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Orthogonal $\perp$

ON THE ASYMPTOTIC BEHAVIOR OF THE FIRST EIGENVALUE OF ROBIN PROBLEM WITH LARGE PARAMETER

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ABSTRACT. We consider the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω with Robin condition $\frac{\partial u}{\partial \nu} + \alpha u = 0$ on $\partial\Omega$ where $\Omega \subset R^n$, $n \geq 2$, is a bounded domain with a smooth boundary, ν is the outward unit normal, α is a real parameter. We obtain two terms of the asymptotic expansion of the first eigenvalue of this problem when $\alpha \to +\infty$. We also obtain an estimate for strong solutions of the non-homogeneous Robin problem for large positive values of the parameter.

1. INTRODUCTION

Let us consider the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} + \alpha u = 0 \quad \text{on } \Gamma, \tag{1.2}$$

where $\Omega \subset R^n$, $n \geq 2$, is a bounded domain with boundary $\Gamma = \partial\Omega \in C^3$. Here ν is the outward unit normal vector to Γ, α is a real parameter. The problem (1.1), (1.2) is usually called a Robin problem ([6], Ch. 7, Par. 7.2).

There is a sequence of eigenvalues $\lambda_1(\alpha) < \lambda_2(\alpha) \leq \ldots$ of the problem (1.1) – (1.2) enumerated according to their multiplicities such that $\lim\limits_{k\to\infty} \lambda_k(\alpha) = +\infty$. Note that $\lambda_1(\alpha)$ is simple with a positive eigenfunction. Let $0 < \lambda_1^D < \lambda_2^D \leq \ldots$, $\lim\limits_{k\to\infty} \lambda_k^D = +\infty$ be the sequence of eigenvalues of the Dirichlet eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega, \tag{1.3}$$

$$u = 0 \quad \text{on } \Gamma. \tag{1.4}$$

By variational principle ([8], Ch. 4, Par. 1, no. 4) we have

$$\lambda_1(\alpha) = \inf_{v \in H^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 dx + \alpha \int_\Gamma v^2 ds}{\int_\Omega v^2 dx}, \tag{1.5}$$

$$\lambda_1^D = \inf_{v \in \overset{o}{H}{}^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega v^2 dx}. \tag{1.6}$$

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We are interested in the behavior of $\lambda_1(\alpha)$ when $\alpha \to +\infty$. In [10], for $n = 2$, the following two-side estimate was obtained:

$$\lambda_1^D \left(1 + \frac{\lambda_1^D}{\alpha q_1}\right)^{-1} \le \lambda_1(\alpha) \le \lambda_1^D \left(1 + \frac{4\pi}{\alpha |\Gamma|}\right)^{-1}, \quad \alpha > 0,$$

where q_1 is the first eigenvalue of the Steklov problem

$$\begin{aligned} \Delta^2 u &= 0 && \text{in } \Omega, \\ u = 0, \quad \Delta u - q\frac{\partial u}{\partial \nu} &= 0 && \text{on } \Gamma. \end{aligned}$$

One can prove that $\lambda_k(\alpha) \le \lambda_k^D$, $k = 1, 2, \dots$, which gives an upper bound of $\lambda_k(\alpha)$ for all values of α. The behavior of higher order eigenvalues of the problem (1.1), (1.2) for large positive α is considered in [1] for $n = 2$ and a smooth boundary Γ. It was noticed in ([1], Ch. 6, Par. 2, no. 1) that $\lim\limits_{\alpha \to +\infty} \lambda_k(\alpha) = \lambda_k^D$. In [4] the following inequalities were obtained for $\lambda_k(\alpha)$:

$$\lambda_k^D - C_1 \frac{\left(\lambda_k^D\right)^2}{\sqrt{\alpha}} \le \lambda_k(\alpha) \le \lambda_k^D, \quad \alpha > \alpha_1 > 0, \quad k = 1, 2, \dots. \tag{1.7}$$

The inequalities (1.7) were improved in [5]: the eigenvalues $\lambda_k(\alpha)$, $k = 1, 2, \dots$ satisfy indeed the estimates

$$\lambda_k^D - C_1 \frac{\left(\lambda_k^D\right)^2}{\alpha} \le \lambda_k(\alpha) \le \lambda_k^D, \quad \alpha > \alpha_1 > 0, \tag{1.8}$$

where the constants C_1 and α_1 do not depend on k.

2. Results

The main result of this paper is the following.

Main Theorem. *Let $n \ge 2$. Then the eigenvalue $\lambda_1(\alpha)$ satisfies*

$$\lambda_1(\alpha) = \lambda_1^D - \frac{\int_\Gamma \left(\frac{\partial u_1^D}{\partial \nu}\right)^2 ds}{\int_\Omega \left(u_1^D\right)^2 dx} \frac{1}{\alpha} + o\left(\frac{1}{\alpha}\right), \quad \textit{when } \alpha \to +\infty, \tag{2.1}$$

where u_1^D is the first eigenfunction of the Dirichlet problem (1.3), (1.4).

