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A VARIATIONAL APPROACH TO THE EIGENFUNCTIONS OF THE ONE PARTICLE RELATIVISTIC HAMILTONIAN

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Abstract. In this note we give a variational characterization of the eigenvalues and eigenvectors for the operator

$$H = H_0 + V = \sqrt{-c^2\Delta + m^2c^4} + V,$$

where H_0 is the relativistic (free) Hamiltonian operator and V is a real valued potential. Our results hold when $V(x) = -\frac{1}{|x|}$ and H describe a relativistic atom.

The characterization we give for the eigenvectors is useful in proving regularity and exponential decay of the solutions — properties which have been object of investigation by B. Simon with different techniques.

1. Introduction and main results

In this note we give a variational characterization of the eigenvalues and eigenvectors (see Theorem 1) for the operator

$$H = H_0 + V = \sqrt{-c^2\Delta + m^2c^4} + V,$$

where H_0 is the relativistic (free) Hamiltonian operator – which has been used to study models where relativistic effects became relevant – and V is a real valued potential. Our results hold when $V(x) = -\frac{1}{|x|}$ and H describe a relativistic atom.

The characterization we give for the eigenvectors is useful in proving properties — such as regularity (see Theorem 3) and exponential decay of the solutions (see Theorem 2) — which have been object of investigation by B. Simon with different techniques in [16].

In order to describe our results, let us recall that to the operator H_0 can be defined for all $f \in H^1(\mathbb{R}^3)$ as the inverse Fourier transform of the L^2 function $\sqrt{c^2|p|^2 + m^2c^4}\,\hat{f}(p)$ (where $\hat{f}$ denotes the Fourier transform of f). To H_0 we can associate the following quadratic form

$$\mathcal{Q}(f, g) = \int_{\mathbb{R}^3} \sqrt{c^2|p|^2 + m^2c^4}\ \hat{f}(p)\hat{g}(p)\,dp$$

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which can be extended to all functions $f, g \in H^{1/2}(\mathbb{R}^3)$ where

$$H^{1/2}(\mathbb{R}^3) = \{ f \in L^2(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} (1+|p|)|\hat{f}(p)|^2 \, dp < +\infty \}.$$

see for example [13] for more details.

On the potential V we assume

(h1) $V \in L^3_w(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, $V \in L^\infty(\mathbb{R}^3 \setminus B_{R_0})$ for some $R_0 > 0$ and

$$(i) \lim_{R\to+\infty} \|V\|_{L^\infty(|x|>R)} = 0;$$

$$(ii) \lim_{R\to+\infty} \operatorname{sup\,ess}_{|x|>R} V(x)|x|^2 = -\infty.$$

(h2) V is H_0 - form bounded with bound less than 1, i.e. there exists $a \in (0,1)$ such that

$$|(\phi, V\phi)_{L^2}| \le a(\phi, H_0\phi)_{L^2}$$

for all $\phi \in H^{1/2}(\mathbb{R}^3; \mathbb{C})$;

Remark 1. The above assumptions are similar to those used in the study of the characterization and computation of the eigenvalues for the Dirac-Coulomb Hamiltonian, to which our problem is related, see [8, 9] and references therein.

Remark 2. We recall that $L^q_w(\mathbb{R}^N)$, the weak L^q space, is the space of all measurable functions f such that

$$\sup_{\alpha>0} \alpha |\{ x \mid |f(x)| > \alpha \}|^{1/q} < +\infty,$$

where $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^N$. Note that $f(x) = |x|^{-1}$ does not belong to any L^q-space but it belongs to $L^3_w(\mathbb{R}^3)$. (see e.g. [13] for more details).

Remark 3. The validity of **(h2)** when V is the Coulomb potential of a nucleus with Z protons

$$V(x) = -\frac{Ze^2}{|x|} \quad \textit{(in cgs units)} \tag{1.1}$$

follows from important inequalities. Let us recall them here.

Hardy: for all $\psi \in H^1(\mathbb{R}^3)$

$$\| |x|^{-1}\psi \|_{L^2} \le 2\|\nabla\psi\|_{L^2} \le \frac{2}{c\hbar} \|\sqrt{-c^2\hbar^2\Delta + m^2c^4}\psi\|_{L^2}$$

Kato, Herbst [10]**:** for all $\psi \in H^{1/2}(\mathbb{R}^3)$

$$\left(\psi, |x|^{-1}\psi\right)_{L^2} \le \frac{\pi}{2} \left(\psi, \sqrt{-\Delta}\psi\right)_{L^2} \le \frac{\pi}{2c\hbar} \left(\psi, \sqrt{-c^2\hbar^2\Delta + m^2c^4}\psi\right)_{L^2}$$

Note that **(h2)** is satisfied for the electrostatic potential provided $0 < Z < 68$ by Hardy and provided $0 < Z < 87$ by Kato.

Let us recall that the operator $\sqrt{-c^2\Delta + m^2c^4}$, exactly as for the fractional Laplacian, can be related, following [3], to a Dirichlet to Neumann operator (see also [2] and [5, 6, 7] for more closely related models).

To show this, we take a given function $u \in \mathcal{S}(\mathbb{R}^3)$ with Fourier transform $\hat{u}$ and let

$$v(x,y) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \mathrm{e}^{ip\cdot y} \hat{u}(p) \mathrm{e}^{-\sqrt{c^2|p|^2+m^2c^4}x} \, dp.$$

be the solution of the Dirichlet boundary problem

$$\begin{cases} -\partial_x^2 v - c^2\Delta_y v + m^2c^4 v = 0 & \text{in } \mathbb{R}^4_+ = \{ (x,y) \in \mathbb{R}\times\mathbb{R}^3 \mid x>0 \} \\ v(0,y) = u(y) & \text{for } y \in \mathbb{R}^3 = \partial\mathbb{R}^4_+. \end{cases}$$

Setting

$$\mathcal{T}u(y) = \frac{\partial v}{\partial \nu}(0,y) = -\frac{\partial v}{\partial x}(0,y);$$

we have that

$$\mathcal{T}u(y) = -\frac{\partial v}{\partial x}(0,y) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \mathrm{e}^{ip\cdot y} \sqrt{c^2|p|^2 + m^2c^4}\, \hat{u}(p)\, dp$$

namely $\mathcal{T} = \sqrt{-c^2\Delta_y + m^2c^4} = H_0$ on the dense domain $\mathcal{S}(\mathbb{R}^3)$.

We consider the functional $\mathcal{I}(\phi)$ defined on $H^1(\mathbb{R}^4_+, \mathbb{C})$

$$\mathcal{I}(\phi) = \iint_{\mathbb{R}^4_+} (|\partial_x\phi|^2 + c^2|\nabla_y\phi|^2 + m^2c^4|\phi|^2)\, dx\, dy + \int_{\mathbb{R}^3} (\phi_{tr}, V\phi_{tr})\, dy \tag{1.2}$$

where $\phi_{tr} \in H^{1/2}$ denotes the trace of $\phi \in H^1$ on $\partial\mathbb{R}^4_+ = \mathbb{R}^3$.

We have the following existence and characterization results for the eigenvalues and eigenfunctions, where we always assume $m > 0$.

Theorem 1. *Let $m > 0$ and (h1)-(h2) hold. Then there exist $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \le \ldots$ and $\phi_1, \phi_2, \ldots, \phi_k, \ldots \in H^1(\mathbb{R}^4_+, \mathbb{C})$ such that, for all $k \in \mathbb{N}$*

$$\lambda_k = \mathcal{I}(\phi_k) = \inf_{X_k} \mathcal{I}(\phi)$$

where

$$X_1 = \{ \phi \in H^1 \mid |\phi_{tr}|_{L^2} = 1 \}.$$

and, for $1 < k \in \mathbb{N}$

$$X_k = \{ \phi \in H^1 \mid |\phi_{tr}|_{L^2} = 1,\ (\phi_{tr}, (\phi_i)_{tr})_{L^2} = 0,\ i = 1, \ldots, k-1 \}.$$

Moreover $\{\lambda_k\}_{k\ge1} \in \sigma_{disc}(H_0 + V)$ and

$$0 < \lambda_1 \le \ldots \le \lambda_k \le \lambda_{k+1} \to \inf\{\sigma_{ess}(H_0+V)\} = mc^2 \quad \text{for } k \to +\infty.$$

The functions $\varphi_k = (\phi_k)_{tr} \in H^{1/2}(\mathbb{R}^3, \mathbb{C})$ are the eigenfunctions of the operator $H_0 + V$, and the functions $\phi_k \in H^1(\mathbb{R}^4_+, \mathbb{C})$ are weak solution of the Neumann problem

$$\begin{cases} -\partial_x^2\phi_k - c^2\Delta_y\phi_k + m^2c^4\phi_k = 0 & \text{in } \mathbb{R}^4_+ \\ \dfrac{\partial\phi_k}{\partial\nu} + V\varphi_k = \lambda_k\varphi_k & \text{on } \partial\mathbb{R}^4_+ = \mathbb{R}^3. \end{cases} \tag{$\mathcal{E}_k$}$$

The following Theorems give some properties of the eigenfunctions: regularity and exponential decay.

Theorem 2 (exponential decay)**.** *Let $m > 0$ and (h1)-(h2) hold. Let $\phi_k \in H^1(\mathbb{R}^4_+, \mathbb{C})$ (and $\varphi_k = (\phi_k)_{tr}$) be the functions given by the Theorem 1.*

Then for all $0 \le \beta < \sqrt{m^2c^4 - \lambda_k^2}$ there exists $R > 0$ such that $e^{\frac{\beta}{c}|y|}\varphi_k \in L^2(\mathbb{R}^3 \setminus B_R)$.

Remark 4. Several authors have investigated the asymptotic behaviour of eigenfunctions. Let us recall here the classical book by Agmon [1] and [14, 15, 4].

Theorem 3 (regularity). *Let $\phi_k \in H^1(\mathbb{R}^4_+, \mathbb{C})$ (and $\varphi_k = (\phi_k)_{tr}$) be the functions given by the Theorem 1 and R_0 be given by (**h1**).*

Then we have

(i) $\phi_k \in W^{1,q}([0,r) \times (\mathbb{R}^3 \setminus B_{R_0}))$ *for any* $q \in [2,\infty]$, $r > 0$;
(ii) $\phi_k \in C^{0,\alpha}([0,+\infty) \times (\mathbb{R}^3 \setminus B_{R_0}))$ *for any* $\alpha \in [0,1]$ *and* $\varphi_k \in C^{0,\alpha}(\mathbb{R}^3 \setminus B_{R_0})$;
(iii) *if in addition* $V \in L^3_{loc}(\mathcal{U})$ *for some* $\mathcal{U} \subset \mathbb{R}^3$ *then for every* $\mathcal{V} \subset\subset \mathcal{U}$ *(i.e. such that its closure is compact in* $\mathcal{U}$*)* $\phi_k \in W^{1,p}([0,r) \times \mathcal{V})$ *for any* $p \in [2,\infty)$ *and* $r > 0$ *and* $\varphi_k \in C^{0,\alpha}(\mathcal{V})$ *for any* $\alpha \in [0,1)$.

