JEPE Vol 1, 2015, p. 109-121

*Orthogonal* ⊥

# REGULARITY RESULTS FOR QUASILINEAR ELLIPTIC EQUATIONS IN THE PLANE

LINDA MARIA DE CAVE, LUIGI D'ONOFRIO AND ROBERTA SCHIATTARELLA

ABSTRACT. For a planar domain $\Omega$, we study the Dirichlet problem for the quasilinear elliptic equation

$$-\operatorname{div} A(x,\nabla v) = f$$

when $f$ belongs to the Zygmund space $L(\log L)^{\frac{1}{2}}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)$, $\beta \geq 0$. We prove that there exists a unique solution $v \in W_0^{1,2}(\Omega)$ with $|\nabla v| \in L^2(\log\log\log L)^{\beta}(\Omega)$.

## 1. INTRODUCTION

In this paper we consider the following Dirichlet problem on a bounded open set $\Omega \subset \mathbb{R}^2$ with $\mathcal{C}^1$ boundary

$$\begin{cases} \mathcal{A}v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $\mathcal{A}$ is the differential operator defined by

$$\mathcal{A}v = -\operatorname{div} A(x,\nabla v)$$

Here $A : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a Charathéodory mapping, that is

$$\begin{aligned} &A(\cdot,\xi) \text{ is measurable for all } \xi \in \mathbb{R}^2 \\ &A(x,\cdot) \text{ is continuous for almost every } x \in \Omega. \end{aligned} \tag{1.2}$$

Furthermore, we assume that $A$ satisfies the Leray–Lions type conditions, i.e. there exists $K \geq 1$ such that, for almost every $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^2$, it results

$$\begin{aligned} i)&\quad |A(x,\xi) - A(x,\eta)| \leq K|\xi - \eta| \\ ii)&\quad |\xi - \eta|^2 \leq K\langle A(x,\xi) - A(x,\eta), \xi - \eta\rangle \\ iii)&\quad A(x,0) = 0. \end{aligned} \tag{1.3}$$

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2010 *Mathematics Subject Classification.* Primary 35J62; Secondary 35B65.

*Key words and phrases.* Gradient regularity, Quasilinear elliptic equations, Zygmund spaces.

Received 09/10/2014, accepted 14/05/2015.

Research supported by the Gruppo Nazionale per l' Analisi Matematica, la Probabilita' e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

The first Author was partially supported by *L.R. n. 5/2008*, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II".

In [17], under the assumptions (1.2) and (1.3), the Authors proved for the problem (1.1) with $f \in L^1(\Omega)$ an existence and uniqueness theorem of the solution in the Grand Sobolev space $W_0^{1,2)}(\Omega)$. This is the space of functions $v \in W_0^{1,1}(\Omega)$ whose gradient satisfies

$$\sup_{0<\varepsilon\le 1}\left[\varepsilon\int_\Omega |\nabla v|^{2-\varepsilon}\,dx\right]^{\frac{1}{2-\varepsilon}} = \|v\|_{W^{1,2)}(\Omega)} < \infty.$$

We emphasize that $W_0^{1,2)}(\Omega)$ is a function space slightly larger than $W_0^{1,2}(\Omega)$.

The critical Zygmund class "close"to $L^1$ for $f$ such that the solution $v$ has finite energy, i.e. $v \in W_0^{1,2}(\Omega)$, is $L(\log L)^{\frac12}(\Omega)$. This derives from the Trudinger embedding (see [23] and Section 2 for definitions)

$$W_0^{1,2}(\Omega) \quad \hookrightarrow \quad \exp_2(\Omega)$$

that implies

$$L(\log L)^{\frac12}(\Omega) \hookrightarrow W^{-1,2}(\Omega)\,,$$

as follows by a duality relation in the usual topological sense (see [22]).
Further regularity derives from the stronger assumption $f \in L\log L(\Omega)$. Precisely, (see [2], [7]):

$$f \in L\log L(\Omega) \;\Rightarrow\; |\nabla v| \in L^2\log L(\Omega).$$

By the embedding theorems for Orlicz-Sobolev spaces (see [8], [14]), the solution $v$ belongs to the double exponential space $L^\Phi(\Omega)$ with $\Phi(t) = \exp\left(\exp\left(t^2\right)\right) - e$. In [3] the Authors covered previous results by proving, for $\frac12 \le \delta \le 1$, the following estimate:

$$\|\nabla v\|_{L^2(\log L)^{2\delta-1}(\Omega)} \le C(K,\delta)\|f\|_{L(\log L)^\delta(\Omega)}.$$

If $f$ belongs to a space slightly smaller than $L(\log L)^{\frac12}(\Omega)$, given by $L(\log L)^{\frac12}(\log\log L)^{\frac{\beta}{2}}(\Omega)$ with $0 \le \beta < 2$, there exists a unique solution $v$ to the Dirichlet problem (1.1) such that $|\nabla v| \in L^2(\log\log L)^\beta(\Omega)$ with the estimate

$$\|\nabla v\|_{L^2(\log\log L)^\beta(\Omega)} \le C(K,\beta)\|f\|_{L(\log L)^{\frac12}(\log\log L)^{\frac{\beta}{2}}(\Omega)}.$$

(see [13]). It generalizes a result of [24] obtained for $\beta = 1$.

In this paper we prove the following

**Main Theorem.** *Let $A = A(x,\xi)$ satisfy (1.2) and (1.3) and let $\beta \ge 0$. Then, for $f \in L(\log L)^{\frac12}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)$ there exists a unique solution $v \in W_0^{1,2}(\Omega)$ to the Dirichlet problem (1.1) with $|\nabla v| \in L^2(\log\log\log L)^\beta(\Omega)$ and the following estimate holds*

$$\|\nabla v\|_{L^2(\log\log\log L)^\beta(\Omega)} \le C(K,\beta)\;\|f\|_{L(\log L)^{\frac12}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)}.$$

For the proof, one of the main tool is a regularity result for elliptic equations with right hand side in divergence form

$$-\mathcal{A}\varphi = \operatorname{div}\chi$$

slightly below the natural space $\chi \in L^2$. Following an idea of [16], we use the well known estimate

$$\|\nabla\varphi\|_{L^{2-\varepsilon}(\Omega)} \le c(K)\|\chi\|_{L^{2-\varepsilon}(\Omega)} \qquad |\varepsilon| \le \varepsilon_0$$

to deduce

$$\|\nabla\varphi\|_{L^2(\log\log\log L)^{-\beta}(\Omega)} \leq c(K,\beta)\|\chi\|_{L^2(\log\log\log L)^{-\beta}(\Omega)}.$$

Similar results are proved in [12] for $f \in L(\log L)^{\delta}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)$ for $\delta > \frac{1}{2}$ and $\beta > 2\delta - 1$. When the datum is a measure we refer the interested reader to [6], [20], [21] and the reference therein.