Remark 1. Formula (2.1) is valid for $n = 1$ with an appropriate correction. For $\Omega = (a, b)$ we have

$$\lambda_1(\alpha) = \lambda_1^D - \frac{([u_1^D]'(a))^2 + ([u_1^D]'(b))^2}{\int_a^b \left(u_1^D\right)^2 dx} \frac{1}{\alpha} + o\left(\frac{1}{\alpha}\right), \quad \alpha \to +\infty. \tag{2.2}$$

One can obtain the equality (2.2) also by analysis of the asymptotic behavior of the eigenfunction of a Sturm-Liouville problem with Robin condition for large positive α's.

The expansion (2.1) was announced in [3]. Let us note, that thanks to the properties of the function u_1^D one has

$$\int_\Gamma \left(\frac{\partial u_1^D}{\partial \nu}\right)^2 ds > 0. \tag{2.3}$$

Moreover, the relations (2.1), (2.3) show that the power 1 of α in the denominator in (1.8) cannot be replaced by $1+\delta$ with $\delta > 0$.

The proof of the expansion (1.8) uses uniform (with respect to large positive values of α) estimates of the strong solution of an elliptic boundary value problem with Robin boundary condition.

Let $h(x) \in L_2(\Omega)$ and $u(x) \in H^1(\Omega)$ be a weak solution of the boundary value problem with parameter

$$-\Delta u + u = h \quad \text{in } \Omega, \tag{2.4}$$

$$\frac{\partial u}{\partial \nu} + \alpha u = 0 \quad \text{on } \Gamma, \quad \alpha > 0. \tag{2.5}$$

In domains with C^2 boundary the weak solution of the problem (2.4), (2.5) belongs to $H^2(\Omega)$ and is a strong solution ([8], Ch. 4, Par. 2, Th. 4).

Theorem 1. *The solution of the problem (2.4), (2.5) satisfies*

$$\|u\|_{H^2(\Omega)} \leq C_2 \|h\|_{L_2(\Omega)}, \quad \alpha > \alpha_1 > 0, \tag{2.6}$$

with the constant C_2 independent of α.

Remark 2. Let us note that the estimate (2.6) for the solution of the problem (2.4), (2.5) is known for fixed α (see, for example, [8]). But we need the estimate (2.6) to be valid with a constant C_2 for $\alpha \to +\infty$.

3. Estimates for the problem with parameter

For $h(x) \in L_2(\Omega)$ a weak solution $u(x) \in H^1(\Omega)$ of the problem (2.4), (2.5) satisfies the integral identity

$$\int_\Omega ((\nabla u, \nabla v) + uv)dx + \alpha \int_\Gamma uv\, ds = \int_\Omega hv\, dx \tag{3.1}$$

for all $v \in H^1(\Omega)$.

Proof of Theorem 1. At first we obtain some auxiliary estimates for the solution of the problem (2.4), (2.5) for $\alpha > 0$. Taking $v = u$ in (3.1), we obtain

$$\int_\Omega (|\nabla u|^2 + u^2)dx + \alpha \int_\Gamma u^2 ds = \int_\Omega hu\, dx. \tag{3.2}$$

It follows from (3.2) that

$$\int_\Omega (|\nabla u|^2 + u^2)dx + \alpha \int_\Gamma u^2 ds \leq \frac{1}{2}\int_\Omega u^2 dx + \frac{1}{2}\int_\Omega h^2 dx. \tag{3.3}$$

Due to $\Gamma \in C^2$ a weak solution of the problem (2.4), (2.5) is a strong solution in $H^2(\Omega)$. So, $\frac{\partial u}{\partial \nu} \in H^1(\Omega)$ and we have a trace $\frac{\partial u}{\partial \nu}\big|_\Gamma \in L_2(\Gamma)$. By the boundary condition (2.5) we obtain the equalities

$$u = -\frac{1}{\alpha}\frac{\partial u}{\partial \nu} \quad \text{on } \Gamma$$

and

$$\alpha \int_\Gamma u^2 ds = \frac{1}{\alpha}\int_\Gamma \left(\frac{\partial u}{\partial \nu}\right)^2 ds. \tag{3.4}$$

Combining (3.3) and (3.4), we have the estimate

$$\int_\Omega (|\nabla u|^2 + u^2)dx + \alpha \int_\Gamma u^2 ds + \frac{1}{\alpha}\int_\Gamma \left(\frac{\partial u}{\partial \nu}\right)^2 ds \le \int_\Omega h^2 dx. \tag{3.5}$$

Now we suppose that $u(x) \in C^2(\overline{\Omega})$ and that u satisfies the boundary condition (2.5). Since $\Gamma \in C^2$ we consider u as extended to $R^n \setminus \Omega$ such that $u \in C^2(R^n)$. A direct computation gives

$$(\Delta u)^2 = |\nabla^2 u|^2 + \operatorname{div}\Big(\Delta u \nabla u - \frac{1}{2}\nabla\left(|\nabla u|^2\right)\Big), \tag{3.6}$$

where

$$|\nabla^2 u|^2 = \sum_{i,j=1}^n u^2_{x_i x_j}.$$

Integrating the relation (3.6) on Ω and applying the Gauss-Ostrogradskiy formula, we have

$$\int_\Omega (\Delta u)^2 dx = \int_\Omega |\nabla^2 u|^2 dx + \int_\Gamma \left(\Delta u \frac{\partial u}{\partial \nu} - \frac{1}{2}\frac{\partial}{\partial \nu}\left(|\nabla u|^2\right)\right) ds. \tag{3.7}$$

