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CRITICAL GROUP ESTIMATES FOR NONREGULAR CRITICAL POINTS OF FUNCTIONALS ASSOCIATED WITH QUASILINEAR ELLIPTIC EQUATIONS

SILVIA CINGOLANI, MARCO DEGIOVANNI, AND GIUSEPPINA VANNELLA

ABSTRACT. We consider a class of quasilinear elliptic equations whose principal part includes the p-area and the p-Laplace operators, when p lies in a suitable left neighborhood of 2. For the critical points of the associated functional, we provide estimates of the corresponding critical groups, under assumptions that do not guarantee any further regularity of the critical point.

1. INTRODUCTION

Consider the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}\left[\left(\kappa^2+|\nabla u|^2\right)^{\frac{p-2}{2}}\nabla u\right]+g(x,u)=\Lambda & \text{in } \Omega\,,\\ u=0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, while $p>1$ and $\kappa\geq 0$ are real numbers.

Under suitable assumptions on g and Λ, weak solutions u of (1.1) correspond to critical points of the C^1-functional $f: W_0^{1,p}(\Omega)\to\mathbb{R}$ defined as

$$f(u)=\int_\Omega \Psi_{p,\kappa}(\nabla u)\,dx+\int_\Omega G(x,u)\,dx-\langle\Lambda,u\rangle\,, \tag{1.2}$$

where

$$\Psi_{p,\kappa}(\xi)=\frac{1}{p}\left[\left(\kappa^2+|\xi|^2\right)^{\frac{p}{2}}-\kappa^p\right]\,,\qquad G(x,s)=\int_0^s g(x,t)\,dt\,.$$

About the principal part of the equation, the reference cases are the *p-area operator* for $\kappa=1$ and the *p-Laplace operator* for $\kappa=0$. In the case $p=2$ the value of κ is irrelevant.

In the recent years, several works have been devoted to the description of the critical groups of f at a solution u_0 of (1.1) via Hessian-type notions.

For functionals defined on Banach spaces, serious difficulties arise in extending Morse theory (see [23, 22, 5, 6, 7]). More precisely, by standard deformation results, which hold

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also in general Banach spaces, one can prove the so-called Morse relations, which can be written as

$$\sum_{m=0}^{\infty} C_m t^m = \sum_{m=0}^{\infty} \beta_m t^m + (1+t)Q(t)\,,$$

where (β_m) is the sequence of the Betti numbers of a pair of sublevels $(\{f \leq b\}, \{f < a\})$ and (C_m) is a sequence related to the critical groups of the critical points u of f with $a \leq f(u) \leq b$ (see e.g. [6, Theorem I.4.3]). The problem, in the extension from Hilbert to Banach spaces, concerns the estimate of (C_m), hence of critical groups, by the Hessian of f or some related concept. In a Hilbert setting, the classical Morse lemma and the generalized Morse lemma [16] provide a satisfactory answer. For Banach spaces, a similar general result is so far not known.

In the specific case of the functional defined by (1.2), for $p > 2$ and $\kappa > 0$ the first and the last author have proved an extension of the Morse Lemma and established a connection between the critical groups and the Morse index (see [11, 12, 13]), taking advantage of the fact that, under suitable assumptions on g and Λ, the functional f is actually of class C^2 on $W^{1,p}_0(\Omega)$ and that

$$\Psi''_{p,\kappa}(\eta)[\xi]^2 \geq \nu_{p,\kappa}|\xi|^2 \qquad \text{with } \nu_{p,\kappa} > 0\,.$$

Moreover, an approximation result of Marino-Prodi type is proved in [14].

The results of [11] have been extended in [9, 10], in order to cover the whole case $1 < p < \infty$, when $\kappa > 0$, and the case $1 < p \leq 2$ when $\kappa = 0$.

If Ω is a ball centered at 0, for any $p > 1$ estimates of critical groups associated to p-Laplacian equations have been obtained by Aftalion and Pacella [2] at the positive radial solutions u_0 such that $|\nabla u_0(x)| \neq 0$ for $x \neq 0$.

In [2, 9, 10], the assumptions on g and Λ imply that any solution of (1.1) is of class $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in]0,1]$. This is crucial for the arguments used in those papers.