2. Proof of Theorem 1

We divide the proof of Theorem 1 in several steps.

2.1. **Notation and preliminary results.** With $\|u\|_p$ we will denote the norm of $u \in L^p(\mathbb{R}^4_+)$ and with $|v|_p$ the norm of $v \in L^p(\mathbb{R}^3)$.

We introduce the following (equivalent) norm in $H^1(\mathbb{R}^4_+, \mathbb{C})$

$$\|\phi\|^2_{H^1} = \iint_{\mathbb{R}^4_+} (|\partial_x \phi|^2 + c^2 |\nabla_y \phi|^2 + m^2 c^4 |\phi|^2)\, dx\, dy.$$

and the following norm in the weak L^q-space:

$$|f|_{L^q_w} = \sup\Big\{ |A|^{-1/r} \int_A |f(x)|\, dx \;\Big|\; A \subset \mathbb{R}^3, \text{measurable},\ 0 < |A| < +\infty \Big\}$$

where $1/q + 1/r = 1$. For the weak L^q spaces the following generalization of the weak Young inequality holds:

Proposition 1 (see [11, thm. 2.10]). *Let $f \in L^q_w(\mathbb{R}^N)$, $g \in L^{q'}_w(\mathbb{R}^N)$ and $h \in L^p(\mathbb{R}^N)$ with $\frac{1}{q} + \frac{1}{q'} = 1$ and $1 < p < q$. Then*

$$\|f(g * h)\|_p \le C \|f\|_{q,w} \|g\|_{q',w} \|h\|_p. \tag{2.1}$$

From this we deduce the following result:

Lemma 1. *Let $V \in L^3_w(\mathbb{R}^3)$ and $f \in H^{1/2}(\mathbb{R}^3)$.*

Then

$$||V|^{1/2} f|_{L^2} \le C |V|^{1/2}_{L^3_w} |f|_{H^{1/2}}. \tag{2.2}$$

Proof. Follows from [12, (42)] that the Green function G^μ_α of $(-\Delta + \mu^2)^{\alpha/2}$ belongs to $L^{3/(3-\alpha)}_w(\mathbb{R}^3)$ if $\mu \ge 0$ and $0 < \alpha < 3$.

Then, given $f \in H^{1/2}(\mathbb{R}^3)$, let $h = (-\Delta + \mu^2)^{1/4} f \in L^2(\mathbb{R}^3)$, $f = G^\mu_{1/2} * h$. From the weak Young's inequality above (2.1), we deduce

$$\begin{aligned} ||V|^{1/2} f|_{L^2} &= ||V|^{1/2} (G^\mu_{1/2} * h)|_{L^2} \le C ||V|^{1/2}|_{L^6_w} |G^\mu_{1/2}|_{L^{6/5}_w} |h|_{L^2} \\ &\le C |V|_{L^3_w} |(-\Delta + \mu^2)^{1/4} f|_{L^2} \le C |V|^{1/2}_{L^3_w} |f|_{H^{1/2}}. \end{aligned}$$

□

We also recall that for all $v \in C_0^\infty(\mathbb{R}^4)$

$$\int_{\mathbb{R}^3} |v(0,y)|^2 dy = \int_{\mathbb{R}^3} dy \int_{+\infty}^0 \partial_x |v|^2 dx \leq 2\|v\|_{L^2(\mathbb{R}^4_+)} \|\partial_x v\|_{L^2(\mathbb{R}^4_+)}$$

and by density we get for all $\phi \in H^1$ and any $\alpha > 0$

$$\alpha \int_{\mathbb{R}^3} |\phi_{\mathrm{tr}}|^2 \, dy \leq \iint_{\mathbb{R}^4_+} (|\partial_x \phi|^2 + \alpha^2 |\phi|^2) \, dx \, dy. \tag{2.3}$$

This implies in particular that the quadratic form (kinetic energy)

$$\mathcal{T}(\phi) = \iint_{\mathbb{R}^4_+} (|\partial_x \phi|^2 + m^2 c^4 |\phi|^2) \, dx dy - mc^2 |\phi_{tr}|^2_{L^2} \geq 0 \tag{2.4}$$

is positive definite.

We introduce the differential $d\mathcal{I}(\phi)\colon H^1 \to \mathbb{R}$ of the functional $\mathcal{I}$

$$d\mathcal{I}(\phi)[h] = 2\,\mathrm{Re} \iint_{\mathbb{R}^4_+} \big((\partial_x \phi, \partial_x h) + c^2 \, (\nabla_y \phi, \nabla_y h) + m^2 c^4 \, (\phi, h) \big) \, dx \, dy \\ + 2\,\mathrm{Re}(\phi_{tr}, V h_{tr})_{L^2}$$

The following property can be easily verified.

Lemma 2. *For $w \in H^1(\mathbb{R}^4_+)$, let $u = w_{tr} \in H^{1/2}(\mathbb{R}^3)$ be the trace of w, $\hat{u} = \mathcal{F}(u)$ and*

$$v(x,y) = \mathcal{F}_y^{-1} \big[\hat{u}(p) e^{-\sqrt{c^2|p|^2 + m^2 c^4} x} \big].$$

Then $v \in H^1(\mathbb{R}^4_+)$, $\|v\|_{H^1(\mathbb{R}^4)} = \|u\|_{H^{1/2}(\mathbb{R}^3)}$, and

$$\int_{\mathbb{R}^3} \sqrt{c^2|p|^2 + m^2c^4} \, |\hat{u}|^2 \, dp = \iint_{\mathbb{R}^4_+} (|\partial_x v|^2 + c^2 |\nabla_y v|^2 + m^2 c^4 |v|^2) \, dx \, dy \\ \leq \iint_{\mathbb{R}^4_+} (|\partial_x w|^2 + c^2 |\nabla_y w|^2 + m^2 c^4 |w|^2) \, dx \, dy.$$

In other words

$$\|w_{tr}\|^2_{H^{1/2}} = (w_{tr}, H_0 w_{tr})_{L^2} \leq \|w\|^2_{H^1} \qquad \textit{for every } w \in H^1(\mathbb{R}^4_+) \tag{2.5}$$

2.2. **Existence of the ground state.** We consider the following minimization problem :

$$(\mathcal{P}_1) \qquad \lambda_1 = \inf_{\phi \in S} \mathcal{I}(\phi).$$

where $S = \{ \phi \in H^1 \mid |\phi_{tr}|^2_{L^2} = 1 \}$.

Lemma 3. *The following holds:*

(i) *$\mathcal{I}(\phi)$ is bounded by below and coercive on H^1,*
(ii) *$0 < \lambda_1 < mc^2$.*

Proof. (i) Let $\phi \in H^1$, $\varphi = \phi_{tr}$. From (**h2**) and (2.5), there exists $a \in (0,1)$ such that

$$(\varphi, V\varphi)_{L^2} \geq -a(\varphi, \sqrt{-c^2\Delta + m^2c^4}\, \varphi)_{L^2} \\ \geq -a \iint_{\mathbb{R}^4_+} (|\partial_x \phi|^2 + c^2 |\nabla_y \phi|^2 + m^2 c^4 |\phi|^2) \, dx \, dy$$

Therefore, we may conclude that there exists $\delta > 0$ such that $\mathcal{I}(\phi) \geq \delta \|\phi\|^2_{H^1}$.

(ii) From (i) immediately follows that $\lambda_1 > 0$. Now take $\phi(x,y) = \mathrm{e}^{-mc^2 x}\varphi(y)$, with $\varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C})$, and $|\varphi|_{L^2} = 1$, we have

$$\mathcal{I}(\phi) - mc^2 = \frac{1}{2m}\int_{\mathbb{R}^3} |\nabla\varphi|^2 + \int_{\mathbb{R}^3} V|\varphi|^2\, dy = \mathcal{E}(\varphi)$$

Take, now $\varphi_\eta(y) = \eta^{3/2}\varphi(\eta y)$, we have $|\varphi_\eta|_{L^2} = 1$, for any $\eta > 0$ and setting $\phi_\eta(x,y) = \mathrm{e}^{-mc^2 x}\varphi_\eta(y)$

$$\begin{aligned}\lambda_1 - mc^2 &\leq \inf_{\eta>0} \mathcal{I}(\phi_\eta) - mc^2 = \inf_{\eta>0} \mathcal{E}(\varphi_\eta) = \\ &= \inf_{\eta>0} \eta^2 \frac{1}{2m}\int_{\mathbb{R}^3} |\nabla\varphi|^2\, dy + \int_{\mathbb{R}^3} V(\eta^{-1}y)|\varphi|^2\, dy.\end{aligned}$$

By (**h1**), for any $K > 0$ there exists $R > 0$ such that for any $|y| > R$ we have $V(y) \leq -K/|y|^2$ a.e.. Hence

$$\begin{aligned}(\varphi, V(\eta^{-1}y)\varphi)_{L^2} &= \int_{\{\eta^{-1}|y|\leq R\}} V(\eta^{-1}y)|\varphi|^2 + \int_{\{\eta^{-1}|y|>R\}} V(\eta^{-1}y)|\varphi|^2 \\ &\leq \eta^3 \sup_{|y|\leq \eta R} |\varphi(y)|^2 \int_{\{|y|\leq R\}} |V(y)| - K\eta^2 \int_{\{|y|>\eta R\}} \frac{1}{|y|^2}|\varphi|^2 \\ &\leq C(\eta^3 - K\eta^2)\end{aligned}$$

where the constant $C > 0$ depends on φ and R, and $K > 0$ is arbitrarily large.