To estimate the surface integral in (3.7) consider a local orthogonal coordinate system $(y_1, \dots, y_n) = (y_1(x), \dots, y_n(x))$ around an arbitrary point $x \in \Gamma$ such that n-th axis direction coincides with the outer normal vector ν to Γ with origin in x. The first $n-1$ coordinate axes lie in the tangential hyperplane to Γ. Now, for any $x \in \Gamma$ there exists a neighborhood $B_\varepsilon(0)$ such that the surface $\Gamma \cap B_\varepsilon(0)$ is determined by the equation $y_n = \omega(y') \in C^2(D)$, $y' = (y_1, \dots, y_{n-1}) \in D \subset R^{n-1}$. Note that

$$\omega_{y_i}(0) = 0, \quad i = 1, \dots, n-1. \tag{3.8}$$

In this local coordinates system we have

$$\begin{aligned}
\int_\Gamma \left(\Delta u \frac{\partial u}{\partial \nu} - \frac{1}{2}\frac{\partial}{\partial \nu}\left(|\nabla u|^2\right)\right) ds &= \int_\Gamma \sum_{j=1}^n \left(u_{y_j y_j} u_{y_n} - u_{y_j} u_{y_j y_n}\right) ds \\
&= \int_\Gamma \sum_{j=1}^{n-1} \left(u_{y_j y_j} u_{y_n} - u_{y_j} u_{y_j y_n}\right) ds \\
&= \int_\Gamma \sum_{j=1}^{n-1} \left(u_{y_j y_j} u_{y_n} - u_{y_j} u_{y_n y_j}\right) ds = I_1(\alpha) + I_2(\alpha),
\end{aligned} \tag{3.9}$$

$$I_1(\alpha) = -\int_\Gamma \sum_{j=1}^{n-1} u_{y_j} u_{y_n y_j} ds = -\frac{1}{2}\int_\Gamma \frac{\partial}{\partial \nu}|\nabla_\tau u|^2 ds, \tag{3.10}$$

$$I_2(\alpha) = \int_\Gamma \sum_{j=1}^{n-1} u_{y_j y_j} u_{y_n} ds = \int_\Gamma \Delta_\tau u u_\nu ds, \tag{3.11}$$

where the vector

$$\nabla_\tau u = \nabla u - \frac{\partial u}{\partial \nu}\nu \tag{3.12}$$

is a tangential gradient of the function u on Γ and

$$\Delta_\tau u = \sum_{j=1}^{n-1} u_{y_j y_j} \tag{3.13}$$

is the Laplace operator in the $(n-1)$-dimensional tangential hyperplane. Due to (3.10) – (3.13) the values of $I_1(\alpha)$ and $I_2(\alpha)$ do not depend on the position of the $y_1, \dots, y_{n-1}$ axes in the tangential hyperplane.

At first we consider the integral $I_1(\alpha)$. The normal vector to Γ in $\Gamma \cap B_\varepsilon(0)$ is

$$\nu = \frac{1}{(1+|\nabla\omega|^2)^{1/2}} \left(-\omega_{y_1}, \dots, -\omega_{y_{n-1}}, 1\right).$$

The boundary condition (2.5) in the local coordinates system is

$$\begin{aligned} &u_{y_n}(y', \omega(y')) - \sum_{j=1}^{n-1} u_{y_j}(y', \omega(y'))\omega_{y_j}(y') \\ &\qquad + \alpha(1+|\nabla\omega|^2)^{1/2} u(y', \omega(y')) = 0, \quad y' \in D. \end{aligned} \tag{3.14}$$

Differentiating the equality (3.14) on y_i, $i = 1, \dots, n-1$ we obtain

$$\begin{aligned} &u_{y_n y_i} + u_{y_n y_n}\omega_{y_i} - \sum_{j=1}^{n-1} \left(\left(u_{y_j y_i} + u_{y_j y_n}\omega_{y_i}\right)\omega_{y_j} + u_{y_j}\omega_{y_j y_i}\right) \\ &\quad + \alpha\Big(\sum_{j=1}^{n-1} \frac{\omega_{y_j}\omega_{y_j y_i} u}{(1+|\nabla\omega|^2)^{1/2}} + (1+|\nabla\omega|^2)^{1/2}\left(u_{y_i} + u_{y_n}\omega_{y_i}\right)\Big) = 0. \end{aligned} \tag{3.15}$$

Consider the relation (3.15) at $y' = 0$. We have by (3.8) the following equality

$$u_{y_n y_i} - \sum_{j=1}^{n-1} u_{y_j}\omega_{y_j y_i} + \alpha u_{y_i} = 0 \quad \text{on } \Gamma.$$

Consequently,

$$u_{y_n y_i} = \sum_{j=1}^{n-1} u_{y_j}\omega_{y_j y_i} - \alpha u_{y_i} = 0 \quad \text{on } \Gamma \tag{3.16}$$

and

$$\begin{aligned} I_1(\alpha) &= \int_\Gamma \sum_{i=1}^{n-1} u_{y_i}\Big(\alpha u_{y_i} - \sum_{j=1}^{n-1} u_{y_j}\omega_{y_j y_i}\Big)\, ds \\ &= \alpha \int_\Gamma |\nabla_\tau u|^2 ds - \int_\Gamma \sum_{i,j=1}^{n-1} \omega_{y_i y_j} u_{y_i} u_{y_j}\, ds. \end{aligned} \tag{3.17}$$