Our purpose is to consider a class of functionals including (1.2) for a certain range of $p \leq 2$, under assumptions that do not guarantee any further regularity of the critical point u_0. More precisely, define

$$f(u) = \int_\Omega \Psi(\nabla u)\,dx + \int_\Omega G(x,u)\,dx - \langle \Lambda, u\rangle\,. \tag{1.3}$$

Throughout the paper, we will assume that:

(Ψ_1) the function $\Psi : \mathbb{R}^N \to \mathbb{R}$ is of class C^1 with $\Psi(0) = 0$ and $\nabla\Psi(0) = 0$; moreover, there exist

$$\max\left\{1, \frac{2N}{N+2}\right\} < p \leq 2\,,$$

$\kappa \geq 0$ and $0 < \nu \leq C$ such that the functions $(\Psi - \nu\,\Psi_{p,\kappa})$ and $(C\,\Psi_{p,\kappa} - \Psi)$ are both convex;

(Ψ_2) if $\kappa = 0$ and $p < 2$, then Ψ is of class C^2 on $\mathbb{R}^N \setminus \{0\}$; otherwise, Ψ is of class C^2 on $\mathbb{R}^N$;

(g) the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that $g(\cdot, s)$ is measurable for every $s \in \mathbb{R}$ and $g(x,\cdot)$ is of class C^1 for a.e. $x \in \Omega$; moreover:
- if $p < N$, then $g(x,0) \in L^{(p^*)'}(\Omega)$ and there exist $a \in L^{(p^*/2)'}(\Omega)$ and $b \geq 0$ such that

$$|D_s g(x,s)| \leq a(x) + b|s|^{p^*-2}\,,$$

where $p^* = \frac{Np}{N-p}$;

– if $p = N$, then there exist $q, r > 1$, $a \in L^q(\Omega)$ and $b \geq 0$ such that $g(x, 0) \in L^q(\Omega)$ and

$$|D_s g(x, s)| \leq a(x) + b|s|^r\,;$$

– if $p > N$, then $g(x, 0) \in L^1(\Omega)$ and, for every $M > 0$, there exists $a_M \in L^1(\Omega)$ such that

$$|s| \leq M \implies |D_s g(x, s)| \leq a_M(x)\,;$$

(Λ) we have $\Lambda \in W^{-1,p'}(\Omega)$.

Under these assumptions, it is easily seen that $f : W_0^{1,p}(\Omega) \to \mathbb{R}$ is of class C^1.

Let us recall the first ingredient we need from [6, 15, 19].

Definition 1. Let $\mathbb{G}$ be an abelian group, $c = f(u_0)$ and $f^c = \left\{u \in W_0^{1,p}(\Omega) \colon f(u) \leq c\right\}$. The *$m$-th critical group of f at u_0 with coefficients in* $\mathbb{G}$ is defined by

$$C_m(f, u_0; \mathbb{G}) = H^m\left(f^c, f^c \setminus \{u_0\}; \mathbb{G}\right)\,,$$

where H^* stands for Alexander-Spanier cohomology [21]. We will simply write $C_m(f, u_0)$, if no confusion can arise.

In general, it may happen that $C_m(f, u_0)$ is not finitely generated some m and that $C_m(f, u_0) \neq \{0\}$ for infinitely many m's. If however u_0 is an isolated critical point of f, under assumptions (Ψ_1), (g) and (Λ) it follows from [8, Theorem 1.1] and [3, Theorem 3.4] that $C_*(f, u_0)$ is of finite type.

Now, as a second ingredient, we need a notion of Morse index, which is not standard, as f is not of class C^2 on $W_0^{1,p}(\Omega)$. More precisely, because of (Ψ_1), (g) and (Λ), the functional

$$\left\{u \mapsto \int_\Omega G(x, u)\, dx - \langle \Lambda, u\rangle\right\}$$

is actually of class C^2 on $W_0^{1,p}(\Omega)$. On the other hand, the principal part

$$\left\{u \mapsto \int_\Omega \Psi(\nabla u)\, dx\right\}$$

is never of class C^2 for $p < 2$ and is of class C^2 in the case $p = 2$ iff Ψ is a quadratic form on $\mathbb{R}^N$ (see [1, Proposition 3.2]).