## 2. Preliminaries

In the present Section we will treat some function spaces and related associate spaces. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ and $X(\Omega)$ be a Banach function space endowed with the norm $\|\cdot\|_{X(\Omega)}$. The Banach function space $(X(\Omega))'$ whose norm is given by

$$\|g\|_{(X(\Omega))'} = \sup\left\{\left|\int_{\Omega} fg\,dx\right| \text{ s.t. } f \in X(\Omega),\ \|f\|_{X(\Omega)} \leq 1\right\}$$

is called the *associate space* of $X(\Omega)$.

A function $u$ belongs to the Lebesgue space $L^p(\Omega)$ with $1 \leq p < \infty$ if, and only if,

$$\|u\|_{L^p(\Omega)} = \left(\fint_{\Omega} |u|^p\,dx\right)^{\frac{1}{p}} < +\infty$$

where $\fint_{\Omega} = \frac{1}{|\Omega|}\int_{\Omega}$.

Now we recall some useful function spaces slightly larger than classical Lebesgue spaces.

**2.1. Grand Lebesgue spaces.** For $1 < p < \infty$, let us consider the class, denoted by $L^{p)}(\Omega)$, consisting of all measurable functions $u \in \bigcap_{1\leq q<p} L^q(\Omega)$ such that

$$\sup_{0<\varepsilon\leq p-1}\left\{\varepsilon\fint_{\Omega}|u(x)|^{p-\varepsilon}\right\}^{\frac{1}{p-\varepsilon}} < +\infty$$

which was introduced in [18]. $L^{p)}(\Omega)$ becomes a Banach space, the *Grand Lebesgue space* $L^{p)}(\Omega)$, equipped with the norm

$$\|u\|_{L^{p)}(\Omega)} = \sup_{0<\varepsilon\leq p-1}\varepsilon^{\frac{1}{p}}\left\{\fint_{\Omega}|u(x)|^{p-\varepsilon}\right\}^{\frac{1}{p-\varepsilon}}.$$

Moreover, $\|u\|_{L^{p)}(\Omega)}$ is equivalent to

$$\sup_{0<\varepsilon\leq p-1}\left\{\varepsilon\fint_{\Omega}|u(x)|^{p-\varepsilon}\right\}^{\frac{1}{p-\varepsilon}}.$$

In general, if $0 < \alpha < \infty$, we can define the space $L^{\alpha,p)}(\Omega)$ as the space of all measurable functions $u \in \bigcap_{1\leq q<p} L^q(\Omega)$ such that

$$\|u\|_{L^{\alpha,p)}(\Omega)} = \sup_{0<\varepsilon\leq p-1}\left\{\varepsilon^{\frac{\alpha}{p}}\|u\|_{p-\varepsilon}\right\} < +\infty.$$

2.2. **Orlicz spaces.** Let $\Omega$ be an open set in $\mathbb{R}^n$, with $n \geq 2$. A function $\Phi : [0,+\infty) \to [0,+\infty)$ will be called a *Young function* if it is convex, left-continuous and vanishes at 0; thus any Young function $\Phi$ admits the representation

$$\Phi(t) = \int_0^t \phi(s)\,ds \qquad \text{for } t \geq 0$$

where $\phi : [0,+\infty) \to [0,+\infty)$ is a non decreasing, left– continuous function, which is neither identically equal to 0 nor to $\infty$. The *Orlicz space* associated to $\Phi$, named $L^\Phi(\Omega)$, consists of all Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \Phi(\lambda|f|) < \infty \qquad \text{for some } \lambda = \lambda(f) > 0.$$

$L^\Phi(\Omega)$ is a Banach space equipped with the Luxemburg norm

$$\|f\|_{L^\Phi(\Omega)} = \inf\left\{\frac{1}{\lambda} : \int_\Omega \Phi(\lambda|f|) \leq 1\right\}.$$

**Examples of Orlicz spaces:**

1) If $\Phi(t) = t^p$ for $1 \leq p < \infty$ then $L^\Phi(\Omega)$ is the classical Lebesgue space $L^p(\Omega)$.
2) If $\Phi(t) = t^p\,(\log(a+t))^q$ with either $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q \geq 0$, where $a \geq e$ is a suitable large constant, then $L^\Phi(\Omega)$ is the Zygmund space denoted by $L^p(\log L)^q(\Omega)$.
3) If $\Phi(t) = t^p\,(\log\log(a+t))^q$ with either $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q \geq 0$, where $a \geq e^e$, then $L^\Phi(\Omega)$ is the space $L^p(\log\log L)^q(\Omega)$.
4) If $\Phi(t) = t^p\,(\log\log\log(a+t))^q$ with either $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q \geq 0$ where $a \geq e^{e^e}$, then $L^\Phi(\Omega)$ is the space $L^p(\log\log\log L)^q(\Omega)$.
5) If $\Phi(t) = e^{t^a} - 1$ and $a > 0$, then $L^\Phi(\Omega)$ is the space of $a$-exponentially integrable functions $\mathrm{EXP}_a(\Omega)$. We denote by $\exp_a(\Omega)$ the closure of $L^\infty(\Omega)$ in $\mathrm{EXP}_a(\Omega)$.