Since $\Gamma \in C^2$ we have

$$\sup_{x\in\Gamma} |\,\omega_{y_i y_j}| \le K$$

and

$$\Big|\int_\Gamma \sum_{i,j=1}^{n-1} \omega_{y_i y_j} u_{y_i} u_{y_j}\, ds\Big| \le (n-1)K \int_\Gamma |\nabla_\tau u|^2 ds. \tag{3.18}$$

Combining (3.17), (3.18), we obtain the inequality

$$I_1(\alpha) \ge (\alpha - (n-1)K) \int_\Gamma |\nabla_\tau u|^2 ds. \tag{3.19}$$

Now, consider the integral $I_2(\alpha)$. By the relation $u_{y_j y_j} u = -u_{y_j}^2 + \big(u u_{y_j}\big)_{y_j}$ we have

$$I_2(\alpha) = \alpha \int_\Gamma |\nabla_\tau u|^2 ds - \alpha \int_\Gamma \sum_{j=1}^{n-1} \big(u u_{y_j}\big)_{y_j}\, ds. \tag{3.20}$$

To estimate the second integral in (3.20) we need some results about differentiable functions on closed surfaces ([9], see also [7], Ch. 1, Par. 7).

Let $\Gamma \in C^2$ be a closed surface in R^n and Θ be the tangential hyperplane to Γ at the point x. Let $p(x) \in C^1(\Gamma)$ be a vector function such that for all $x \in \Gamma$ we have $p(x) \in \Theta$. For any $x \in \Gamma$ and $x' \in \Gamma \cap B_\varepsilon(x)$ we consider the projection $\tilde{x}'$ of the point x on Θ and the projection $\tilde{p}(x')$ of the vector function $p(x)$ on Θ. Denote by $\operatorname{div}_\Theta p(x)$ the value of the divergence of the function $\tilde{p}(x')$ in the $(n-1)$-dimensional space Θ at the point $x' = x$. Then

$$\int_\Gamma \operatorname{div}_\Theta p(x)\, ds = 0. \tag{3.21}$$

Now we set $p = u\nabla_\tau u$. Consequently, in the local coordinate system $(y', y_n)\big|_x$ we have

$$p = u\Big(u_{y_1} - \frac{\omega_{y_1}}{1+|\nabla\omega|^2}\Big(\sum_{j=1}^{n-1} \omega_{y_j} u_{y_j} - u_{y_n}\Big), \dots, u_{y_{n-1}} - \frac{\omega_{y_{n-1}}}{1+|\nabla\omega|^2}\Big(\sum_{j=1}^{n-1} \omega_{y_j} u_{y_j} - u_{y_n}\Big),$$
$$u_{y_n} + \frac{1}{1+|\nabla\omega|^2}\Big(\sum_{j=1}^{n-1} \omega_{y_j} u_{y_j} - u_{y_n}\Big)\Big), \quad y' \in D,$$

and

$$\tilde{p} = \Big(u u_{y_1} - \frac{\omega_{y_1} u}{1+|\nabla\omega|^2}\Big(\sum_{j=1}^{n-1} \omega_{y_j} u_{y_j} - u_{y_n}\Big), \dots, u u_{y_{n-1}} - \frac{\omega_{y_{n-1}} u}{1+|\nabla\omega|^2}\Big(\sum_{j=1}^{n-1} \omega_{y_j} u_{y_j} - u_{y_n}\Big), 0\Big).$$

So,

$$\begin{aligned} &\sum_{i=1}^{n-1}\Big(u u_{y_i} - \frac{\omega_{y_i} u}{1+|\nabla\omega|^2}\Big(\sum_{j=1}^{n-1} \omega_{y_j} u_{y_j} - u_{y_n}\Big)\Big)_{y_i} \\ &\qquad = \sum_{i=1}^{n-1}\big(u u_{y_i}\big)_{y_i} - \frac{u}{1+|\nabla\omega|^2} \sum_{i=1}^{n-1} \omega_{y_i y_i}\Big(\sum_{j=1}^{n-1} \omega_{y_j} u_{y_j} - u_{y_n}\Big) \\ &\qquad - \sum_{i=1}^{n-1} \omega_{y_i}\Big(\frac{u}{1+|\nabla\omega|^2}\Big(\sum_{j=1}^{n-1} \omega_{y_j} u_{y_j} - u_{y_n}\Big)\Big)_{y_i} \end{aligned} \tag{3.22}$$

and by (3.8) we obtain

$$\begin{aligned}
\operatorname{div}_{\Theta} p(x) &= \sum_{i=1}^{n-1} \big(u u_{y_i}\big)_{y_i} + u u_{y_n} \sum_{j=1}^{n-1} \omega_{y_j y_j} \\
&= \sum_{i=1}^{n-1} \big(u u_{y_i}\big)_{y_i} + (n-1) H(x) u u_{y_n}.
\end{aligned}$$

where $H(x)$ is the mean curvature of the surface Γ oriented by the outer normal ν at the point x. Therefore,

$$\begin{aligned}
\sum_{i=1}^{n-1} \big(u u_{y_i}\big)_{y_i} &= \operatorname{div}_{\Theta} p(x) - (n-1) H(x) u u_{y_n} \\
&= \operatorname{div}_{\Theta} p(x) - (n-1) H(x) u \frac{\partial u}{\partial \nu}\,. \qquad (3.23)
\end{aligned}$$