We immediately conclude that for any given $\varphi \in C_0^\infty(\mathbb{R}^3; \mathbb{C})$

$$\limsup_{\eta\to 0^+} \frac{1}{\eta^2}(\varphi, V(\eta^{-1}y)\varphi)_{L^2} = -\infty$$

which implies that $\lambda_1 - mc^2 < 0$. □

Letting $\mathcal{G}(\phi) = |\phi_{tr}|^2_{L^2}$ we have that $S = \{\, \phi \in H^1 \mid \mathcal{G}(\phi) = 1 \,\}$ and the tangent space at S at the point $\phi \in S$ is the set

$$T_\phi S = \{\, h \in H^1 \mid d\mathcal{G}(\phi)[h] = 2\,\mathrm{Re}(\phi_{tr}, h_{tr})_{L^2} = 0 \,\}$$

and that $\nabla_S \mathcal{I}(\phi)$, the projection of the gradient on the tangent space $T_\phi S$ to S at the point ϕ is given by

$$\nabla_S \mathcal{I}(\phi) = \nabla\mathcal{I}(\phi) - \mu(\phi)\nabla\mathcal{G}(\phi)$$

where $\nabla\mathcal{I}(\phi) \in H^1$ is such that

$$(\nabla\mathcal{I}(\phi), h)_{H^1} = d\mathcal{I}(\phi)[h] = 2\,\mathrm{Re}(\phi, h)_{H^1} + 2\,\mathrm{Re}(\phi_{tr}, Vh_{tr})_{L^2} \qquad \text{for all } h \in H^1,$$

$\nabla\mathcal{G}(\phi) \in H^1$ is such that

$$(\nabla\mathcal{G}(\phi), h)_{H^1} = d\mathcal{G}(\phi)[h] = 2\,\mathrm{Re}(\phi_{tr}, h_{tr})_{L^2} \qquad \text{for all } h \in H^1,$$

and $\mu(\phi) \in \mathbb{R}$ is such that $\nabla_S\mathcal{I}(\phi) \in T_\phi S$. Then

$$0 = (\nabla\mathcal{G}(\phi), \nabla_S\mathcal{I}(\phi))_{H^1} = (\nabla\mathcal{G}(\phi), \nabla\mathcal{I}(\phi))_{H^1} - \mu(\phi)\|\nabla\mathcal{G}(\phi)\|^2_{H^1}$$

and

$$\mu(\phi) = \frac{(\nabla\mathcal{G}(\phi), \nabla_S\mathcal{I}(\phi))_{H^1}}{\|\nabla\mathcal{G}(\phi)\|^2_{H^1}}$$

From

$$\begin{aligned}(\nabla_S\mathcal{I}(\phi),\phi)_{H^1} &= (\nabla\mathcal{I}(\phi),\phi)_{H^1} - \mu(\phi)(\nabla\mathcal{G}(\phi),\phi)_{H^1}\\ &= 2\mathcal{I}(\phi) - 2\mu(\phi)\mathcal{G}(\phi) = 2\mathcal{I}(\phi) - 2\mu(\phi)\end{aligned}$$

we also deduce that

$$\mu(\phi) = \mathcal{I}(\phi) - \frac{1}{2}(\nabla_S\mathcal{I}(\phi),\phi)_{H^1} \tag{2.6}$$

We now recall the following well known result

Lemma 4. *There exists a Palais-Smale minimizing sequence ϕ_n for $\mathcal{I}$ on the set $S = \{\,\phi \mid |\phi_{tr}|^2_{L^2} = 1\,\}$, that is a sequence such that, denoting $\varphi_n = (\phi_n)_{tr}$,*

$$\mathcal{I}(\phi_n) \to \lambda_1, \qquad \nabla_S\mathcal{I}(\phi_n) \to 0, \qquad |\varphi_n|^2_{L^2} = 1$$

Proof. Assuming that the result does not hold, one deduces that there exist $\epsilon > 0$, $\delta > 0$ such that $\|\nabla_S\mathcal{I}(\phi)\| \geq \delta > 0$ for all $\phi \in S$ such that $\lambda_1 - \epsilon < \mathcal{I}(\phi) < \lambda_1 + \epsilon$. Then one can build a gradient flow $\eta' = \nabla_S\mathcal{I}(\eta)$, which leaves S invariant and pushes $\{\mathcal{I} < \lambda_1 + \epsilon\} \cap S$ into $\{\mathcal{I} < \lambda_1 - \epsilon\} \cap S$, a contradiction.

The Lemma also follows from Ekeland's variational principle. □

Lemma 5. *Let ϕ_n be a Palais Smale sequence at some level $\lambda \geq 0$ for $\mathcal{I}$ on S. Let $\varphi_n = (\phi_n)_{tr}$.*

If $\varphi_n \rightharpoonup 0$ in $H^{1/2}$ then

$$(\varphi_n, V\varphi_n)_{L^2} \to 0.$$

Proof. Since $\mathcal{I}$ is coercive, ϕ_n is bounded H^1, φ_n is bounded in $H^{1/2}$ and, by Sobolev embedding, relatively compact in $L^p_{\rm loc}$ for $p \in [2,3)$. From (2.6) follows that also μ_n is bounded.

By **(h1)** $V \in L^\infty(\mathbb{R}^3 \setminus B_{R_0})$ and for any $\varepsilon > 0$, the set $A_\varepsilon = \{\, y \in \mathbb{R}^3 \setminus B_{R_0} \mid |V(y)| \geq \varepsilon\,\}$ is bounded.

Take a radial function $\chi \in C_0^\infty(\mathbb{R}^3)$, with values in $[0,1]$ such that $\chi(y) = 1$ for $y \in B_1$ and $\chi(y) = 0$ for $y \in \mathbb{R}^3 \setminus B_2$ and let $\chi_R(y) = \chi(R^{-1}y)$.

Taking $R > R_0$ in such a way that $A_\varepsilon \subset B_R$ we have

$$|(\varphi_n, (1-\chi_R^2)V\varphi_n)_{L^2}| \leq \epsilon|\varphi_n|^2_{L^2} \leq \varepsilon.$$

We have, by assumption, $\mathcal{I}(\phi_n) \to \lambda$, $\mathcal{G}(\phi_n) = |\varphi_n|^2_{L^2} = 1$ and

$$\|\nabla_S\mathcal{I}(\phi_n)\| = \|d\mathcal{I}(\phi_n) - \mu_n d\mathcal{G}(\phi_n)\| \to 0.$$

where $\mu_n = \mu(\phi_n)$ and also, by (2.6)

$$\mu_n = \mathcal{I}(\phi_n) - \frac{1}{2}(\nabla_S\mathcal{I}(\phi_n),\phi_n)_{H^1} \to \lambda. \tag{2.7}$$

Since χ depends only on y we have that

$$d\mathcal{I}(\phi_n)[\chi_R^2\phi_n] = d\mathcal{I}(\chi_R\phi_n)[\chi_R\phi_n] - 2c^2\|\varphi_n\nabla_y\chi_R\|^2_{L^2} \tag{2.8}$$

and since $C\|\phi_n\|_{H^1} \geq \|\chi_R^2\phi_n\|_{H^1}$ we have that

$$
\begin{aligned}
o_n(1) = C\|\nabla\mathcal{I}(\phi_n) - \mu_n\nabla\mathcal{G}(\phi_n)\|\|\phi_n\|_{H^1} &\geq |(\nabla\mathcal{I}(\phi_n) - \mu_n\nabla\mathcal{G}(\phi_n), \chi_R^2\phi_n)_{H^1}| \\
&\geq |d\mathcal{I}(\phi_n)[\chi_R^2\phi_n]| - |\mu_n 2\,\mathrm{Re}(\varphi_n, \chi_R^2\varphi_n)_{L^2}| \\
&\geq 2\mathcal{I}(\chi_R\phi_n) - 2c^2\|\phi_n\nabla_y\chi_R\|_{L^2}^2 - 2|\mu_n||\chi_R\varphi_n|_{L^2}^2
\end{aligned}
$$

Now, by Sobolev compact embedding, for any given $R > 0$,

$$|\chi_R\varphi_n|_{L^2} \to 0 \qquad \text{as} \quad n \to +\infty.$$

Moreover,

$$\|\phi_n\nabla\chi_R\|_{L^2}^2 \leq C \sup_{y\in\mathbb{R}^3} |\nabla\chi_R|^2 \leq \frac{C}{R^2}$$

Since by Lemma 3-(i) we have

$$\mathcal{I}(\chi_R\phi_n) \geq \delta\|\chi_R\phi_n\|_{H^1}^2$$

we may conclude (recalling that μ_n is bounded) that

$$\|\chi_R\phi_n\|_{H^1}^2 \leq \epsilon_n + \frac{C}{R}.$$

and hence by **(h2)** and (2.5) we get

$$|(\chi_R\varphi_n, V\chi_R\varphi_n)_{L^2}| \leq a|(\chi_R\varphi_n, H_0\chi_R\varphi_n)_{L^2}| \leq a\|\chi_R\phi_n\|_{H^1}^2 \leq \epsilon_n + \frac{C}{R}$$

for some $\epsilon_n \to 0$ and R arbitrarily large. □

Now we may conclude the existence of a minimizer for $\mathcal{P}_1$. We have the following Proposition:

Proposition 2. *Let ϕ_n be a minimizing Palais Smale sequence at level $\lambda_1 > 0$ for $\mathcal{I}$ with $|(\phi_n)_{tr}|_{L^2} = 1$ (as in Lemma 5).*

Then $\phi_n \rightharpoonup \phi \not\equiv 0$ in H^1 and $\tilde{\phi} = |\phi_{tr}|_{L^2}^{-1}\phi$ is a minimizer for $\mathcal{I}$ on S, that is

$$\mathcal{I}(\tilde{\phi}) = \lambda_1, \qquad |\tilde{\phi}_{tr}|_{L^2} = 1.$$

Moreover $\tilde{\phi}$ (and hence also ϕ) is a weak solution of the Neumann problem $(\mathcal{E}_1)$.

Proof. Since $\mathcal{I}$ is coercive, ϕ_n is bounded (and weakly convergent) in H^1, $\varphi_n = (\phi_n)_{tr}$ is bounded (and weakly convergent) in $H^{1/2}$.

If by contradiction $\varphi_n \rightharpoonup \varphi \equiv 0$, then by Lemma 5 we have

$$(\varphi_n, V\varphi_n)_{L^2} \to 0.$$

Now, by (2.4) we get

$$\mathcal{I}(\phi_n) - mc^2|\varphi_n|_{L^2}^2 \geq (\varphi_n, V\varphi_n)_{L^2} \to 0.$$

On the other hand, by Lemma 3-(ii)

$$\mathcal{I}(\phi_n) - mc^2|\varphi_n|_{L^2}^2 = \mathcal{I}(\phi_n) - mc^2 \to \lambda_1 - mc^2 < 0$$

a contradiction, that is $\varphi_n \rightharpoonup \varphi \not\equiv 0$.