We have the following relations between Grand Lebesgue and Orlicz spaces:

$$L^p(\Omega) \subset \frac{L^p}{\log L}(\Omega) \subset L^{p)}(\Omega) \subset \bigcap_{\alpha>1} \frac{L^p}{(\log L)^\alpha}(\Omega).$$

The *Young complementary function* is given by

$$\tilde{\Phi}(t) = \sup\{st - \Phi(s) : s > 0\} = \int_0^t \phi^{-1}(s)\,ds \tag{2.1}$$

where

$$\phi^{-1}(s) = \sup\{r : \phi(r) \leq s\}.$$

Moreover, the following Hölder's type inequality holds

$$\left|\int_\Omega f(x)g(x)dx\right| \leq 2\,\|f\|_{L^\Phi(\Omega)}\|g\|_{L^{\tilde{\Phi}}(\Omega)}$$

for $f \in L^\Phi(\Omega)$ and $g \in L^{\tilde{\Phi}}(\Omega)$.

Given two Young functions $\Phi$ and $\Psi$, we will say that $\Psi$ *dominates* $\Phi$ *globally* (respectively *near infinity*), if there exists a constant $k > 0$ such that

$$\Phi(t) \leq \Psi(kt) \text{ for all } t \geq 0 \ \ (\text{respectively for all } t \geq t_0 \ \text{ for some } t_0 > 0)\,;$$

moreover $\Phi$ and $\Psi$ are *equivalent globally* (respectively *near infinity*, $\Phi \cong \Psi$ ) if each dominates the other globally (respectively near infinity). If $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are the complementary

Young functions of, respectively, $\Phi$ and $\Psi$, then $\Psi$ dominates $\Phi$ globally (or near infinity) if and only if $\widetilde{\Phi}$ dominates $\widetilde{\Psi}$ globally (or near infinity). Similarly, $\Phi$ and $\Psi$ are equivalent if and only if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are equivalent. We have the following result.

**Theorem 1.** *The continuous embedding $L^{\Psi}(\Omega) \hookrightarrow L^{\Phi}(\Omega)$ holds if and only if either $\Psi$ dominates $\Phi$ globally or $\Psi$ dominates $\Phi$ near infinity and $\Omega$ has finite measure.*

Here below we recall the explicit expression of the associate of some Orlicz spaces (see [4], [14], [15]).

**Theorem 2.** *Let $\Omega \subset \mathbb{R}^n$ an open set. If $1 < p < \infty$, $q \in \mathbb{R}$, then*

- $(L^p(\log L)^q(\Omega))' \cong L^{p'}(\log L)^{-\frac{q}{p-1}}(\Omega)$
- $(L^p(\log\log L)^q(\Omega))' \cong L^{p'}(\log\log L)^{-\frac{q}{p-1}}(\Omega)$
- $(L^p(\log\log\log L)^q(\Omega))' \cong L^{p'}(\log\log\log L)^{-\frac{q}{p-1}}(\Omega)$

*where $p'$ is the conjugate exponent of $p$, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.*

*If $p = 1$ and $q > 0$ then*

- $(L(\log L)^q(\Omega))' \cong \mathrm{EXP}_{\frac{1}{q}}(\Omega)$.

Finally we recall the definition of the *Orlicz-Sobolev* spaces $W^{1,\Psi}(\Omega)$ and $W_0^{1,\Psi}(\Omega)$ (see [1], [8], [9], [22]). The space $W^{1,\Psi}(\Omega)$ consists of the equivalence classes of functions $u$ in $L^{\Psi}(\Omega)$ such that the length of the distributional gradient $|\nabla u|$ belongs to $L^{\Psi}(\Omega)$. It is a Banach space with respect to the norm given by

$$\|u\|_{W^{1,\Psi}(\Omega)} = \|u\|_{L^{\Psi}(\Omega)} + \|\nabla u\|_{L^{\Psi}(\Omega)}.$$

As in the case of the ordinary Sobolev space, $W_0^{1,\Psi}(\Omega)$ coincides with the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\Psi}(\Omega)$.

2.3. **Orlicz-Sobolev embeddings.** For the embedding Theorem in the setting of Orlicz Sobolev space, we need the following:

**Lemma 1.** *Let $\Phi(t) = \exp\left\{\frac{t^2}{(\log(e+\log(e+t)))^{\beta}}\right\} - 1$ with $\beta \in \mathbb{R}$. Then*

$$\tilde{\Phi}(t) \cong t(\log t)^{\frac{1}{2}}(\log\log\log t)^{\frac{\beta}{2}}.$$

*Proof.* As $\Phi$ is a Young function, by definition we have

$$\Phi(t) = \int_0^t \phi(s)\,ds$$

where

$$\phi(s) \cong \exp\left\{\frac{s^2}{(\log\log s)^{\beta}}\right\} \cdot \left[\frac{2s}{(\log\log s)^{\beta}} - \frac{\beta s}{(\log s)\cdot(\log\log s)^{\beta+1}}\right].$$

For large $s$ we have

$$\phi(s) \cong \exp\left\{\frac{s^2}{(\log\log s)^{\beta}}\right\} \cdot \frac{2s}{(\log\log s)^{\beta}}$$

and there exists a suitable constant $c > 1$ such that

$$\begin{aligned}\exp\left\{\frac{s^2}{(\log\log s)^\beta}\right\} &\le \exp\left\{\frac{s^2}{(\log\log s)^\beta}\right\}\cdot\frac{2s}{(\log\log s)^\beta}\\ &\le \exp\left\{\frac{(cs)^2}{(\log\log(cs))^\beta}\right\}.\end{aligned}$$

Then it is not difficult to check that near infinity it results

$$\phi^{-1}(r) \cong (\log r)^{\frac{1}{2}} (\log\log\log r)^{\frac{\beta}{2}}.$$

By (2.1), we obtain that

$$\tilde{\Phi}(y) = \int_0^y \phi^{-1}(r)\, dr \cong y(\log y)^{\frac{1}{2}} (\log\log\log y)^{\frac{\beta}{2}}.$$

□

Given a Young function $\Psi$ such that

$$\int_0 \left(\frac{r}{\Psi(r)}\right) dr < \infty,$$

we define $\Phi : [0, +\infty) \to [0, +\infty)$ as

$$\Phi(s) = \Psi \circ H_2^{-1}(s) \qquad \text{for } s \ge 0, \tag{2.2}$$

where $H_2^{-1}(s)$ is the (generalized) left continuous inverse of the function $H_2 : [0, +\infty) \to [0, +\infty)$ given by

$$H_2(r) = \left(\int_0^r \left(\frac{t}{\Psi(t)}\right) dt\right)^{\frac{1}{2}} \qquad \text{for } r \ge 0. \tag{2.3}$$

In [10] and in [11], the Author showed that $\Phi$ is a Young function and that the following form of Sobolev embedding theorem holds

$$\|u\|_{L^\Phi(\Omega)} \le C\|\nabla u\|_{L^\Psi(\Omega)}$$

for every function $u$ in the Orlicz-Sobolev space $W_0^{1,\Psi}(\Omega)$. As an application, we have the following result.