Combining the equalities (3.20), (3.21) and (3.23), we have

$$\begin{aligned}
I_2(\alpha) &= \alpha \int_\Gamma |\nabla_\tau u|^2 ds - \alpha \int_\Gamma \left(\operatorname{div}_{\Theta} p(x) - (n-1) H(x) u \frac{\partial u}{\partial \nu} \right) ds \\
&= \alpha \int_\Gamma |\nabla_\tau u|^2 ds + \alpha(n-1) \int_\Gamma H(x) u \frac{\partial u}{\partial \nu}\, ds \\
&= \alpha \int_\Gamma |\nabla_\tau u|^2 ds - (n-1) \int_\Gamma H(x) \Big(\frac{\partial u}{\partial \nu} \Big)^2 ds. \qquad (3.24)
\end{aligned}$$

Now, by

$$\begin{aligned}
\int_\Omega (\Delta u)^2 dx &= \int_\Omega |\nabla^2 u|^2 dx + I_1(\alpha) + I_2(\alpha) \\
&\geq \int_\Omega |\nabla^2 u|^2 dx + (2\alpha - (n-1)K) \int_\Gamma |\nabla_\tau u|^2 ds - (n-1) \int_\Gamma H(x) \Big(\frac{\partial u}{\partial \nu} \Big)^2 ds
\end{aligned}$$

for $\alpha > (n-1)K/2$ we obtain the inequality

$$\int_\Omega |\nabla^2 u|^2 dx \leq \int_\Omega (\Delta u)^2 dx + (n-1) \int_\Gamma H(x) \Big(\frac{\partial u}{\partial \nu} \Big)^2 ds. \qquad (3.25)$$

Let us note that for convex domains Ω we have $H(x) \leq 0$ and the inequality (3.25) provides the estimate

$$\int_\Omega |\nabla^2 u|^2 dx \leq \int_\Omega (\Delta u)^2 dx.$$

In general case, it follows from (3.25) that

$$\int_\Omega |\nabla^2 u|^2 dx \leq \int_\Omega (\Delta u)^2 dx + (n-1) H_1 \int_\Gamma \Big(\frac{\partial u}{\partial \nu} \Big)^2 ds, \qquad (3.26)$$

where $H_1 = \sup\limits_{x \in \Gamma} |H(x)| > 0$. Using the inequality

$$\left| \frac{\partial u}{\partial \nu} \right| \leq |\nabla u| \quad \text{on} \quad \Gamma, \qquad (3.27)$$

we obtain

$$\int_\Omega |\nabla^2 u|^2 dx \le \int_\Omega (\Delta u)^2 dx + (n-1)H_1 \int_\Gamma |\nabla u|^2\, ds.$$

Now we apply the inequality in [8], Ch. 3, Par. 5, Formula 19,

$$\|v\|^2_{L_2(\Gamma)} \le \varepsilon \|\nabla v\|^2_{L_2(\Omega)} + \frac{C_3}{\varepsilon}\|v\|^2_{L_2(\Omega)}, \tag{3.28}$$

valid for $v(x) \in H^1(\Omega)$ with an arbitrary $\varepsilon > 0$. Hence,

$$\|\nabla u\|^2_{L_2(\Gamma)} \le \varepsilon \|\nabla^2 u\|^2_{L_2(\Omega)} + \frac{C_3}{\varepsilon}\|\nabla u\|^2_{L_2(\Omega)} \tag{3.29}$$

and

$$\int_\Omega |\nabla^2 u|^2 dx \le \int_\Omega (\Delta u)^2 dx + (n-1)H_1 \Big(\varepsilon \int_\Omega |\nabla^2 u|^2 dx + \frac{C_3}{\varepsilon}\int_\Omega |\nabla u|^2 dx\Big). \tag{3.30}$$

Taking $\varepsilon = 1\,/(2(n-1)H_1)$, in (3.30) we obtain

$$\frac{1}{2}\int_\Omega |\nabla^2 u|^2 dx \le \int_\Omega (\Delta u)^2 dx + C_4 \int_\Omega |\nabla u|^2 dx, \quad \alpha \ge \alpha_1 > 0. \tag{3.31}$$

It follows from (2.4) and Green's formula that

$$\begin{aligned}\int_\Omega h^2 dx &= \int_\Omega (-\Delta u + u)^2 dx \\ &= \int_\Omega ((\Delta u)^2 + u^2 - 2u\Delta u) dx \\ &= \int_\Omega ((\Delta u)^2 + 2|\nabla u|^2 + u^2) dx - 2\int_\Gamma u \frac{\partial u}{\partial \nu} ds.\end{aligned}$$

Therefore,

$$\int_\Omega ((\Delta u)^2 + 2|\nabla u|^2 + u^2) dx \le \int_\Omega h^2 dx + 2\Big(\int_\Gamma u^2 ds\Big)^{1/2}\Big(\int_\Gamma \Big(\frac{\partial u}{\partial \nu}\Big)^2 ds\Big)^{1/2}. \tag{3.32}$$