It follows from (2.6) that

$$\mu_n = \mathcal{I}(\phi_n) - \frac{1}{2}(\nabla_S \mathcal{I}(\phi_n), \phi_n)_{H^1} \to \lambda_1$$

and hence, by weak convergence, we have

$$d\mathcal{I}(\phi_n)[h] - \mu_n d\mathcal{G}(\phi_n)[h] \to d\mathcal{I}(\phi)[h] - \lambda_1 d\mathcal{G}(\phi)[h] = 0 \quad \forall h \in H^1$$

hence in particular

$$0 = d\mathcal{I}(\phi)[\phi] - \lambda_1 d\mathcal{G}(\phi)[\phi] = 2\mathcal{I}(\phi) - 2\lambda_1 \mathcal{G}(\phi)$$

and we may conclude that $\tilde{\phi} = \mathcal{G}(\phi)^{-1/2}\phi$ is a minimizer for $\mathcal{I}$ on S, namely

$$\lambda_1 = \frac{\mathcal{I}(\phi)}{\mathcal{G}(\phi)} = \mathcal{I}(\mathcal{G}(\phi)^{-1/2}\phi) = \mathcal{I}(\tilde{\phi})$$
$$\mathcal{G}(\tilde{\phi}) = \mathcal{G}(\mathcal{G}(\phi)^{-1/2}\phi) = 1$$

□

Now, we look for the existence of higher eigenvalues and corresponding eigenfunctions. We proceed by induction.

Let λ_1 be defined by $(\mathcal{P}_1)$ and ϕ_1 be the corresponding minimizer given by Proposition 2.

Assume we have defined, for $j = 1, \dots, k-1$, $\lambda_1 \le \dots \le \lambda_j \le \dots \le \lambda_{k-1} < mc^2$ and $\phi_j \in H^1$, $\varphi_j = (\phi_j)_{tr} \in H^{1/2}$ such that

$$(\varphi_i, \varphi_j)_{L^2} = \delta_{ij}, \qquad i, j = 1, \dots, k-1,$$

and

$$(\mathcal{P}_j) \qquad \lambda_j = \mathcal{I}(\phi_j) = \inf_{\phi \in X_j} \mathcal{I}(\phi) \qquad j = 1, \dots, k-1$$

where,

$$X_j = \left\{ \phi \in H^1 \;\middle|\; \mathcal{G}(\phi) = |\phi_{tr}|^2_{L^2} = 1,\ (\phi_{tr}, \varphi_i)_{L^2} = 0 \quad \text{for} \quad i = 1, \dots, j-1 \right\}.$$

We define

$$(\mathcal{P}_k) \qquad \lambda_k = \inf_{\phi \in X_k} \mathcal{I}(\phi)$$

Remark 5. Setting $\mathcal{G}_j(\phi) = (\varphi_j, \phi_{tr})_{L^2}$, for $j \ge 1$, we have that the linear functionals $\mathcal{G}_j$ are bounded on H^1 and for any $\phi, h \in H^1$

$$d\mathcal{G}_j(\phi)[h] = (\nabla \mathcal{G}_j(\phi), h)_{H^1} = (\varphi_j, h_{tr})_{L^2} = \mathcal{G}_j(h) \qquad j = 1, \dots k-1.$$

Then $X_k = \left\{ \phi \in H^1 \;\middle|\; \mathcal{G}(\phi) = 1,\ \mathcal{G}_j(\phi) = 0,\ j = 1, \dots, k-1 \right\}$,

$$T_\phi X_k = \left\{ h \in H^1 \;\middle|\; (\nabla \mathcal{G}(\phi), h)_{H^1} = 0,\ \mathcal{G}_j(h) = 0,\ j = 1, \dots, k-1 \right\}$$

and the constrained gradient (i.e. the projection of the gradient of $\mathcal{I}$ on the tangent space $T_\phi X_k$) is given by

$$\nabla_{X_k} \mathcal{I}(\phi) = \nabla \mathcal{I}(\phi) - \mu_0(\phi) \nabla \mathcal{G}(\phi) - \sum_{j=1}^{k-1} \mu_j(\phi) \nabla \mathcal{G}_j(\phi).$$

Taking $\phi \in X_k$ we have that

$$\begin{aligned}(\nabla_{X_k}\mathcal{I}(\phi),\phi)_{H^1} &= (\nabla\mathcal{I}(\phi),\phi)_{H^1} - \mu_0(\phi)(\nabla\mathcal{G}(\phi),\phi)_{H^1} - \sum_{j=1}^{k-1}\mu_j(\phi)(\nabla\mathcal{G}_j(\phi),\phi)_{H^1} \\ &= 2\mathcal{I}(\phi) - 2\mu_0(\phi)\mathcal{G}(\phi) - \sum_{j=1}^{k-1}\mu_j(\phi)\mathcal{G}_j(\phi) = 2\mathcal{I}(\phi) - 2\mu_0(\phi)\end{aligned}$$

and we deduce that

$$\mu_0(\phi) = \mathcal{I}(\phi) - \frac{1}{2}(\nabla_{X_k}\mathcal{I}(\phi),\phi)_{H^1}. \tag{2.9}$$

Taking again $\phi \in X_k$ and $i = 1, \dots, k-1$, from

$$\begin{aligned}(\nabla_{X_k}\mathcal{I}(\phi),\phi_i)_{H^1} =&(\nabla\mathcal{I}(\phi),\phi_i)_{H^1} - \mu_0(\phi)(\nabla\mathcal{G}(\phi),\phi_i)_{H^1} - \sum_{j=1}^{k-1}\mu_j(\phi)(\nabla\mathcal{G}_j(\phi),\phi_i)_{H^1} \\ =&d\mathcal{I}(\phi)[\phi_i] - \mu_0(\phi)2\,\mathrm{Re}(\phi_{tr},\varphi_i)_{L^2} - \sum_{j=1}^{k-1}\mu_j(\phi)(\varphi_j,\varphi_i)_{L^2} \\ =&d\mathcal{I}(\phi)[\phi_i] - \mu_i(\phi)\end{aligned}$$

we have that

$$\mu_i(\phi) = d\mathcal{I}(\phi)[\phi_i] - (\nabla_{X_k}\mathcal{I}(\phi),\phi_i)_{H^1} \tag{2.10}$$

We say that $\phi_n \in X_k$ is a (constrained) Palais Smale sequence for $\mathcal{I}$ on X_k at level λ_k if $\phi_n \in X_k$,

$$\mathcal{I}(\phi_n) \to \lambda_k \quad \text{and} \quad \|\nabla_{X_k}\mathcal{I}(\phi_n)\| \to 0.$$

The proof of existence of a minimizer for $(\mathcal{P}_k)$ proceeds as the proof of the existence of the ground state ϕ_1. The key points are the following two Lemmas.

Lemma 6. $\lambda_1 \le \lambda_k < mc^2$.

Proof. Let us consider any k-dimensional linear subspace $G_k \subset C_0^\infty(\mathbb{R}^3;\mathbb{C})$.

For $\varphi \in G_k \cap S$ and $\eta > 0$ we let $\varphi_\eta(y) = \eta^{3/2}\varphi(\eta y) \in S$ and

$$F_k^\eta = \left\{\, \phi_\eta \in H^1 \;\middle|\; \phi_\eta(x,y) = \mathrm{e}^{-mc^2x}\,\varphi_\eta(y), \quad \varphi \in G_k \cap S \,\right\}.$$

Then, for any $\phi_\eta \in F_k^\eta$

$$\begin{aligned}\mathcal{I}(\phi_\eta) - mc^2 &= \frac{1}{2m}\int_{\mathbb{R}^3}|\nabla\varphi_\eta|^2 + \int_{\mathbb{R}^3}(\varphi_\eta, V\varphi_\eta) \\ &= \frac{\eta^2}{2m}\int_{\mathbb{R}^3}|\nabla\varphi|^2 + \int_{\mathbb{R}^3}V(\eta^{-1}y)|\varphi|^2\end{aligned}$$

Arguing as in Lemma 3-(ii) and by compactness of the set $G_k \cap S$, there exists $\bar\eta > 0$ such that for any $\phi_{\bar\eta} \in F_k^{\bar\eta}$, we have

$$\bar\eta^2\frac{1}{2m}\int_{\mathbb{R}^3}|\nabla\varphi|^2 + \int_{\mathbb{R}^3}V(y/\bar\eta)|\varphi|^2 < 0$$

Since $X_k \cap F_k^{\bar\eta} \neq \emptyset$, we have $\lambda_k \le \sup_{F_k^{\bar\eta}}\mathcal{I}(\phi_{\bar\eta}) < mc^2$. ∎

Lemma 7. *Let $\zeta_n \in X_k$ be a (constrained) Palais Smale sequence at level λ_k for $\mathcal{I}$ on X_k, with gradient*

$$\nabla_{X_k}\mathcal{I}(\zeta_n) = \nabla\mathcal{I}(\zeta_n) - \mu_0(\zeta_n)\nabla\mathcal{G}(\zeta_n) - \sum_{j=1}^{k-1}\mu_j(\zeta_n)\nabla\mathcal{G}_j(\zeta_n).$$

Then, as $n \to +\infty$

$$\mu_0(\zeta_n) \to \lambda_k \qquad \mu_j(\zeta_n) \to 0 \qquad (j = 1, \dots, k-1)$$

Moreover, if $\xi_n = (\zeta_n)_{tr} \rightharpoonup 0$ in $H^{1/2}$ then

$$\left(\xi_n, V\xi_n\right)_{L^2} \to 0.$$

Proof. We have that $\zeta_n \in X_k$ is such that

$$\mathcal{I}(\zeta_n) \to \lambda_k \quad \text{and} \quad \|\nabla_{X_k}\mathcal{I}(\zeta_n)\| \to 0.$$

Then ζ_n is bounded in H^1 and from (2.9) we have, as $n \to +\infty$

$$\mu_0(\zeta_n) = \mathcal{I}(\zeta_n) - \frac{1}{2}\left(\nabla_{X_k}\mathcal{I}(\zeta_n), \zeta_n\right)_{H^1} \to \lambda_k.$$

Remark that, for all $j \in \{1, \dots, k-1\}$

$$0 = \nabla_{X_j}\mathcal{I}(\phi_j) = \nabla\mathcal{I}(\phi_j) - \mu_0(\phi_j)\nabla\mathcal{G}(\phi_j) - \sum_{i=1}^{j-1}\mu_i(\phi_j)\nabla\mathcal{G}_i(\phi_j).$$

and hence, for all $\zeta_n \in X_k$ and $j \in \{1, \dots, k-1\}$ we have that

$$\begin{aligned} d\mathcal{I}(\zeta_n)[\phi_j] = d\mathcal{I}(\phi_j)[\zeta_n] &= \left(\nabla\mathcal{I}(\phi_j), \zeta_n\right)_{H^1} \\ &= \mu_0(\phi_j)\left(\nabla\mathcal{G}(\phi_j), \zeta_n\right)_{H^1} + \sum_{i=1}^{j-1}\mu_i(\phi_j)\left(\nabla\mathcal{G}_i(\phi_j), \zeta_n\right)_{H^1} = 0. \end{aligned}$$

From this we conclude, using (2.10), that

$$\mu_j(\zeta_n) = d\mathcal{I}(\zeta_n)[\phi_j] - \left(\nabla_{X_k}\mathcal{I}(\zeta_n), \phi_j\right)_{H^1} = -\left(\nabla_{X_k}\mathcal{I}(\zeta_n), \phi_j\right)_{H^1} \to 0$$

for $j = 1, \dots, k-1$.