So, let $u_0 \in W_0^{1,p}(\Omega)$ be a critical point of the functional f, namely a weak solution of

$$\begin{cases} -\operatorname{div}\left[\nabla\Psi(\nabla u)\right] + g(x, u) = \Lambda & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Because of the term $\Lambda \in W^{-1,p'}(\Omega)$, we cannot expect any further regularity on u_0, even in the case $g = 0$.

In the case $\kappa > 0$, observe that

$$\frac{(p-1)\nu}{(\kappa^2 + |\eta|^2)^{\frac{2-p}{2}}}\,|\xi|^2 \leq \Psi''(\eta)[\xi]^2 \leq \frac{C}{(\kappa^2 + |\eta|^2)^{\frac{2-p}{2}}}\,|\xi|^2 \qquad \text{for any } \eta, \xi \in \mathbb{R}^N\,, \tag{1.4}$$

as $(\Psi - \nu\,\Psi_{p,\kappa})$ and $(C\,\Psi_{p,\kappa} - \Psi)$ are both convex. Let us define a generalized quadratic form $Q_{u_0} : W_0^{1,p}(\Omega) \to]-\infty, +\infty]$ by

$$Q_{u_0}(v) = \int_\Omega \Psi''(\nabla u_0)[\nabla v]^2\,dx + \int_\Omega D_s g(x, u_0) v^2\,dx\,.$$

In the case $\kappa = 0$ and $p < 2$, observe that

$$\frac{(p-1)\nu}{|\eta|^{2-p}}\,|\xi|^2 \le \Psi''(\eta)[\xi]^2 \le \frac{C}{|\eta|^{2-p}}\,|\xi|^2 \qquad \text{for any } \eta, \xi \in \mathbb{R}^N \text{ with } \eta \neq 0\,. \tag{1.5}$$

Then set

$$Z_{u_0} = \{x \in \Omega :\ \nabla u_0(x) = 0\} \tag{1.6}$$

and define $Q_{u_0} : W_0^{1,p}(\Omega) \to]-\infty, +\infty]$ by

$$Q_{u_0}(v) = \begin{cases} \displaystyle\int_{\Omega\setminus Z_{u_0}} \Psi''(\nabla u_0)[\nabla v]^2\,dx + \int_\Omega D_s g(x, u_0) v^2\,dx & \text{if } \nabla v(x) = 0 \text{ a.e. in } Z_{u_0}\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Finally, define the *Morse index of f at u_0* (denoted by $m(f, u_0)$) as the supremum of the dimensions of the linear subspaces V of $C_c^1(\Omega)$ such that

$$Q_{u_0}(v) < 0 \qquad \text{for any } v \in V \setminus \{0\}$$

and the *large Morse index of f at u_0* (denoted by $m^*(f, u_0)$) as the supremum of the dimensions of the linear subspaces V of $W_0^{1,p}(\Omega)$ such that

$$Q_{u_0}(v) \le 0 \qquad \text{for any } v \in V\,.$$

We clearly have $m(f, u_0) \le m^*(f, u_0)$.

Now we can state our main results.

Theorem 1. *Let $\kappa > 0$ and let $u_0 \in W_0^{1,p}(\Omega)$ be a critical point of the functional f defined in* (1.3)*. Then we have $m^*(f, u_0) < +\infty$ and*

$$C_m(f, u_0) = \{0\} \qquad \text{whenever } m < m(f, u_0) \text{ or } m > m^*(f, u_0)\,.$$

Remark 1. Since the value of κ is irrelevant in the case $p = 2$, Theorem 1 covers also the case $\kappa = 0$ with $p = 2$.

Remark 2. If $u_0 \in W^{1,\infty}(\Omega)$, from inequality (1.4) we infer that

$$\left\{ v \mapsto \int_\Omega \Psi''(\nabla u_0)[\nabla v]^2\,dx \right\}$$

is well behaved in the space $W_0^{1,2}(\Omega)$, which is compactly embedded in $L^2(\Omega)$, and this is on the basis of the technique used in [9, 10].

As we have already observed, in our case $u_0 \in W_0^{1,p}(\Omega)$ and we cannot expect any further regularity, because of the presence of the term $\Lambda \in W^{-1,p'}(\Omega)$. As a substitute, we will prove in Proposition 2 a $W_0^{1,p}(\Omega)$-coercivity which allows, in combination with the compact embedding of $W_0^{1,p}(\Omega)$ in $L^2(\Omega)$ due to assumption (Ψ_1), to recover enough information to prove our results.