**Lemma 2.** *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with $\mathcal{C}^1$ boundary. If we consider Young functions $\Psi(t)$*

$$\Psi(t) \cong t^2 (\log\log\log t)^{-\beta}$$

*with $\beta \in \mathbb{R}$, then*

$$W^{1,\Psi}(\Omega) \hookrightarrow L^\Phi(\Omega)$$

*where*

$$\Phi(s) \cong e^{s^2(\log\log s)^{-\beta}}.$$

*Proof.* By (2.3) we have that

$$H_2(r) = \left( \int_0^r \frac{(\log\log\log t)^\beta}{t}\, dt \right)^{\frac{1}{2}} \cong (\log r)^{\frac{1}{2}} (\log\log\log r)^{\frac{\beta}{2}}.$$

Moreover, as showed in the proof of Lemma 1, the inverse function $H_2^{-1}(s)$ is equivalent near infinity to

$$e^{s^2 (\log\log s)^{-\beta}}.$$

By (2.2), we obtain that

$$\Phi(s) \cong e^{2s^2 (\log\log s)^{-\beta}} (\log\log s)^{-\beta} \cong e^{s^2 (\log\log s)^{-\beta}}.$$

and we conclude that

$$W^{1,\Psi}(\Omega) \hookrightarrow L^{\Phi}(\Omega).$$

□

## 3. Equivalent norm on the Zygmund spaces $L^q(\log\log\log L)^{-\beta}(\Omega)$

We shall introduce an equivalent norm on $L^q(\log\log\log L)^{-\beta}(\Omega)$ with $\beta > 0$, which involves the norms in $L^{q-\varepsilon}(\Omega)$, for $1 < q < \infty$ and $0 < \varepsilon \le q-1$. This is based on a method recently suggested by L. Greco et al. (see [16]). If $f$ is a measurable function on $\Omega$, we set

$$|||f|||_{L^q(\log\log\log L)^{-\beta}(\Omega)} = \left\{ \int_0^{\varepsilon_0} (\varepsilon|\log\varepsilon|)^{-1} \big(1 + \log|\log\varepsilon|\big)^{-(\beta+1)} \|f\|_{q-\varepsilon}^q \, d\varepsilon \right\}^{\frac{1}{q}}. \tag{3.1}$$

Here $\varepsilon_0 \in ]0, q-1]$ is fixed.

**Theorem 3.** *We have* $f \in L^q (\log\log\log L)^{-\beta} (\Omega)$ *if and only if*

$$|||f|||_{L^q(\log\log\log L)^{-\beta}(\Omega)} < +\infty.$$

*Moreover,* $|||\cdot|||_{L^q(\log\log\log L)^{-\beta}(\Omega)}$ *is a norm equivalent to the Luxemburg one, that is, there exist constants* $C_i = C_i(q,\beta,\varepsilon_0)$, $i = 1,2$ *such that for all* $f \in L^q (\log\log\log L)^{-\beta} (\Omega)$ *it results*

$$C_1 \|f\|_{L^q(\log\log\log L)^{-\beta}(\Omega)} \le |||f|||_{L^q(\log\log\log L)^{-\beta}(\Omega)} \le C_2 \|f\|_{L^q(\log\log\log L)^{-\beta}(\Omega)}. \tag{3.2}$$

Now we recall the following standard inequalities (see [16], [13]).

**Lemma 3.** *If* $b \ge e$, $\beta > 0$, *we have*

$$(\log b)^{-\beta} \frac{\varepsilon_0^\beta}{\beta} e^{-\varepsilon_0} \le \int_0^{\varepsilon_0} \varepsilon^{\beta-1} b^{-\varepsilon}\, d\varepsilon \le (\log b)^{-\beta} \frac{\Gamma(\beta+1)}{\beta},$$

*where* $\Gamma$ *is the Euler Gamma function.*

As a consequence of the previous Lemma, the following results hold.

**Corollary 1.** *Let* $0 < \delta < \frac{1}{e}$, $\beta > 0$. *Then we have*

$$\frac{1}{\Gamma(\beta+1)} \int_0^{\varepsilon_0} \varepsilon^{\beta-1} \delta^\varepsilon \, d\varepsilon \le \frac{|\log\delta|^{-\beta}}{\beta} \le \left( \int_0^{\varepsilon_0} \varepsilon^{\beta-1} \delta^\varepsilon \, d\varepsilon \right) \frac{e^{\varepsilon_0}}{\varepsilon_0^\beta}.$$

**Corollary 2.** *Let $a \geq e^e$, $M, \beta > 0$, then there exist constants $C_i = C_i(\beta, \varepsilon_0)$, $i = 3, 4$ such that*

$$\frac{C_3}{\beta\Gamma(\beta+1)}\left(\log\log(a+M)\right)^{-\beta} \leq \int_0^{\varepsilon_0} |\log\sigma|^{-(\beta+1)}\sigma^{-1}(a+M)^{-\sigma}d\sigma$$
$$\leq C_4\left(\log\log(a+M)\right)^{-\beta}\frac{(\beta+1)}{\beta}\Gamma(\beta+1)\,.$$

We shall need the following Lemma.