By the inequality (3.5)

$$\int_\Gamma u^2 ds \le \frac{1}{\alpha}\int_\Omega h^2 dx, \tag{3.33}$$

$$\int_\Gamma \Big(\frac{\partial u}{\partial \nu}\Big)^2 ds \le \alpha \int_\Omega h^2 dx. \tag{3.34}$$

Now, combining the inequalities (3.32) – (3.34), we have

$$\int_\Omega ((\Delta u)^2 + |\nabla u|^2 + u^2) dx \le 3\int_\Omega h^2 dx, \quad \alpha > 0, \tag{3.35}$$

and, finally, it follows from (3.31) and (3.35) that

$$\int_\Omega \big(|\nabla^2 u|^2 + |\nabla u|^2 + u^2\big)\, dx \le C_2 \int_\Omega h^2 dx, \quad \alpha \ge \alpha_1, \quad C_2 = 3(1 + C_4). \tag{3.36}$$

We prove now the estimate (2.6) for all functions $u \in C^2(\overline{\Omega})$ satisfying the boundary condition (2.5). To prove the estimate (2.6) for solutions of (2.4), (2.5) in $H^2(\Omega)$ we take a sequence of functions $u_m \in C^2(\overline{\Omega})$ satisfying (2.5) such that $\|u - u_m\|_{H^2(\Omega)} \to 0$, $m \to \infty$. Applying the estimate (2.6) to the functions u_m we have

$$\|u_m\|_{H^2(\Omega)} \le C_2 \|h_m\|_{L_2(\Omega)}, \quad \alpha \ge \alpha_1, \tag{3.37}$$

where $h_m = -\Delta u_m + u_m$. Therefore, $\|h_m\|_{L_2(\Omega)} \to \|h\|_{L_2(\Omega)}$, $m \to \infty$. Taking a limit in (3.37), we obtain the inequality (2.6) for all functions $u \in H^2(\Omega)$ satisfying (2.5).

Theorem 1 is proved.

4. L_2-CONVERGENCE OF EIGENFUNCTION

Let $u_\alpha \in H^1(\Omega)$ be the first eigenfunction of the problem (1.1), (1.2) such that $u_\alpha \ge 0$ in Ω and $\|u_\alpha\|_{L_2(\Omega)} = 1$. This eigenfunction satisfies the integral identity

$$\int_\Omega (\nabla u_\alpha, \nabla v)\, dx + \alpha \int_\Gamma u_\alpha v\, ds = \lambda_1(\alpha) \int_\Omega u_\alpha v\, dx \tag{4.1}$$

for all $v \in H^1(\Omega)$. Therefore, taking $v = u_\alpha$ in (4.1), we obtain:

$$\int_\Omega |\nabla u_\alpha|^2 dx + \alpha \int_\Gamma u_\alpha^2 dx = \lambda_1(\alpha) \le \lambda_1^D. \tag{4.2}$$

and

$$\|u_\alpha\|^2_{H^1(\Omega)} + \alpha \int_\Gamma u_\alpha^2 ds \le \lambda_1^D + 1. \tag{4.3}$$

Consider a sequence $\alpha_k \to +\infty$, $k \to \infty$. By (4.3) the sequence u_{α_k} is bounded in $H^1(\Omega)$ and

$$\|u_{\alpha_k}\|_{H^1(\Omega)} \le \sqrt{\lambda_1^D + 1}. \tag{4.4}$$

Take a subsequence (denoted also by $\{u_{\alpha_k}\}$) such that

$$u_{\alpha_k} \to \tilde{u} \text{ weakly in } H^1(\Omega), \text{ strongly in } L_2(\Omega) \text{ and } L_2(\Gamma). \tag{4.5}$$

Let $v \in \overset{o}{H}{}^1(\Omega)$, then by (4.1) we have

$$\int_\Omega (\nabla u_{\alpha_k}, \nabla v)\, dx = \lambda_1(\alpha_k) \int_\Omega u_{\alpha_k} v\, dx. \tag{4.6}$$

Note that by (1.8)

$$\lim_{\alpha \to +\infty} \lambda_1(\alpha) = \lambda_1^D. \tag{4.7}$$

Now, it follows from (4.5), (4.6) and (4.7) that

$$\int_\Omega (\nabla \tilde{u}, \nabla v)\, dx = \lambda_1^D \int_\Omega \tilde{u} v\, dx \tag{4.8}$$

for any $v \in \overset{o}{H}{}^1(\Omega)$. Applying the estimate (4.2), we obtain the inequality

$$\int_\Gamma u_{\alpha_k}^2 ds \le \frac{\lambda_1^D}{\alpha_k},$$

so

$$\int_\Gamma \tilde{u}^2 ds = 0.$$

Hence, we obtain that $\tilde{u} \in \overset{o}{H}{}^1(\Omega)$. So, the function $\tilde{u}$ is a weak solution of the boundary value problem

$$\Delta \tilde{u} + \lambda_1^D \tilde{u} = 0 \quad \text{in } \Omega, \tag{4.9}$$

$$\tilde{u} = 0 \quad \text{on } \Gamma. \tag{4.10}$$

Moreover, it follows from (4.5) that $\tilde{u} \geq 0$ in Ω and $\|\tilde{u}\|_{L_2(\Omega)} = 1$. Therefore, $\tilde{u} = u^D$, where u^D is a positive normalized eigenfunction of the Dirichlet problem (1.3), (1.4) and

$$\|u^D - u_{\alpha_k}\|_{L_2(\Omega)} \to 0, \quad k \to \infty. \tag{4.11}$$

Now, we want to show that

$$\|u^D - u_{\alpha}\|_{L_2(\Omega)} \to 0, \quad \alpha \to +\infty. \tag{4.12}$$