We then proceed as in the proof of Lemma 5. Since ζ_n is a constrained Palais Smale sequence, we have

$$\begin{aligned} o_n(1) = C\|\nabla_{X_k}\mathcal{I}(\zeta_n)\|_{H^1}\|\zeta_n\|_{H^1} &\geq |(\nabla_{X_k}\mathcal{I}(\zeta_n), \chi_R^2\zeta_n)|_{H^1} \\ &\geq |d\mathcal{I}(\zeta_n)[\chi_R^2\zeta_n]| - |\mu_0(\zeta_n)(\nabla\mathcal{G}(\zeta_n), \chi_R^2\zeta_n)_{H^1}| \\ &\quad - |\sum_{j=1}^{k-1}\mu_j(\zeta_n)(\nabla\mathcal{G}_j(\zeta_n), \chi_R^2\zeta_n)_{H^1}| \\ &\geq 2\mathcal{I}(\chi_R\zeta_n) - 2c^2\|\zeta_n\nabla\chi_R\|_{L^2}^2 - 2|\mu_0(\zeta_n)||\chi_R\xi_n|_{L^2}^2 - \sum_{j=1}^{k-1}|\mu_j(\zeta_n)||\chi_R\xi_n|_{L^2} \end{aligned}$$

where

$$\|\zeta_n\nabla\chi_R\|_{L^2}^2 \leq C\sup_{y\in\mathbb{R}^3}|\nabla\chi_R|^2 \leq \frac{C}{R^2}$$

and by Sobolev compact embedding, for any given $R > 0$,

$$|\chi_R \xi_n|_{L^2} \to 0 \quad \text{as} \quad n \to +\infty.$$

Moreover, $|\mu_j(\zeta_n)| \leq C$ for $j = 0, \dots, k-1$.

Now, since $\mathcal{I}$ is coercive, exactly as in Lemma 5 we may conclude

$$\|\chi_R \zeta_n\|^2_{H^1} \leq \epsilon_n + \frac{C}{R}$$

and by **(h2)** and Lemma 2,

$$|(V\chi_R \xi_n, \chi_R \xi_n)_{L^2}| \leq a\|\chi_R \zeta_n\|^2_{H^1} \leq \epsilon_n + \frac{C}{R}$$

for $\epsilon_n \to 0$ as $n \to +\infty$, R arbitrary large, and the Lemma follows. □

We are now ready to prove the following Proposition for the existence of a minimizer for $(\mathcal{P}_k)$.

Proposition 3. *Let $\zeta_n \in X_k$ be a minimizing Palais Smale sequence for $(\mathcal{P}_k)$.*

Then $\zeta_n \rightharpoonup \phi_k$ in H^1 and $|(\phi_k)_{tr}|^{-1}_{L^2}\phi_k \in X_k$ is a minimizer for problem $(\mathcal{P}_k)$, and a weak solution of the Neumann problem $(\mathcal{E}_k)$.

Proof. We proceed as in the proof of Lemma 2 to conclude that $\zeta_n \rightharpoonup \phi_k \not\equiv 0$.

We clearly have that $\mathcal{G}_j(\phi_k) = 0$ for $j = 1, \dots, k-1$. We do not know if $|\varphi_k|_{L^2} = 1$ (where $\varphi_k = (\phi_k)_{tr}$).

By Lemma 7 we have that

$$\mu_0(\zeta_n) \to \lambda_k \qquad \mu_j(\zeta_n) \to 0 \qquad (j = 1, \dots, k-1)$$

then by weak convergence we then have that for all $h \in H^1$, as $n \to +\infty$

$$\begin{aligned}(\nabla_{X_k}\mathcal{I}(\zeta_n), h)_{H^1} &= d\mathcal{I}(\zeta_n)[h] - 2\mu_0(\zeta_n)\operatorname{Re}(\xi_n, h_{tr})_{L^2} - \sum_{j=1}^{k-1}\mu_j(\zeta_n)(\varphi_j, h_{tr})_{L^2} \\ &\to d\mathcal{I}(\phi_k)[h] - 2\lambda_k \operatorname{Re}(\varphi_k, h_{tr})_{L^2} = 0.\end{aligned}$$

We deduce, taking $h = \phi_k$

$$0 = d\mathcal{I}(\phi_k)[\phi_k] - 2\lambda_k|\varphi_k|^2_{L^2} = 2\mathcal{I}(\phi_k) - 2\lambda_k|\varphi_k|^2_{L^2}$$

and we conclude that $|\varphi_k|^{-1}_{L^2}\phi_k \in X_k$ is a minimizer for $(\mathcal{P}_k)$. □

Remark 6. It follows from the above Theorem that

$$\nabla_{X_k}\mathcal{I}(\phi_k) = \nabla\mathcal{I}(\phi_k) - \lambda_k\nabla\mathcal{G}(\phi_k) = 0. \tag{2.11}$$

To conclude the proof of Theorem 1 we prove that $\{\lambda_k\}_{k\geq 1} \in \sigma_{disc}(H_0 + V)$ namely that λ_k has finite multiplicity.

Indeed suppose that there exists an eigenvalue λ_k with infinite multiplicity. Then there exists a corresponding sequence $\{\varphi_n^{(k)}\}_{n\in\mathbb{N}} \subset H^{1/2}$ of eigenfunctions corresponding to the same eigenvalue λ_k. We will assume that $|\varphi_n^{(k)}|_{L^2} = 1$ for all $n \in \mathbb{N}$. Letting

$$\phi_n^{(k)} = \mathcal{F}_y^{-1}\left[e^{-x\sqrt{m^2c^4+c^2|p|^2}}\mathcal{F}[\varphi_n^{(k)}]\right] \in X_k,$$

by Lemma 2 we have $\mathcal{I}(\phi_n^{(k)}) = \lambda_k$ and $\nabla_{X_k}\mathcal{I}(\phi_n^{(k)}) = 0$. We deduce from this that φ_n^k is a bounded sequence in $H^{1/2}$, since by orthogonality $\varphi_n^{(k)} \rightharpoonup 0$ in L^2, we have $\varphi_n^{(k)} \rightharpoonup 0$ in $H^{1/2}$, therefore by Lemma 5 we get

$$(\varphi_n^{(k)}, V\varphi_n^{(k)})_{L^2} \to 0 \qquad \text{as } n \to +\infty$$

and from this we get a contradiction, namely $\lambda_k = \mathcal{I}(\phi_n^{(k)}) \geq mc^2$.

Finally since eigenvalues can accumulate only on the essential spectrum, we may conclude that

$$0 < \lambda_1 \leq .... \leq \lambda_{k-1} \leq \lambda_k \ \to \ \inf\{\sigma_{\text{ess}}(v)\} = mc^2 \quad \text{for } k \to +\infty.$$

3. Proof of Theorem 2

Take ϕ_k (and $\varphi_k = (\phi_k)_{tr}$) and λ_k as in Theorem 1, and take $R > 0$ and $T > 0$, we set $\chi_T(y) = \xi_R(y) g_T(y)$ where $\xi_R(y) = \min\{(|y| - R)_+, 1\}$ and $g_T(y) = \min\{\mathrm{e}^{\frac{\beta}{c}|y|}, T\}$, we introduce also the sets $\mathcal{C}_R = \{ (x,y) \in \mathbb{R}^4_+ \mid R < |y| < R+1 \}$ and $\mathcal{D}_T = \{ (x,y) \in \mathbb{R}^4_+ \mid \mathrm{e}^{\frac{\beta}{c}|y|} < T \}$ where ξ_R and g_T are respectively not constants.

From (2.11), (2.8) and (2.3) we have

$$\begin{aligned}
0 &= \frac{1}{2} d\mathcal{I}(\phi_k)[\chi_T^2 \phi_k] - \lambda_k \operatorname{Re}\left(\varphi_k, (\chi_T^2 \phi_k)_{tr}\right)_{L^2} \\
&= \frac{1}{2} d\mathcal{I}(\chi_T \phi_k)[\chi_T \phi_k] - c^2 \|\phi_k \nabla_y \chi_T\|^2_{L^2} - \lambda_k |\chi_T \varphi_k|^2_{L^2} \\
&= \mathcal{I}(\chi_T \phi_k) - c^2 \|\phi_k \nabla_y \chi_T\|^2_{L^2} - \lambda_k |\chi_T \varphi_k|^2_{L^2} \\
&= \mathcal{I}(\chi_T \phi_k) - \lambda_k |\chi_T \varphi_k|^2_{L^2} \\
&\qquad - \beta^2 \iint_{\mathcal{D}_T} |\chi_T \phi_k|^2 - c^2 \iint_{\mathcal{C}_R} |\nabla_y \xi_R|^2 |g_T \phi_k|^2 \\
&\qquad - 2c\beta \iint_{\mathcal{D}_T \cap \mathcal{C}_R} \frac{y}{|y|} \cdot (\nabla_y \xi_R) \xi_R |g_T \phi_k|^2 \\
&\geq \iint_{\mathbb{R}^4_+} |\partial_x (\chi_T \phi_k)|^2 + c^2 \iint_{\mathbb{R}^4_+} |\nabla_y (\chi_T \phi_k)|^2 + (m^2c^4 - \beta^2) \iint_{\mathbb{R}^4_+} |\chi_T \phi_k|^2 \\
&\qquad - \int_{\mathbb{R}^3} |V| |\chi_T \varphi_k|^2 - \lambda_k \int_{\mathbb{R}^3} |\chi_T \varphi_k|^2 \\
&\qquad - c^2 \iint_{\mathcal{C}_R} |g_T \phi_k|^2 - 2c\beta \iint_{\mathcal{D}_T \cap \mathcal{C}_R} |g_T \phi_k|^2 \\
&\geq \left(\sqrt{m^2c^4 - \beta^2} - \lambda_k - \sup_{|y| \geq R} |V(y)| \right) \int_{\mathbb{R}^3} |\chi_T \varphi_k|^2 - C(R)
\end{aligned}$$

Then, given $\beta < \sqrt{m^2c^4 - \lambda_k^2}$ there exists $R > 0$ such that

$$\sqrt{m^2c^4 - \beta^2} - \lambda_k - \sup_{|y| \geq R} |V(y)| > 0$$

and hence

$$\int_{\mathbb{R}^3} |\chi_T \varphi_k|^2 \leq C$$

with C independent on T. Using monotone convergence we can pass to the limit as $T \to +\infty$ to get

$$\int_{|y|\geq R} |e^{\frac{\beta}{c}|y|}\varphi_k|^2 \leq C$$

Namely, $e^{\frac{\beta}{c}|y|}\varphi_k \in L^2(\mathbb{R}^3 \setminus B_R)$.