**Lemma 4.** *Let $a \geq e^{e^e}$, $\beta > 0$. Then there exist constants $C_i = C_i(\beta, \varepsilon_0)$, $i = 5, 6$ such that*

$$\begin{aligned}C_5\left(\log\log\log(a+M)\right)^{-\beta} &\leq \int_0^{\varepsilon_0}\left(1+\log|\log\sigma|\right)^{-(\beta+1)}(\sigma|\log\sigma|)^{-1}(a+M)^{-\sigma}d\sigma\\ &\leq C_6\left(\log\log\log(a+M)\right)^{-\beta}\end{aligned} \tag{3.3}$$

*for every measurable function $f : \Omega \to \mathbb{R}$.*

*Proof.* We start by proving the following

$$\int_0^{\varepsilon_0}\left(1+\log|\log\sigma|\right)^{-(\beta+1)}(\sigma|\log\sigma|)^{-1}(a+M)^{-\sigma}d\sigma \leq C_6\left(\log\log\log(a+M)\right)^{-\beta}\,. \tag{3.4}$$

Since $\sigma \leq \frac{1}{e}$, we can apply Lemma 3 with the choice $b = |\log\sigma^e|$ and so, applying also Corollary 2, we obtain

$$\int_0^{\varepsilon_0}\left(1+\log|\log\sigma|\right)^{-(\beta+1)}\sigma^{-1}|\log\sigma|^{-1}(a+M)^{-\sigma}d\sigma \leq$$
$$\leq \frac{(\beta+1)e^{\varepsilon_0}}{\varepsilon_0^{(\beta+1)}}\int_0^{\varepsilon_0}\varepsilon^{\beta}\left(\int_0^{\varepsilon_0}e^{-\varepsilon}|\log\sigma|^{-(\varepsilon+1)}\sigma^{-1}(a+M)^{-\sigma}d\sigma\right)d\varepsilon \leq$$
$$\leq C_4\frac{(\beta+1)e^{\varepsilon_0}}{\varepsilon_0^{(\beta+1)}}\int_0^{\varepsilon_0}\varepsilon^{\beta}\frac{(\varepsilon+1)\Gamma(\varepsilon+1)}{\varepsilon}\left(\log\log(a+M)\right)^{-\varepsilon}d\varepsilon$$

Since $\varepsilon+1 < \varepsilon_0+1$ and $\lim_{\varepsilon\to 0}\Gamma(\varepsilon+1) = 1$, we have

$$\int_0^{\varepsilon_0}\left(1+\log|\log\sigma|\right)^{-(\beta+1)}\sigma^{-1}|\log\sigma|^{-1}(a+M)^{-\sigma}d\sigma \leq$$
$$\leq C(\varepsilon_0)\frac{(\beta+1)e^{\varepsilon_0}}{\varepsilon_0^{(\beta+1)}}\int_0^{\varepsilon_0}\varepsilon^{\beta-1}\left(\log\log(a+M)\right)^{-\varepsilon}d\varepsilon$$

Applying the first inequality of Corollary 1 with the choice $\delta = \left(\log\log(a+M)\right)^{-1}$, we obtain (3.4).

To prove the first inequality in (3.3), we apply Lemma 3 with the choice $b = \log\log(a+M)$ and the first inequality of Corollary 2, obtaining

$$\left(\log\log\log(a+M)\right)^{-\beta} \leq \frac{\beta e^{\varepsilon_0}}{\varepsilon_0^{\beta}}\int_0^{\varepsilon_0}\varepsilon^{\beta-1}\left(\log\log(a+M)\right)^{-\varepsilon}d\varepsilon \leq$$
$$\leq \frac{\beta e^{\varepsilon_0}}{\varepsilon_0^{\beta}}\frac{1}{C_3}\int_0^{\varepsilon_0}\varepsilon^{\beta}\Gamma(\varepsilon+1)\left(\int_0^{\varepsilon_0}|\log\sigma|^{-(\varepsilon+1)}\sigma^{-1}(a+M)^{-\sigma}d\sigma\right)d\varepsilon \leq$$
$$\leq C(\varepsilon_0)\frac{\beta e^{\varepsilon_0}}{\varepsilon_0^{\beta}}\int_0^{\varepsilon_0}|\log\sigma|^{-1}\sigma^{-1}(a+M)^{-\sigma}\left(\int_0^{\varepsilon_0}\varepsilon^{\beta}|\log\sigma|^{-\varepsilon}d\varepsilon\right)d\sigma$$

Applying again Lemma 3 with the choice $b = |\log \sigma^e|$, we find

$$(\log\log\log(a+M))^{-\beta} \leq$$
$$\leq C(\varepsilon_0)\frac{\beta e^{2\varepsilon_0}\Gamma(\beta+2)}{\varepsilon_0^{\beta}(\beta+1)}\int_0^{\varepsilon_0}(1+\log|\log\sigma|)^{-(\beta+1)}|\log\sigma|^{-1}\sigma^{-1}(a+M)^{-\sigma}d\sigma$$

□

Now we are in position to prove Theorem 3.

*Proof of Theorem 3.* It is easy to check that $|||f|||_{L^q(\log\log\log L)^{-\beta}(\Omega)}$, defined by (3.1), is a norm on $L^q(\log\log\log L)^{-\beta}(\Omega)$.
Moreover, for any measurable function $f$ and for a.e. $x \in \Omega$, if $a \geq e^{e^e}$ we have

$$|f|^q(a+|f|)^{-\varepsilon} \leq |f|^{q-\varepsilon} \leq 2^{q-1}\left[a^q + |f|^q(a+|f|)^{-\varepsilon}\right].$$

Integrating over $\Omega$ we get

$$⨍_\Omega |f|^q(a+|f|)^{-\varepsilon}dx \leq \|f\|_{q-\varepsilon}^{q-\varepsilon} \leq 2^{q-1}a^q + 2^{q-1}⨍_\Omega |f|^q(a+|f|)^{-\varepsilon}dx\,.$$

This implies

$$\int_0^{\varepsilon_0}(1+\log|\log\varepsilon|)^{-(\beta+1)}(\varepsilon|\log\varepsilon|)^{-1}\left[⨍_\Omega |f|^q(a+|f|)^{-\varepsilon}dx\right]d\varepsilon \leq$$
$$\leq \int_0^{\varepsilon_0}(1+\log|\log\varepsilon|)^{-(\beta+1)}(\varepsilon|\log\varepsilon|)^{-1}\|f\|_{q-\varepsilon}^{q-\varepsilon}d\varepsilon$$
$$\leq 2^{q-1}a^q\int_0^{\varepsilon_0}(1+\log|\log\varepsilon|)^{-(\beta+1)}(\varepsilon|\log\varepsilon|)^{-1}d\varepsilon +$$
$$+ 2^{q-1}\int_0^{\varepsilon_0}(1+\log|\log\varepsilon|)^{-(\beta+1)}(\varepsilon|\log\varepsilon|)^{-1}\left[⨍_\Omega |f|^q(a+|f|)^{-\varepsilon}dx\right]d\varepsilon$$