Let us suppose that (4.12) is not true. It means that there exists an $\varepsilon > 0$ and a sequence $\alpha_k \to +\infty$, $k \to \infty$ such that

$$\|u^D - u_{\alpha_k}\|_{L_2(\Omega)} > \varepsilon, \quad k = 1, 2, \ldots . \tag{4.13}$$

Let us take a subsequence (denoted also by $\{u_{\alpha_k}\}$) such that (4.5) holds for some $\tilde{u} \in H^1(\Omega)$. But this means that $\tilde{u} = u^D$ and (4.11) holds, which contradicts (4.13). The relation (4.12) is proved.

5. H^2-convergence of eigenfunction

Let u^D and u_α be nonnegative first normalized Dirichlet and Robin eigenfunctions respectively. Therefore,

$$\Delta u_\alpha + \lambda_1(\alpha) u_\alpha = 0 \quad \text{in } \Omega, \tag{5.1}$$

$$\frac{\partial u_\alpha}{\partial \nu} + \alpha u_\alpha = 0 \quad \text{on } \Gamma, \tag{5.2}$$

$$\Delta u^D + \lambda_1^D u^D = 0 \quad \text{in } \Omega, \tag{5.3}$$

$$u^D = 0 \quad \text{on } \Gamma. \tag{5.4}$$

By (5.1) – (5.4) the function $w = u^D - u_\alpha$ is a solution of the boundary value problem

$$-\Delta w + w = \left(\lambda_1^D + 1\right)\left(u^D - u_\alpha\right) + \left(\lambda_1^D - \lambda_1(\alpha)\right) u_\alpha \quad \text{in } \Omega, \tag{5.5}$$

$$w = \frac{1}{\alpha}\frac{\partial u_\alpha}{\partial \nu} \quad \text{on } \Gamma. \tag{5.6}$$

Consider a function $b(x) = (b_1(x), \ldots, b_n(x)) \in C^2(\overline{\Omega})$ such that $b = \nu$ on Γ. Therefore, the function

$$\tilde{w} = w - \frac{1}{\alpha}(b, \nabla u_\alpha)$$

is a solution of the boundary value problem

$$-\Delta \tilde{w} + \tilde{w} = h_\alpha(x) \quad \text{in } \Omega, \tag{5.7}$$

$$\tilde{w} = 0 \quad \text{on } \Gamma, \tag{5.8}$$

where

$$
\begin{aligned}
h_\alpha &= \left(\lambda_1^D + 1\right)\left(u^D - u_\alpha\right) + \left(\lambda_1^D - \lambda_1(\alpha)\right) u_\alpha + \frac{1}{\alpha}\left((b, \nabla u_\alpha) - \Delta(b, \nabla u_\alpha)\right) \\
&= \left(\lambda_1^D + 1\right)\left(u^D - u_\alpha\right) + \left(\lambda_1^D - \lambda_1(\alpha)\right) u_\alpha \\
&+ \frac{1}{\alpha}\Big((b, \nabla u_\alpha) - (\Delta b, \nabla u_\alpha) - 2\sum_{i,j=1}^{n} (b_j)_{x_i}(u_\alpha)_{x_j} - (b, \nabla \Delta u_\alpha)\Big) \\
&= \left(\lambda_1^D + 1\right)\left(u^D - u_\alpha\right) + \left(\lambda_1^D - \lambda_1(\alpha)\right) u_\alpha \\
&+ \frac{1}{\alpha}\Big((b - \Delta b, \nabla u_\alpha) - 2\sum_{i,j=1}^{n} (b_j)_{x_i}(u_\alpha)_{x_i x_j} + \lambda_1(\alpha)(b, \nabla u_\alpha)\Big) \\
&= \left(\lambda_1^D + 1\right)\left(u^D - u_\alpha\right) + \left(\lambda_1^D - \lambda_1(\alpha)\right) u_\alpha \\
&+ \frac{1}{\alpha}\Big(((1 + \lambda_1(\alpha))b - \Delta b, \nabla u_\alpha) - 2\sum_{i,j=1}^{n} (b_j)_{x_i}(u_\alpha)_{x_i x_j}\Big).
\end{aligned} \tag{5.9}
$$

The first eigenfunction u_α is solution of the boundary value problem

$$
\begin{aligned}
-\Delta u_\alpha + u_\alpha &= (\lambda_1(\alpha) + 1) u_\alpha \quad && \text{in } \Omega, \\
\frac{\partial u_\alpha}{\partial \nu} + \alpha u_\alpha &= 0 && \text{on } \Gamma,
\end{aligned}
$$

and by (2.6) satisfies

$$
\|u_\alpha\|_{H^2(\Omega)} \le C_2(\lambda_1(\alpha) + 1), \quad \alpha > \alpha_1. \tag{5.10}
$$