4. Proof of Theorem 3

We need the following preliminary results.

Proposition 4. *Let $\phi_k \in H^1(\mathbb{R}^4_+)$ (and $\varphi_k = (\phi_k)_{tr}$) as in Theorem 1 and $V \in L^3_{loc}(\mathcal{U})$. Then $\phi_k \in L^p(\mathbb{R}_+ \times \mathcal{V})$ and $\varphi_k \in L^p(\mathcal{V})$ for any $p \geq 2$ and $\mathcal{V} \subset\subset \mathcal{U}$.*

Proof. Take ϕ_k (and $\varphi_k = (\phi_k)_{tr}$) and λ_k as in Theorem 1, let $v = \operatorname{Re}\phi_k$. Take $r, \delta > 0$ and $y_0 \in \mathcal{U}$ such that the set $B_{r+\delta}(y_0) = \{ y \in \mathbb{R}^3 \mid |y - y_0| \leq r + \delta \} \subset \mathcal{U}$. For $n \in \mathbb{N}$, let $\xi_n(y) \in [0,1]$ a cut off function radial, piecewise linear and such that $\xi_n(y) = 1$ if $|y| \leq r + \delta(\frac{2}{3})^n$ and $\xi_n(y) = 0$ if $|y| \geq r + \delta(\frac{2}{3})^{n-1}$.

Let $T > 0$, we set $v_T = \min\{v_+, T\}$, $\xi^0_n(y) = \xi_n(y - y_0)$ and $\chi_{n,T}(x,y) = \xi^0_n(y) v_T^{\beta_n}(x,y)$ where $\beta_n = (\frac{3}{2})^n - 1$. We introduce also the sets $B^0_n = \{ y \in \mathbb{R}^3 \mid \xi^0_n(y) = 1 \}$, $C^0_n = \{ y \in \mathbb{R}^3 \mid |\nabla_y \xi^0_n(y)| = \frac{2}{\delta}(\frac{3}{2})^n \}$ and $D_T = \{ (x,y) \in \mathbb{R}^4_+ \mid v_+(x,y) < T \}$. We have $B_r(y_0) \subset B^0_{n+1} \subset B^0_n \subset B_{r+\delta}(y_0)$ and $C^0_{n+1} \subset B^0_n$ for any $n \in \mathbb{N}$.

From

$$\begin{aligned}\iint_{\mathbb{R}^4_+} |\partial_x(\chi_{n,T} v)|^2 &= \iint_{\mathbb{R}^4_+} (\xi^0_n)^2 v_T^{2\beta_n} |\partial_x v|^2 + \beta_n^2 \iint_{D_T} (\xi^0_n)^2 v_T^{2\beta_n} |\partial_x v|^2 \\ &\quad + 2\beta_n \iint_{D_T} (\xi^0_n)^2 v_T^{2\beta_n} |\partial_x v|^2 \\ &\geq (1+\beta_n)^2 \iint_{D_T} (\xi^0_n)^2 v_T^{2\beta_n} |\partial_x v|^2\end{aligned}$$

we obtain

$$\beta_n^2 \iint_{D_T} (\xi^0_n)^2 v_T^{2\beta_n} |\partial_x v|^2 \leq \frac{\beta_n^2}{(1+\beta_n)^2} \iint_{\mathbb{R}^4_+} |\partial_x(\chi_{n,T} v)|^2 \tag{4.1}$$

while from

$$
\begin{aligned}
\iint_{\mathbb{R}^4_+} |\nabla_y(\chi_{n,T} v)|^2 &= \iint_{\mathbb{R}^4_+} (\xi_n^0)^2 v_T^{2\beta_n} |\nabla_y v|^2 + \beta_n^2 \iint_{D_T} (\xi_n^0)^2 v_T^{2\beta_n} |\nabla_y v|^2 \\
&\quad + \iint_{\mathbb{R}^4_+} |\nabla_y \xi_n^0|^2 |v_T^{\beta_n} v|^2 + 2\beta_n \iint_{D_T} \xi_n^0 (\nabla_y \xi_n^0)\cdot(\nabla_y v_T) v_T^{2\beta_n-1}|v|^2 \\
&\quad + 2\iint_{\mathbb{R}^4_+} \xi_n^0 v_T^{2\beta_n} v \nabla_y \xi_n^0 \cdot \nabla_y v + 2\beta_n \iint_{D_T} (\xi_n^0)^2 v_T^{2\beta_n} |\nabla_y v|^2 \\
&= \iint_{\mathbb{R}^4_+\setminus D_T} v_T^{2\beta_n} (v\nabla_y \xi_n^0 + \xi_n^0 \nabla_y v)^2 + (\beta_n+1)^2 \iint_{D_T} (\xi_n^0)^2 v_T^{2\beta_n}|\nabla_y v|^2 \\
&\quad + 2(\beta_n+1) \iint_{D_T} \xi_n^0 (\nabla_y \xi_n^0)\cdot(\nabla_y v_T) v_T^{2\beta_n-1}|v|^2 \\
&\geq (\beta_n+1)\Big(\beta_n \iint_{D_T} (\xi_n^0)^2 v_T^{2\beta_n}|\nabla_y v|^2 + 2\iint_{D_T} \xi_n^0(\nabla_y \xi_n^0)\cdot(\nabla_y v_T) v_T^{2\beta_n-1}|v|^2\Big).
\end{aligned}
$$

we deduce

$$
\begin{aligned}
\beta_n^2 \iint_{D_T} (\xi_n^0)^2 v_T^{2\beta_n}|\nabla_y v|^2 + 2\beta_n \iint_{D_T} \xi_n^0 (\nabla_y \xi_n^0)\cdot(\nabla_y v_T) v_T^{2\beta_n-1}|v|^2 \\
\leq \frac{\beta_n}{\beta_n+1} \iint_{\mathbb{R}^4_+} |\nabla_y(\chi_{n,T} v)|^2
\end{aligned}
\tag{4.2}
$$

Computations similar to those at the beginning of section 3, (we recall that $v = \operatorname{Re}\phi_k$), leads to

$$
\begin{aligned}
0 &= \frac{1}{2} d\mathcal{I}(\phi_k)[\chi_{n,T}^2 v] - \lambda_k \operatorname{Re}\Big(\varphi_k, (\chi_{n,T}^2 v)_{tr}\Big)_{L^2} \\
&= \mathcal{I}(\chi_{n,T} v) - \|v\partial_x \chi_{n,T}\|_{L^2}^2 - c^2 \|v\nabla_y \chi_{n,T}\|_{L^2}^2 - \lambda_k |\chi_{n,T} v|_{L^2}^2 \\
&= \mathcal{I}(\chi_{n,T} v) - \lambda_k |\chi_{n,T} v|_{L^2}^2 - \beta_n^2 \iint_{\mathcal{D}_T} (\xi_n^0)^2 v_T^{2\beta_n} |\partial_x v_T|^2 \\
&\quad - c^2 \beta_n^2 \iint_{\mathcal{D}_T} (\xi_n^0)^2 v_T^{2\beta_n} |\nabla_y v_T|^2 - c^2 \iint_{\mathbb{R}^4_+} |\nabla_y \xi_n^0|^2 |v_T^{\beta_n} v|^2 \\
&\quad - 2c^2 \beta_n \iint_{D_T} \xi_n^0 (\nabla_y \xi_n^0)\cdot(\nabla_y v_T) v_T^{2\beta_n-1} v^2
\end{aligned}
$$

Therefore we get, using (4.1) and (4.2)

$$\begin{aligned}
0 =& \iint_{\mathbb{R}^4_+} |\partial_x(\chi_{n,T} v)|^2 + c^2 \iint_{\mathbb{R}^4_+} |\nabla_y(\chi_{n,T} v)|^2 + m^2c^4 \iint_{\mathbb{R}^4_+} |\chi_{n,T} v|^2 \\
&- \beta_n^2 \iint_{D_T} (\xi_n^0)^2 v_T^{2\beta_n} |\partial_x v_T|^2 - c^2\beta_n^2 \iint_{D_T} (\xi_n^0)^2 v_T^{2\beta_n} |\nabla_y v_T|^2 \\
&- c^2 \iint_{\mathbb{R}^4_+} |\nabla_y \xi_n^0|^2 |v_T^{\beta_n} v|^2 - 2c^2\beta_n \iint_{D_T} \xi_n^0 (\nabla_y \xi_n^0)\cdot(\nabla_y v_T) v_T^{2\beta_n - 1} v^2 \\
&+ \int_{\mathbb{R}^3} V|(\chi_{n,T} v)_{tr}|^2 - \lambda_k \int_{\mathbb{R}^3} |(\chi_{n,T} v)_{tr}|^2 \\
\geq& \left(1 - \frac{\beta_n}{\beta_n + 1}\right) \|\chi_{n,T} v\|^2_{H^1} - c^2 \iint_{\mathbb{R}^4_+} |\nabla_y \xi_n^0|^2 |v_T^{\beta_n} v|^2 \\
&- \int_{\mathbb{R}^3} |V| |(\chi_{n,T} v)_{tr}|^2 - \lambda_k \int_{\mathbb{R}^3} |(\chi_{n,T} v)_{tr}|^2
\end{aligned}$$