Applying Lemma 4 with $M = |f|$ we have

$$C_5 ⨍_\Omega |f|^q(\log\log\log(a+|f|))^{-\beta}dx \leq$$
$$\leq ⨍_\Omega |f|^q\left[\int_0^{\varepsilon_0}(1+\log|\log\varepsilon|)^{-(\beta+1)}(\varepsilon|\log\varepsilon|)^{-1}(a+|f|)^{-\varepsilon}d\varepsilon\right] =$$
$$= \int_0^{\varepsilon_0}(1+\log|\log\varepsilon|)^{-(\beta+1)}(\varepsilon|\log\varepsilon|)^{-1}\left[⨍_\Omega |f|^q(a+|f|)^{-\varepsilon}dx\right]d\varepsilon \leq$$
$$\leq \int_0^{\varepsilon_0}(1+\log|\log\varepsilon|)^{-(\beta+1)}(\varepsilon|\log\varepsilon|)^{-1}\|f\|_{q-\varepsilon}^{q-\varepsilon}d\varepsilon$$
$$\leq 2^{q-1}a^q\int_0^{\varepsilon_0}(1+\log|\log\varepsilon|)^{-(\beta+1)}(\varepsilon|\log\varepsilon|)^{-1}d\varepsilon +$$
$$+ 2^{q-1}\int_0^{\varepsilon_0}(1+\log|\log\varepsilon|)^{-(\beta+1)}(\varepsilon|\log\varepsilon|)^{-1}\left[⨍_\Omega |f|^q(a+|f|)^{-\varepsilon}dx\right]d\varepsilon$$
$$\leq 2^{q-1}a^q\frac{(1+\log|\log\varepsilon_0|)^{-\beta}}{\beta} + 2^{q-1}C_6 ⨍_\Omega |f|^q(\log\log\log(a+|f|))^{-\beta}dx$$

Then we get

$$
\begin{aligned}
&C_5 \fint_\Omega |f|^q \left(\log\log\log(a+|f|)\right)^{-\beta} dx \le \\
&\le \int_0^{\varepsilon_0} (1+\log|\log\varepsilon|)^{-(\beta+1)} (\varepsilon|\log\varepsilon|)^{-1} \|f\|_{q-\varepsilon}^{q-\varepsilon} d\varepsilon \le \qquad (3.5)\\
&\le C_7 + C_8 \fint_\Omega |f|^q \left(\log\log\log(a+|f|)\right)^{-\beta} dx
\end{aligned}
$$

Let $f : \Omega \to \mathbb{R}$ be a measurable function, such that $|||f|||_{L^q(\log\log\log L)^{-\beta}(\Omega)}$ is finite. Since

$$
\|f\|_{q-\varepsilon}^{q-\varepsilon} \le \|f\|_{q-\varepsilon}^{q} + 1\,,
$$

by the first inequality in (3.5) we get that $f \in L^q\left(\log\log\log L\right)^{-\beta}(\Omega)$ and, moreover, if $|||f|||_{L^q(\log\log\log L)^{-\beta}(\Omega)} = 1$, it results

$$
\fint_\Omega |f|^q \left(\log\log\log(a+|f|)\right)^{-\beta} dx \le C_9\,,
$$

where $C_9$ is a constant independent on $f$. By homogeneity, for any measurable $f$, we get

$$
\|f\|_{L^q(\log\log\log L)^{-\beta}(\Omega)} \le C_9 |||f|||_{L^q(\log\log\log L)^{-\beta}(\Omega)}\,.
$$

On the other hand, if $f \in L^q\left(\log\log\log L\right)^{-\beta}(\Omega)$, i.e. if $\|f\|_{L^q(\log\log\log L)^{-\beta}(\Omega)}$ is finite, there exists a constant $C_{10}$ such that

$$
\|f\|_{L^{q-\varepsilon}(\Omega)} \le C_{10}\varepsilon^{-\frac{\alpha}{q}} \|f\|_{L^q(\log\log\log L)^{-\beta}(\Omega)} \qquad (3.6)
$$

as the following embeddings hold:

$$
L^q\left(\log\log\log L\right)^{-\beta}(\Omega) \subset L^q\left(\log\log L)\right)^{-\delta}(\Omega) \subset L^q\left(\log L\right)^{-\gamma}(\Omega) \subset L^{\alpha,q)}(\Omega)
$$

for any $\delta, \gamma, \alpha, \beta > 0$. By (3.6) we get

$$
\|f\|_{q-\varepsilon}^{q} \le C_{11} \|f\|_{q-\varepsilon}^{q-\varepsilon} \|f\|^{\varepsilon}_{L^q(\log\log\log L)^{-\beta}(\Omega)}\,, \qquad (3.7)
$$

hence, by (3.5) we obtain that $|||f|||_{L^q(\log\log\log L)^{-\beta}(\Omega)} < +\infty$ and if $\|f\|_{L^q(\log\log\log L)^{-\beta}(\Omega)} = 1$, integrating (3.7) and using (3.5), we deduce that

$$
|||f|||_{L^q(\log\log\log L)^{-\beta}(\Omega)} < C_{12}\,,
$$

where the constant $C_{12}$ is independent on $f$. By homogeneity we conclude the proof, obtaining

$$
|||f|||_{L^q(\log\log\log L)^{-\beta}(\Omega)} < C_{12} \|f\|_{L^q(\log\log\log L)^{-\beta}(\Omega)}\,.
$$

□

## 4. Proof of Main Theorem

In this Section we will prove Main Theorem stated in the Introduction. As already hinted, we will use a regularity result for elliptic equations with right hand side in divergence form that we will apply to a linear problem. Actually we will give a stability estimate for equations of Leray–Lions type whose interest is independent from our context. Exactly, using the equivalence given by Theorem 3 and a well known result contained in [17], we deduce the following.

