V(\epsilon_n x + \epsilon_n x_n) - V(\epsilon_n x_n)) w \frac{\partial \phi_j}{\partial x_i}\, dx \right| = 0.$$

Now, recalling that $\frac{\partial w}{\partial x_i} \in L^2(\mathbb{R}^N)$, we have that $(\frac{\partial \phi_j}{\partial x_i})$ is bounded in $L^2(\mathbb{R}^N)$. Hence,

$$\limsup_{n \to +\infty} \left| \int_{\mathbb{R}^N} (V(\epsilon_n x + \epsilon_n x_n) - V(\epsilon_n x_n)) \phi_j \frac{\partial \phi_j}{\partial x_i}\, dx \right| = o_j(1),$$

and so,

$$\limsup_{n \to +\infty} \left| \frac{1}{2} \int_{\mathbb{R}^N} (V(\epsilon_n x + \epsilon_n x_n) - V(\epsilon_n x_n)) \frac{\partial (\phi_j^2)}{\partial x_i}\, dx \right| = o_j(1).$$

Using Green's Theorem together with the fact that ϕ_j has compact support, we obtain the limit below

$$\limsup_{n \to +\infty} \left| \int_{\mathbb{R}^N} \frac{\partial V}{\partial x_i}(\epsilon_n x + \epsilon_n x_n)\, \phi_j^2 \right| dx = o_j(1),$$

which combined with (A_1) leads to

$$\limsup_{n \to +\infty} \left| \frac{\partial V}{\partial x_i}(\epsilon_n x_n) \int_{\mathbb{R}^N} |\phi_j|^2\, dx \right| = o_j(1).$$

As

$$\int_{\mathbb{R}^2} |\phi_j|^2\, dx \to \int_{\mathbb{R}^N} |w|^2\, dx > 0 \quad \text{as} \quad j \to +\infty,$$

it follows that

$$\limsup_{n \to +\infty} \left| \frac{\partial V}{\partial x_i}(\epsilon_n x_n) \right| = o_j(1), \quad \forall i \in \{1, ..., N\}.$$

Since j is arbitrary, we derive that

$$\nabla V(\epsilon_n x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $(\epsilon_n x_n)$ is a $(PS)_{\alpha_1}$ sequence for V, which is absurd, since by (A_2), V satisfies the (PS) condition and $(\epsilon_n x_n)$ does not have any convergent subsequence in $\mathbb{R}^N$, because

$$|\epsilon_n x_{\epsilon_n}| = R_{\epsilon_n} = \frac{1}{\epsilon_n} \to +\infty, \quad \text{as} \quad n \to +\infty.$$

□

Proof of Theorem 1 (conclusion): From Lemma 2, there exists an $\epsilon_0 > 0$ such that

$$\max_{x \in \partial B_{\frac{R_\epsilon}{\epsilon}}(0)} u_\epsilon(x) < a, \quad \forall \epsilon \in (0, \epsilon_0).$$

Considering the function

$$\tilde{u}_\epsilon(x) = \begin{cases} 0, & x \in \overline{B}_{\frac{R_\epsilon}{\epsilon}}(0), \\ (u_\epsilon - a)^+(x), & x \in \mathbb{R}^N \setminus B_{\frac{R_\epsilon}{\epsilon}}(0), \end{cases}$$

it follows that $\tilde{u}_\epsilon \in H^1(\mathbb{R}^N)$. Thereby, $J'_\epsilon(u_\epsilon)\tilde{u}_\epsilon = 0$, or equivalently,

$$\int_{\mathbb{R}^N} \nabla u_\epsilon \nabla \tilde{u}_\epsilon \, dx + \int_{\mathbb{R}^N} V(\epsilon x) u_\epsilon \tilde{u}_\epsilon \, dx = \int_{\mathbb{R}^N} g_\epsilon(x, u_\epsilon)\tilde{u}_\epsilon \, dx.$$

Now, using the definition of g_ϵ, it is possible to prove that $\tilde{u}_\epsilon \equiv 0$. From this,

$$u_\epsilon(x) \le a, \quad \forall x \in \mathbb{R}^N \setminus B_{\frac{R_\epsilon}{\epsilon}}(0),$$

showing that u_ϵ is a solution for $(P)'_\epsilon$. □

4. Proof of Theorem 1: The Class 2

In this section, we will prove the Theorem 1 for the case that V belongs to Class 2. However, we will use the results showed in Section 2 with

$$\Omega = \Lambda.$$

Then, we also have a solution $u_\epsilon \in H^1(\mathbb{R}^N)$ for $(AP)_\epsilon$.