Namely,

$$\begin{aligned}
&\frac{1}{\beta_n + 1} \|\xi_n^0 v_T^{\beta_n} v\|^2_{H^1} \\
&\qquad \leq c^2 \iint_{\mathbb{R}^4_+} |\nabla_y \xi_n^0|^2 |v_T^{\beta_n} v|^2 + \int_{\mathbb{R}^3} |V| |\xi_n^0 (v_T^{\beta_n} v)_{tr}|^2 + \lambda_k \int_{\mathbb{R}^3} |\xi_n^0 (v_T^{\beta_n} v)_{tr}|^2
\end{aligned}$$

Using Fatou's Lemma and monotone convergence, we can pass to the limit as $T \to +\infty$ to get

$$\begin{aligned}
&\frac{1}{\alpha_n} \|\xi_n^0 v_+^{\alpha_n}\|^2_{H^1} \\
&\qquad \leq c^2 \iint_{\mathbb{R}^4_+} |\nabla_y \xi_n^0|^2 |v_+^{\alpha_n}|^2 + \int_{\mathbb{R}^3} |V| |\xi_n^0 (v_+)^{\alpha_n}_{tr}|^2 + \lambda_k \int_{\mathbb{R}^3} |\xi_n^0 (v_+)^{\alpha_n}_{tr}|^2
\end{aligned} \tag{4.3}$$

where $\alpha_n = \beta_n + 1 = (3/2)^n$

For any $M > 0$, let $A_1 = \{|V| \leq M\} \cap B_{r+\delta}(y_0)$, $A_2 = \{|V| > M\} \cap B_{r+\delta}(y_0)$, then, since $V \in L^3_{loc}(\mathcal{U})$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |V| |\xi_n^0 (v_+)^{\alpha_n}_{tr}|^2 &\leq \int_{A_1} |V| |\xi_n^0 (v_+)^{\alpha_n}_{tr}|^2 + \int_{A_2} |V| |\xi_n^0 (v_+)^{\alpha_n}_{tr}|^2 \\
&\leq M \int_{A_1} |\xi_n^0 (v_+)^{\alpha_n}_{tr}|^2 + \Big(\int_{A_2} |V|^3\Big)^{1/3} \Big(\int_{A_2} |\xi_n^0 (v_+)^{\alpha_n}_{tr}|^3\Big)^{2/3} \\
&\leq M |\xi_n^0 (v_+)^{\alpha_n}_{tr}|_2^2 + \epsilon(M) |\xi_n^0 (v_+)^{\alpha_n}_{tr}|_3^2
\end{aligned}$$

now we take $C \geq \max\{(\frac{2}{\delta} c)^2, M + \lambda_k\}$ and we get

$$\begin{aligned}
\|\xi_n^0 v_+^{\alpha_n}\|^2_{H^1} \leq& C \left(\alpha_n^3 \iint_{\mathbb{R}_+ \times C_n^0} |v_+^{\alpha_n}|^2 + \alpha_n |\xi_n^0 (v_+)^{\alpha_n}_{tr}|_2^2\right) \\
&+ \alpha_n \epsilon(M) |\xi_n^0 (v_+)^{\alpha_n}_{tr}|_3^2
\end{aligned}$$

Taking M sufficiently large, that is $\epsilon(M)$ sufficiently small, by Sobolev inequality we have

$$|(v_+^{\alpha_n})_{tr}|^2_{L^{2^\sharp}(B_n^0)} + \|v_+^{\alpha_n}\|^2_{L^{2^*}(\mathbb{R}_+\times B_n^0)} \le K_n\left(\|v_+^{\alpha_n}\|^2_{L^2(\mathbb{R}_+\times C_n^0)} + |\xi_n^0(v_+^{\alpha_n})_{tr}|_2^2\right)$$

where $2^\sharp = 2N/(N-1) = 3$ (here $N = 3$) and $2^* = 2N/(N-2) = 4$ (here $N = 4$) are the critical Sobolev exponent for the embedding of $H^{1/2}(\mathbb{R}^3)$ in $L^p(\mathbb{R}^3)$ and for the embedding of $H^1(\mathbb{R}^4)$ in $L^p(\mathbb{R}^4)$ and the constant K_n depend on $n \in \mathbb{N}$.

Finally, since $C_n^0 \subset B_{n-1}^0$ we may conclude

$$\begin{cases} |(v_+)_{tr}^{\alpha_n}|^2_{L^{2^\sharp}(B_n^0)} \le K_n\left(\|v_+^{\alpha_n}\|^2_{L^2(\mathbb{R}_+\times B_{n-1}^0)} + |(v_+)_{tr}^{\alpha_n}|^2_{L^2(B_{n-1}^0)}\right) \\ \|v_+^{\alpha_n}\|^2_{L^{2^*}(\mathbb{R}_+\times B_n^0)} \le K_n\left(\|v_+^{\alpha_n}\|^2_{L^2(\mathbb{R}_+\times B_{n-1}^0)} + |(v_+)_{tr}^{\alpha_n}|^2_{L^2(B_{n-1}^0)}\right). \end{cases} \tag{4.4}$$

Then a bootstrap argument can start: since $v_+ \in H^1(\mathbb{R}^4_+)$ we have $v_+ \in L^p(\mathbb{R}^4_+)$ for $p \in [2,4]$ and $(v_+)_{tr} \in L^q(\mathbb{R}^3)$ for $q \in [2,3]$, hence we can apply (4.4) with $n = 1$ to deduce that $(v_+)_{tr} \in L^{2^\sharp\alpha_1}(B_1^0) = L^{3(3/2)}(B_1^0)$ and $v_+ \in L^{2^*\alpha_1}(\mathbb{R}_+ \times B_1^0) = L^6(\mathbb{R}_+ \times B_1^0)$. Since $2\alpha_n = 2^\sharp\alpha_{n-1} < 2^*\alpha_{n-1}$ we can then apply again (4.4) and, after n iterations, we deduce that $(v_+)_{tr} \in L^{3(3/2)^n}(B_n^0)$, $v_+ \in L^{4(3/2)^n}(\mathbb{R}_+ \times B_n^0)$. Hence we may conclude that $(v_+)_{tr} \in L^p(B_r(y_0))$ and $v_+ \in L^p(\mathbb{R}_+ \times B_r(y_0))$ for all $p \in [2,+\infty)$.

The same is clearly true for v_- and hence for $v = \mathrm{Re}\,\phi_k$. Analogously we can argue for $\mathrm{Im}\,\phi_k$ and we get the result for $\varphi_k = (\phi_k)_{tr}$.

□

Proposition 5. *Let $\phi_k \in H^1(\mathbb{R}^4_+)$ (and $\varphi_k = (\phi_k)_{tr}$) as in Theorem 1. Then given any $R > R_0$ (with R_0 given in (h1)) we have $\phi_k \in L^p(\mathbb{R}_+ \times (\mathbb{R}^3 \setminus B_R))$ and $\varphi_k \in L^p(\mathbb{R}^3 \setminus B_R)$ for any $p \in [2,\infty]$.*

Proof. By (**h1**) we have $V \in L^\infty(\mathbb{R}^3 \setminus B_{R_0})$ for some $R_0 > 0$. Take ϕ_k (and $\varphi_k = (\phi_k)_{tr}$) and λ_k as in Theorem 1, let $v = \mathrm{Re}\,\phi_k$.

Take any $\delta > 0$ and for $n \in \mathbb{N}$ let $\xi_n(y) \in [0,1]$ be a cut off function, radial, piecewise linear and such that $\xi_n(y) = 0$ if $|y| \le R_0 + \delta\sum_{k=0}^{n-1}(\frac{2}{3})^k$ and $\xi_n(y) = 1$ if $|y| \ge R_0 + \delta\sum_{k=0}^{n}(\frac{2}{3})^k$.

Let $T > 0$, we set $v_T = \min\{v_+, T\}$ and $\chi_{n,T}(x,y) = \xi_n(y)v_T^{\beta_n}(x,y)$ where $\beta_n = (\frac{3}{2})^n - 1$. We introduce also the sets $F_n = \{\, y \in \mathbb{R}^3 \mid \xi_n(y) = 1 \,\}$, $C_n = \{\, y \in \mathbb{R}^3 \mid |\nabla_y\xi_n^0(y)| = \frac{2}{\delta}(\frac{3}{2})^n \,\}$ and $D_T = \{\, (x,y) \in \mathbb{R}^4_+ \mid v_+(x,y) < T \,\}$. We have $\mathbb{R}^3 \setminus B_{R_0+\delta} \subset F_{n+1} \subset F_n \subset \mathbb{R}^3 \setminus B_{R_0}$ and $C_{n+1} \subset F_n$ for any $n \in \mathbb{N}$.

Now we can repeat the estimates in the proof of Proposition 4 to deduce that also in this case (4.3) holds, namely

$$\frac{1}{\alpha_n}\|\xi_n v_+^{\alpha_n}\|^2_{H^1} \le c^2 \iint_{\mathbb{R}^4_+} |\nabla_y\xi_n|^2|v_+^{\alpha_n}|^2 + \int_{\mathbb{R}^3}|V||\xi_n(v_+^{\alpha_n})_{tr}|^2 + \lambda_k\int_{\mathbb{R}^3}|\xi_n(v_+^{\alpha_n})_{tr}|^2$$

where also here $\alpha_n = \beta_n + 1 = (3/2)^n$.