**Theorem 4.** *Let* $A = A(x,\xi)$ *be a Leray–Lions mapping that satisfies* (1.3). *Then, if* $\beta > 0$, *for* $i = 1,2$ *and for any* $\underline{\chi}_i \in L^2(\log\log\log L)^{-\beta}(\Omega;\mathbb{R}^2)$, *there exists a unique solution* $\varphi_i$ *to the Dirichlet problem*

$$\begin{cases} \operatorname{div} A(x,\nabla\varphi_i) = \operatorname{div}\underline{\chi}_i & \text{in } \Omega \\ \varphi_i \in W_0^{1,1}(\Omega)\,. \end{cases} \tag{4.1}$$

*Moreover it results*

$$\|\nabla\varphi_1 - \nabla\varphi_2\|_{L^2(\log\log\log L)^{-\beta}(\Omega)} \leq C\|\underline{\chi}_1 - \underline{\chi}_2\|_{L^2(\log\log\log L)^{-\beta}(\Omega)}$$

*where* $C = C(\beta,K) > 0$ *is a positive constant that depends on the parameters* $K$ *and* $\beta$.

*Proof.* By Theorem 3.1 in [17] we know that there exists a positive constant $\sigma_0 = \sigma(K)$ such that, if $|\sigma| \leq \sigma_0$, for $i = 1,2$ and for any $\underline{\chi}_i \in L^{2-\sigma}(\Omega;\mathbb{R}^2)$, problem (4.1) admits a unique solution $\varphi_i \in W^{1,2-\sigma}$ and it results

$$\|\nabla\varphi_1 - \nabla\varphi_2\|_{L^{2-\sigma}(\Omega)} \leq C\|\underline{\chi}_1 - \underline{\chi}_2\|_{L^{2-\sigma}(\Omega)}\,, \tag{4.2}$$

where $C = C(K) > 0$ is a positive constant that depends only on the parameter $K$, that is (4.2) is uniform in $\sigma$.
If $\beta > 0$ is fixed, using (4.2) and Theorem 3, we find

$$\begin{aligned}
&\|\nabla\varphi_1 - \nabla\varphi_2\|^2_{L^2(\log\log\log L)^{-\beta}(\Omega)} \leq \\
\leq& C_1(\beta)|||\nabla\varphi_1 - \nabla\varphi_2|||^2_{L^2(\log\log\log L)^{-\beta}(\Omega)} = \\
=& C_1(\beta)\int_0^{\varepsilon_0}(1+\log|\log\varepsilon|)^{-(\beta+1)}(\varepsilon|\log\varepsilon|)^{-1}\|\nabla\varphi_1 - \nabla\varphi_2\|^2_{L^{2-\epsilon}(\Omega)}d\varepsilon \leq \\
\leq& C_2(\beta,K)\int_0^{\varepsilon_0}(1+\log|\log\varepsilon|)^{-(\beta+1)}(\varepsilon|\log\varepsilon|)^{-1}\|\underline{\chi}_1 - \underline{\chi}_2\|^2_{L^{2-\varepsilon}(\Omega)}d\varepsilon = \\
=& C_2(\beta,K)|||\underline{\chi}_1 - \underline{\chi}_2|||^2_{L^2(\log\log\log L)^{-\beta}(\Omega)} \leq \\
\leq& C_3(\beta,K)\|\underline{\chi}_1 - \underline{\chi}_2\|^2_{L^2(\log\log\log L)^{-\beta}(\Omega)}\,.
\end{aligned}$$

□

Now we are in position to prove the main Theorem.

*Proof.* Since $L^{\tilde{\Phi}}(\Omega) = L(\log L)^{\frac{1}{2}}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)$ is a subspace of $L(\log L)^{\frac{1}{2}}(\Omega)$ if $\beta \geq 0$, we can ensure (as already observed) that (1.1) has a unique finite energy solution $v \in W_0^{1,2}(\Omega)$. From now on we will treat only the case $\beta > 0$, since the case $\beta = 0$ was already studied in [3].
In order to prove the Main Theorem, we want to apply the regularity result given by Theorem 4. To do this, as already showed by the papers [3], [13] and [24], we need to linearize problem (1.1).
We will use a linearization procedure introduced in [19] that preserves the ellipticity bounds. For shortness, we do not give all the details of the linearization procedure and we refer, for example, to [13, Proof of Theorem 1.1]. We know that there exists a symmetric, positive definite and measurable matrix valued function $B = B(x)$ such that, the unique finite energy

solution $v \in W_0^{1,2}(\Omega)$ of (1.1) with $f \in L^{\Psi}(\Omega)$ solves also the following linear problem

$$\begin{cases} -\operatorname{div} B(x)\nabla v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{4.3}$$

that is

$$\int_\Omega B(x)\nabla v \nabla\varphi = \int_\Omega f\varphi\,, \quad \forall \varphi \in W_0^{1,2}(\Omega)\,.$$

Now, if $\beta > 0$, we fix $\underline{\chi} \in C^1(\overline{\Omega};\mathbb{R}^2)$ such that $\|\underline{\chi}\|_{L^2(\log\log\log L)^{-\beta}(\Omega;\mathbb{R}^2)} \le 1$ and we consider the unique finite energy solution $\varphi$ to the linear Dirichlet problem

$$\begin{cases} -\operatorname{div} B(x)\nabla \varphi = \operatorname{div}\underline{\chi} & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega\,. \end{cases}$$

where $B(x)$ is the matrix given by the linearization procedure, by Theorem 4 we have:

$$\|\nabla\varphi\|_{L^2(\log\log\log L)^{-\beta}(\Omega)} \le C(\beta,K)\|\underline{\chi}\|_{L^2(\log\log\log L)^{-\beta}(\Omega)} \le C(\beta,K)\,,$$

and so, using Lemma 2, we obtain

$$\|\varphi\|_{L^{\Phi}(\Omega)} \le C_1(\beta,K)\,, \tag{4.4}$$

where $\Phi(s) \cong e^{s^2(\log\log s)^{-\beta}}$ and $C_1(\beta,K)$ is another constant depending only on $\beta$ and $K$. Thanks to the fact that $v$ satisfies the linear problem (4.3) and that $B(x)$ is a symmetric matrix, using Lemma 1 and the Hölder inequality between the complementary spaces $L^{\Phi}(\Omega)$ and $L^{\tilde{\Phi}}(\Omega)$, by (4.4) we obtain that, for any $\underline{\chi} \in C^1(\overline{\Omega};\mathbb{R}^2)$ such that $\|\underline{\chi}\|_{L^2(\log\log\log L)^{-\beta}(\Omega)} \le 1$, it results