Now, using the estimate (5.10) and the boundary flattering procedure for proving the higher regularity of solutions to boundary value problems associated to second-order elliptic operators ([8], Ch 4, Par. 2, No. 3, [2], Ch. 6, Par 6.3), we obtain

$$
\|u_\alpha\|_{H^3(\Omega)} \le C_5 \|u_\alpha\|_{H^1(\Omega)} \le C_6, \quad \alpha > \alpha_1, \tag{5.11}
$$

where C_5, C_6 do not depend on α. Combining (1.8), (4.12), (5.9) and (5.10), we obtain the estimate

$$
\begin{aligned}
\|h_\alpha\|_{L_2(\Omega)} &\le (\lambda_1^D + 1)\|u^D - u_\alpha\|_{L_2(\Omega)} \\
&\quad + (\lambda_1^D - \lambda_1(\alpha))\|u_\alpha\|_{L_2(\Omega)} + \frac{C_7}{\alpha}\|u_\alpha\|_{H^2(\Omega)} \\
&\le C_8\Big(\|u^D - u_\alpha\|_{L_2(\Omega)} + \frac{1}{\alpha}\Big), \quad \alpha > \alpha_1.
\end{aligned} \tag{5.12}
$$

It follows from the inequality ([8], Ch 4, Par. 2, Th. 4) that

$$
\|\tilde{w}\|_{H^2(\Omega)} \le C_9\Big(\|u^D - u_\alpha\|_{L_2(\Omega)} + \frac{1}{\alpha}\Big), \quad \alpha > \alpha_1. \tag{5.13}
$$

Combining (3.5), (5.11), (5.12) with (5.13), we get

$$
\begin{aligned}
\|u^D - u_\alpha\|_{H^2(\Omega)} &= \Big\|\tilde{w} + \frac{1}{\alpha}(b, \nabla u_\alpha)\Big\|_{H^2(\Omega)} \\
&\le \|\tilde{w}\|_{H^2(\Omega)} + \frac{1}{\alpha}\,\|(b, \nabla u_\alpha)\|_{H^2(\Omega)} \\
&\le C_{10}\Big(\Big(\|u^D - u_\alpha\|_{L_2(\Omega)} + \frac{1}{\alpha}\Big) + \frac{1}{\alpha}\|u_\alpha\|_{H^3(\Omega)}\Big) \\
&\le C_{11}\Big(\|u^D - u_\alpha\|_{L_2(\Omega)} + \frac{1}{\alpha}\Big), \quad \alpha > \alpha_1, \qquad (5.14)
\end{aligned}
$$

with the constant C_{11} independent of α.

6. Asymptotic expansion

Proof of the Main Theorem.

For the normalized eigenfunction u^D the relation (2.1) is equivalent to

$$
\lim_{\alpha\to+\infty} \frac{\lambda_1(\alpha) - \lambda_1^D}{\dfrac{1}{\alpha}} = -\int_\Gamma \Big(\frac{\partial u^D}{\partial \nu}\Big)^2 ds. \tag{6.1}
$$

Let us note that, by (1.8), the numerator $\lambda_1(\alpha) - \lambda_1^D$ in the fraction in (6.1) tends to zero when $\alpha \to +\infty$. By the formula ([4], Th. 1, Form. (7)) and the boundary condition (1.2) we have

$$
\lambda_1'(\alpha) = \int_\Gamma u_\alpha^2 ds = \frac{1}{\alpha^2}\int_\Gamma \Big(\frac{\partial u_\alpha}{\partial \nu}\Big)^2 ds,
$$

where u_α is the first normalized eigenfunction of the problem (1.1), (1.2). Therefore,

$$
\lim_{\alpha\to+\infty} \frac{\lambda_1'(\alpha)}{-\dfrac{1}{\alpha^2}} = -\lim_{\alpha\to+\infty}\int_\Gamma \Big(\frac{\partial u_\alpha}{\partial \nu}\Big)^2 ds. \tag{6.2}
$$

Let us prove that

$$
\lim_{\alpha\to+\infty}\int_\Gamma \Big(\frac{\partial u_\alpha}{\partial \nu}\Big)^2 ds = \int_\Gamma \Big(\frac{\partial u^D}{\partial \nu}\Big)^2 ds. \tag{6.3}
$$

By the inequalities (3.27), (3.29), (4.12) and (5.14) we have

$$
\begin{aligned}
\int_\Gamma \Big(\frac{\partial u^D}{\partial \nu} - \frac{\partial u_\alpha}{\partial \nu}\Big)^2 ds &\le \int_\Gamma |\nabla(u^D - u_\alpha)|^2 ds \\
&\le C\left(\big\|\nabla^2(u^D - u_\alpha)\big\|^2_{L_2(\Omega)} + \big\|\nabla(u^D - u_\alpha)\big\|^2_{L_2(\Omega)}\right) \\
&\le C\|u^D - u_\alpha\|_{H^2(\Omega)} \to 0, \quad \alpha \to +\infty. \qquad (6.4)
\end{aligned}
$$

Using (6.4), we obtain the relation (6.3). Now, by L'Hôpital's rule the equality (6.1) follows from (6.2).The proof of the Main Theorem is completed.

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