Then taking a positive constant $C \ge \max\{(\frac{2}{\delta}c)^2, (\sup_{\mathbb{R}^3\setminus B_{R_0}}|V| + \lambda_k)\}$ we get

$$\|\xi_n v_+^{\alpha_n}\|^2_{H^1} \le C\left(\alpha_n^3\iint_{\mathbb{R}_+\times C_n}|v_+^{\alpha_n}|^2 + \alpha_n\int_{\mathbb{R}^3}|\xi_n(v_+^{\alpha_n})_{tr}|^2\right)$$

and again by Sobolev inequality and recalling that $C_n \subset F_{n-1}$

$$|(v_+)_{tr}^{\alpha_n}|^2_{L^{2^\sharp}(F_n)} + \|v_+^{\alpha_n}\|^2_{L^{2^*}(\mathbb{R}_+\times F_n)} \le C\left(\alpha_n^3\|v_+^{\alpha_n}\|^2_{L^2(\mathbb{R}_+\times F_{n-1})} + \alpha_n|(v_+)_{tr}^{\alpha_n}|^2_{L^2(F_{n-1})}\right)$$

Finally, we may conclude

$$\begin{cases} |(v_+)_{tr}^{\alpha_n}|^2_{L^{2^\sharp}(F_n)} \le C\left(\alpha_n^3\|v_+^{\alpha_n}\|^2_{L^2(\mathbb{R}_+\times F_{n-1})} + \alpha_n|(v_+)_{tr}^{\alpha_n}|^2_{L^2(F_{n-1})}\right) \\ \|v_+^{\alpha_n}\|^2_{L^{2^*}(\mathbb{R}_+\times F_n)} \le C\left(\alpha_n^3\|v_+^{\alpha_n}\|^2_{L^2(\mathbb{R}_+\times F_{n-1})} + \alpha_n|(v_+)_{tr}^{\alpha_n}|^2_{L^2(F_{n-1})}\right). \end{cases} \tag{4.5}$$

Then, exactly as in the proof of Proposition 4, a bootstrap argument can start and after n iterations, we deduce that $(v_+)_{tr} \in L^{3(3/2)^n}(F_n)$, $v_+ \in L^{4(3/2)^n}(\mathbb{R}_+\times F_n)$. Hence we may conclude that $(v_+)_{tr} \in L^p(\mathbb{R}^3\setminus B_{R_0+\delta})$ and $v_+ \in L^p(\mathbb{R}_+\times(\mathbb{R}^3\setminus B_{R_0+\delta}))$ for all $p\in[2,\infty)$.

To prove that actually $(v_+)_{tr} \in L^\infty(\mathbb{R}^3\setminus B_{R_0+\delta})$ and $v_+ \in L^\infty(\mathbb{R}_+\times(\mathbb{R}^3\setminus B_{R_0+\delta}))$ we can argue as follows. In view of (4.5) we have

$$\begin{aligned} |(v_+)_{tr}|^{2\alpha_n}_{L^{2^\sharp\alpha_n}(F_n)} \le& C\left(\alpha_n^3\|v_+\|^{2\alpha_n}_{L^{2\alpha_n}(\mathbb{R}_+\times F_{n-1})} + \alpha_n|(v_+)_{tr}|^{2\alpha_n}_{L^{2\alpha_n}(F_{n-1})}\right) \\ \le& M_0^2\,\mathrm{e}^{2\sqrt{\alpha_n}}\left(\max\{\|v_+\|_{L^{2\alpha_n}(\mathbb{R}_+\times F_{n-1})}, |(v_+)_{tr}|_{L^{2\alpha_n}(F_{n-1})}\}\right)^{2\alpha_n} \end{aligned}$$

Moreover, since

$$\|v_+^{\alpha_n}\|_{L^{2^\sharp}} \le \|v_+^{\alpha_n}\|^{1/2}_{L^2}\|v_+^{\alpha_n}\|^{1/2}_{L^{2^*}}$$

and $F_n \subset F_{n-1}$ we have

$$\begin{aligned} \|v_+\|^{2\alpha_n}_{L^{2^\sharp\alpha_n}(\mathbb{R}_+\times F_n)} &\le \|v_+\|^{\alpha_n}_{L^{2\alpha_n}}\|v_+\|^{\alpha_n}_{L^{2^*\alpha_n}} \le \frac{1}{2}\|v_+\|^{2\alpha_n}_{L^{2\alpha_n}} + \frac{1}{2}\|v_+\|^{2\alpha_n}_{L^{2^*\alpha_n}} \\ &\le M_0^2\,\mathrm{e}^{2\sqrt{\alpha_n}}\left(\max\{\|v_+\|_{L^{2\alpha_n}(\mathbb{R}_+\times F_{n-1})}, |(v_+)_{tr}|_{L^{2\alpha_n}(F_{n-1})}\}\right)^{2\alpha_n} \end{aligned}$$

where the positive constant $M_0 > 1$ is independent of n.

Hence, recalling also that $2^\sharp\alpha_n = 2\alpha_{n+1}$, we get

$$\begin{cases} |(v_+)_{tr}|_{L^{2\alpha_{n+1}}(F_n)} & \le M_0^{\frac{1}{\alpha_n}}\,\mathrm{e}^{\frac{1}{\sqrt{\alpha_n}}}\max\{\|v_+\|_{L^{2\alpha_n}(\mathbb{R}_+\times F_{n-1})}, |(v_+)_{tr}|_{L^{2\alpha_n}(F_{n-1})}\} \\ \|v_+\|_{L^{2\alpha_{n+1}}(\mathbb{R}_+\times F_n)} & \le M_0^{\frac{1}{\alpha_n}}\,\mathrm{e}^{\frac{1}{\sqrt{\alpha_n}}}\max\{\|v_+\|_{L^{2\alpha_n}(\mathbb{R}_+\times F_{n-1})}, |(v_+)_{tr}|_{L^{2\alpha_n}(F_{n-1})}\} \end{cases}$$

We set $A_n = \max\{\|v_+\|_{L^{2\alpha_n}(\mathbb{R}_+\times F_{n-1})}, |(v_+)_{tr}|_{L^{2\alpha_n}(F_{n-1})}\}$ then we have

$$A_{n+1} \le M_0^{\frac{1}{\alpha_n}}\,\mathrm{e}^{\frac{1}{\sqrt{\alpha_n}}}A_n \le M_0^{\sum_{i=0}^n\frac{1}{\alpha_i}}\,\mathrm{e}^{\sum_{i=0}^n\frac{1}{\sqrt{\alpha_i}}}A_0.$$

Since

$$\sum_{i=0}^{+\infty}\frac{1}{\sqrt{\alpha_i}} < +\infty$$

then there exists a constant K independent on p such that $|(v_+)_{tr}|_{L^p(\mathbb{R}^3\setminus B_{R_0+\delta})} < K$ and $\|v_+\|_{L^p(\mathbb{R}_+\times(\mathbb{R}^3\setminus B_{R_0+\delta}))} < K$, for any $p\ge 2$ and we deduce that $(v_+)_{tr} \in L^\infty(\mathbb{R}^3\setminus B_{R_0+\delta})$ and $v_+ \in L^\infty(\mathbb{R}_+\times(\mathbb{R}^3\setminus B_{R_0+\delta}))$.

The same is clearly true for v_- and hence for $v = \operatorname{Re}\phi_k$. Analogously we can argue for $\operatorname{Im}\phi_k$ and we get the result for $\varphi_k = (\phi_k)_{tr}$.

□

Now we finally conclude the proof of Theorem 3 as follow

(i) : Recalling that $\phi_k \in H^1(\mathbb{R}^4_+, \mathbb{C})$ is a weak solution of the Neumann problem

$$\begin{cases} -\partial_x^2 \phi_k - c^2 \Delta_y \phi_k + m^2 c^4 \phi_k = 0 & \text{in } \mathbb{R}^4_+ \\ \dfrac{\partial \phi_k}{\partial \nu} + V \varphi_k = \lambda_k \varphi_k & \text{on } \partial \mathbb{R}^4_+ = \mathbb{R}^3. \end{cases}$$

then following [2] we introduce

$$\psi_k(x, y) = \int_0^x \phi_k(t, y)\, dt$$

we clearly have that $\psi_k \in H^1((0, r) \times \mathbb{R}^3, \mathbb{C})$ for any $r > 0$ and we have (see [5, Proposition 3.9] for the details) that ψ_k is a weak solution of the following Dirichlet problem

$$\begin{cases} -\partial_x^2 \psi_k - c^2 \Delta_y \psi_k + m^2 c^4 \psi_k = f(x, y) & \text{in } \mathbb{R}^4_+ \\ \psi_k = 0 & \text{on } \partial \mathbb{R}^4_+ = \mathbb{R}^3. \end{cases}$$

where $f(x, y) = (\lambda_k - V(y))\varphi_k(y)$.

Now let us define

$$(\psi_k)_{odd}(x, y) = \begin{cases} \psi_k(x, y) & x \geq 0 \\ -\psi_k(-x, y) & x < 0 \end{cases} \qquad \text{and} \qquad f_{odd}(x, y) = \begin{cases} f(x, y) & x \geq 0 \\ -f(x.y) & x < 0 \end{cases}$$

It is easy to check that $(\psi_k)_{odd} \in H^1((-r, r) \times \mathbb{R}^3, \mathbb{C})$ is a weak solution of the (linear) second order elliptic problem

$$-\partial_x^2 u - c^2 \Delta_y u + m^2 c^4 u = f_{odd} \qquad \text{in} \quad \mathbb{R}^4.$$

Since by Proposition 5 $f_{odd} \in L^q((-r, r) \times (\mathbb{R}^3 \setminus B_R))$ for any $q \in [2, \infty]$, $r > 0$ and $R > R_0$ we deduce by standard elliptic regularity that $(\psi_k)_{odd} \in W^{2,q}((-r, r) \times (\mathbb{R}^3 \setminus B_R))$ and hence in particular $\phi_k = \partial_x \psi_k \in W^{1,q}((0, r) \times (\mathbb{R}^3 \setminus B_R))$.

(ii) : By Sobolev's embedding $\psi_k \in C^{1,\alpha}([0, +\infty) \times (\mathbb{R}^3 \setminus B_R))$ for all $\alpha \in [0, 1]$. Namely, we get that $\phi_k = \partial_x \psi_k \in C^{0,\alpha}([0, +\infty) \times (\mathbb{R}^3 \setminus B_R))$ and $\varphi_k = \phi_k(0, \cdot\,) \in C^{0,\alpha}(\mathbb{R}^3 \setminus B_R)$ for any $\alpha \in [0, 1]$ and $R > R_0$.

(iii): Since by Proposition 4 $f_{odd} \in L^q((-r, r) \times \mathcal{V})$ for any $q \in [2, \infty)$, $r > 0$ and $\mathcal{V} \subset\subset \mathcal{U}$ we deduce by standard elliptic regularity that $(\psi_k)_{odd} \in W^{2,q}((-r, r) \times \mathcal{V})$ hence in particular $\phi_k = \partial_x \psi_k \in W^{1,q}((0, r) \times \mathcal{V})$. Then by the trace Theorem we get $\varphi_k \in W^{1-\frac{1}{q},q}(\mathcal{V})$ for any $q \in [2, \infty)$ and $\mathcal{V} \subset\subset \mathcal{U}$ and by Sobolev embedding $\varphi_k \in C^{0,\alpha}(\mathcal{V})$ for any $\alpha \in [0, 1)$.

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