$$\begin{aligned} &\left|\int_\Omega \nabla v\cdot\underline{\chi}\right| = \left|\int_\Omega v\operatorname{div}\underline{\chi}\right| = \left|\int_\Omega v\operatorname{div}(B(x)\nabla\varphi)\right| = \left|\int_\Omega B(x)\nabla v\cdot\nabla\varphi\right| = \\ &= \left|\int_\Omega f\varphi\right| \le C_2(\beta)\|\varphi\|_{L^{\Phi}(\Omega)}\|f\|_{L(\log L)^{\frac12}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)} \le \\ &\le C_2(\beta,K)\|f\|_{L(\log L)^{\frac12}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)}\,, \end{aligned} \tag{4.5}$$

where $C_2(\beta,K)$ is a constant that depends only on $\beta$ and $K$.
Since $C^1(\overline{\Omega};\mathbb{R}^2)$ is dense in $L^2(\log\log\log L)^{-\beta}(\Omega)$, taking the supremum in (4.5) under the conditions $\underline{\chi} \in C^1(\overline{\Omega};\mathbb{R}^2)$, $\|\underline{\chi}\|_{L^2(\log\log\log L)^{-\beta}(\Omega;\mathbb{R}^2)} \le 1$ and recalling that $L^2(\log\log\log L)^{\beta}(\Omega)$ is the associate space of $L^2(\log\log\log L)^{-\beta}(\Omega)$, we obtain

$$\|\nabla v\|_{L^2(\log\log\log L)^{\beta}(\Omega)} \le c(\beta,K)\|f\|_{L(\log L)^{\frac12}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)}$$

as desired. □

## References


[1] R. A. Adams: *Sobolev Spaces*, Academic Press, New York, (1975).

[2] A. Alberico, V. Ferone, *Regularity properties of solutions of elliptic equations in* $\mathbb{R}^2$ *in limit cases*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 6 (1995), no.4, 237–250.

[3] A. Alberico, T. Alberico, C. Sbordone: *Planar quasilinear elliptic equations with right-hand side in* $L(logL)^{\delta}$, Discrete Contin. Dyn. Syst. 31 (2011), no. 4, 1053–1067.

[4] C. Bennett, K. Rudnick: *On Lorentz-Zygmund spaces*, Dissert. Math. 175 (1980), 1–67.

[5] C. Bennett, R. Sharpley: *Interpolations of operators*, Pure and Applied Mathematics, 129, Academic Press, 1988.

[6] L. Boccardo, T. Gallouët, L. Orsina: *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*, Ann. Inst. H. Poincarè Anal. Non Linèaire 13(5), (1996) no. 5, 539–551

[7] H. Brezis, F. Merle: *Uniform estimates and blow-up behavior for solutions of* $\Delta u = V(x)e^u$ *in two dimensions*, Comm. Partial Differential Equations 16 (1991), no. 8-9, 1223–1253.

[8] A. Cianchi: *A Sharp Embedding Theorem for Orlicz-Sobolev Spaces*, Indiana Univ. Math. J., 45 (1996), 39–65.

[9] A. Cianchi: *Continuity Properties of Functions from Orlicz-Sobolev Spaces and Embedding Theorems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 23 (1996), 575–608.

[10] A. Cianchi: *Boundedness of solutions to variational problems under general growth conditions*, Comm. Partial Differential Equations 22 (1997), no. 9-10, 1629–1646.

[11] A. Cianchi: *A fully anisotropic Sobolev inequality*, Pacific J. Math. 196(2000), no. 2, 283–295.

[12] L.M. De Cave, L. D' Onofrio, R. Schiattarella: *Orlicz regularity of the gradient of solutions to quasilinear elliptic equations in the plane*, submitted 2014.

[13] L.M. De Cave, C. Sbordone: *Gradient regularity for solutions to quasilinear elliptic equations in the plane*, J. Math. Anal. Appl. 417 (2014), pp. 537–551.

[14] D.E. Edmunds, P. Gurka, B. Opic: *On Embeddings of Logarithmic Bessel Potential Spaces*, J. Funct. Anal., 146 (1997), 116–150.

[15] D.E. Edmunds, H. Triebel: *Function spaces, entropy numbers, differential operators*, Cambridge University Press, Cambridge, 1996.

[16] F. Farroni, L. Greco, G. Moscariello: *Stability for p-Laplace type equation in a borderline case*, Nonlinear Anal. 116 (2015), 100–111.

[17] A. Fiorenza, C. Sbordone: *Existence and uniqueness results for solutions of nonlinear equations with right hand side in* $L^1$, Studia Math. 127 (1998), no. 3, 223–231.

[18] T. Iwaniec, C. Sbordone: *On the integrability of the Jacobian under minimal hypotheses*, Arch. Rat. Mech. Anal., 119 (1992), 129–143.

[19] T. Iwaniec and C. Sbordone, *Quasiharmonic fields*, Ann. Inst. H. Poincaré Anal. Non Linéaire 18 (2001), no. 5, 519–572.

[20] G. Mingione, *The Calderon-Zygmund theory for elliptic problems with measure data*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 6 (2007), 195–261.

[21] G. Mingione, *Gradient estimates below the duality exponent*. Math. Ann. 346(3), (2010) 571–627.

[22] M. Rao, Z. D. Ren: *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 146 (1991).

[23] N. Trudinger: *On imbeddings into Orlicz spaces and some applications.*, J. Math. Mech. 17 (1967), 473–483.

[24] G. Zecca: *Regularity results for planar quasilinear equations*, Rend. Mat. Appl. (7) 30 (2010), 329–343.



Linda Maria De Cave, Dipartimento S.B.A.I., Sapienza Università di Roma, Via A. Scarpa 16, 00161 Roma, Italy

*E-mail address*: linda.decave@sbai.uniroma1.it, lindamdecave@gmail.com

Luigi D'Onofrio, Università degli Studi di Napoli "Parthenope", Via Parisi 13, 80100 Napoli, Italy

*E-mail address*: donofrio@uniparthenope.it

Roberta Schiattarella, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Via Cintia, 80126 Napoli, Italy

*E-mail address*: roberta.schiattarella@unina.it