Next, we will show that there exists an $\epsilon_0 > 0$ such that

$$u_\epsilon(x) < a, \quad \forall x \in \mathbb{R}^N \setminus \Lambda_\epsilon \quad \text{and} \quad \forall \epsilon \in (0, \epsilon_0).$$

Lemma 3. *The function u_ϵ verifies the following convergence*

$$\max_{x \in \partial\Lambda_\epsilon} u_\epsilon(x) \to 0, \quad \epsilon \to 0.$$

Proof: Using the same type of arguments explored in the proof of Lemma 2, we find a sequence $(x_n) \subset \partial\Lambda_{\epsilon_n}$, with $\epsilon_n \to 0$, satisfying

$$\nabla V(\epsilon_n x_n) \to 0.$$

Since $(\epsilon_n x_n) \subset \partial\Lambda$, and $\partial\Lambda$ is a compact set in $\mathbb{R}^N$, we can assume that there is $x_0 \in \partial\Lambda$ such that

$$\epsilon_n x_n \to x_0 \quad \text{in} \quad \mathbb{R}^N.$$

Gathering the above limits with the fact that $V \in C^1(\mathbb{R}^N, \mathbb{R})$, we get

$$x_0 \in \partial\Lambda \quad \text{and} \quad \nabla V(x_0) = 0,$$

contradicting (A_3). □

Proof of Theorem 1 (conclusion): The conclusion of the proof follows as in Section 3.

5. Final considerations

In this section, we would like to point out that the arguments explored in the present paper can be applied to study the existence of solution for elliptic problems with critical growth for $N \geq 3$ and exponential critical growth for $N = 2$, with the following hypotheses:

Critical growth for $N \geq 3$:

$$\left\{\begin{array}{l} -\epsilon^2 \Delta u + V(x)u = \lambda |u|^{q-2}u + |u|^{2^*-2}u \ \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{array}\right. \qquad (P_\epsilon)_*$$

where $\epsilon, \lambda > 0$ are positive parameters, $q \in (2, 2^*)$ and $2^* = \frac{2N}{N-2}$.

The main result associated with this class of problem is the following

Theorem 2. *Assume that V belongs to Class 1 or 2. Then, there is $\epsilon_0 > 0$ such that*
a) If $N \geq 4$, $(P_\epsilon)_*$ has a positive solution for all $\epsilon \in (0, \epsilon_0]$ and $\lambda > 0$.
b) If $N = 3$, there exists an $\lambda^* > 0$, which is independent of $\epsilon_0 > 0$, such that $(P_\epsilon)_*$ has a positive solution for all $\epsilon \in (0, \epsilon_0]$ and $\lambda \geq \lambda^*$.

In the proof of this result, we adapt the arguments found in [4], because in that paper, the problem $(P_\epsilon)_*$ has been considered with the same condition on V, as considered in [6].

Critical growth for $N = 2$:

Hereafter, $f \in C^1(\mathbb{R})$ and it satisfies the following conditions:

(f_1) There is $C > 0$ such that
$$|f(s)| \leq Ce^{4\pi |s|^2} \quad \text{for all} \quad s \in \mathbb{R}.$$

(f_2) $\displaystyle\lim_{s \to 0} \frac{f(s)}{s} = 0.$

(f_3) There is $\theta > 2$ such that
$$0 < \theta F(s) := \theta \int_0^s f(t)dt \leq sf(s), \quad \text{for all} \quad s \in \mathbb{R} \setminus \{0\}.$$

(f_4) There exist constants $p > 2$ and $C_p > 0$ such that
$$f(s) \geq C_p s^{p-1} \quad \text{for all} \quad s > 0,$$
where
$$C_p > \left[\beta_p \left(\frac{2\theta}{\theta - 2}\right) \frac{1}{\min\{1, V_0\}}\right]^{(p-2)/2},$$
with
$$\beta_p = \inf_{\mathcal{N}_\infty} J_\infty,$$
$$\mathcal{N}_\infty = \{u \in H^1(\mathbb{R}^2) \setminus \{0\} : \ J'_\infty(u)u = 0\}$$
and
$$J_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + |V|_\infty |u|^2\right) dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx.$$

The main result related to above hypotheses is the following

Theorem 3. *Assume that V belongs to Class 1 or 2 and f verifies $(f_1) - (f_4)$. Then, problem $(P)_\epsilon$ has a positive solution for $\epsilon > 0$ small enough.*

In the proof of Theorem 3, we follow the ideas found in [9], because in that paper, problem $(P)_\epsilon$ has been considered with V verifying the same conditions, as explored in [6] and f satisfying the above assumptions.

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