In the case $\kappa = 0$ and $p < 2$, we can prove that critical groups of high dimension are trivial.

Theorem 2. *Let $\kappa = 0$ and $p < 2$. Let $u_0 \in W_0^{1,p}(\Omega)$ be a critical point of the functional f defined in* (1.3)*. Then we have $m^*(f, u_0) < +\infty$ and*

$$C_m(f, u_0) = \{0\} \qquad \textit{whenever } m > m^*(f, u_0)\,.$$

2. Parametric minimization

First of all, let us recall some basic facts. The next concept is taken from [4, 20].

Definition 2. Let X be a reflexive Banach space and $D \subseteq X$. A map $F : D \to X'$ is said to be *of class* $(S)_+$ if, for every sequence (u_k) in D weakly convergent to u in X with

$$\limsup_k \langle F(u_k), u_k - u\rangle \le 0\,,$$

we have $\|u_k - u\| \to 0$.

The next result is contained for instance in [10, Proposition 2.5].

Proposition 1. *Let X be a reflexive Banach space, let $f : X \to \mathbb{R}$ be a function of class C^1 and let C be a closed and convex subset of X. Assume that f' is of class $(S)_+$ on C.*

Then the following facts hold:

(*a*) *f is sequentially lower semicontinuos on C with respect to the weak topology;*
(*b*) *if (u_k) is a sequence in C weakly convergent to u with*

$$\limsup_k f(u_k) \le f(u)\,,$$

we have $\|u_k - u\| \to 0$.

Throughout this section, u_0 will denote a critical point of the functional f defined in (1.3). We will also denote by $\|\ \|_p$ the usual norm of L^p.

Given a continuous function $\Phi : \mathbb{R}^N \to \mathbb{R}$, for any $x, v \in \mathbb{R}^N$ we set

$$\underline{\Phi}''(x)[v]^2 = \liminf_{\substack{y\to x\\ t\to 0\\ w\to v}} \frac{\Phi(y+tw) + \Phi(y-tw) - 2\Phi(y)}{t^2}\,.$$

Then the function $\left\{(x, v) \mapsto \underline{\Phi}''(x)[v]^2\right\}$ is lower semicontinuous. If Φ is convex, it is also clear that $\underline{\Phi}''(x)[v]^2 \in [0, +\infty]$ and that $\underline{\Phi}''(x)[0]^2 = 0$. In particular, it is easily seen that

$$\kappa = 0 \text{ and } p < 2 \implies \underline{\Psi}''_{p,\kappa}(\eta)[\xi]^2 = \begin{cases} +\infty & \text{if } \eta = 0 \text{ and } \xi \ne 0\,,\\ 0 & \text{if } \eta = \xi = 0\,.\end{cases}$$

Since $(\Psi - \nu\,\Psi_{p,\kappa})$ is convex, we also have

$$\kappa = 0 \text{ and } p < 2 \implies \underline{\Psi}''(\eta)[\xi]^2 = \begin{cases} +\infty & \text{if } \eta = 0 \text{ and } \xi \ne 0\,,\\ 0 & \text{if } \eta = \xi = 0\,,\end{cases}$$

while $\underline{\Psi}''(\eta)[\xi]^2 = \Psi''(\eta)[\xi]^2$ in the other cases. In particular, the function $\left\{\xi \mapsto \underline{\Psi}''(\eta)[\xi]^2\right\}$ is convex for any $\eta \in \mathbb{R}^N$.

Proposition 2. *For every $u, v \in W_0^{1,p}(\Omega)$, we have*

$$(p-1)\nu\|\nabla v\|_p^2 \leq \left(\int_\Omega \left(\kappa^2 + |\nabla u|^2\right)^{\frac{p}{2}}\, dx\right)^{\frac{2-p}{p}} \int_\Omega \underline{\Psi}''(\nabla u)[\nabla v]^2\, dx$$

with the convention $0 \cdot (+\infty) = +\infty$.

Proof. If $\kappa > 0$ and $p < 2$, we have

$$|\nabla v|^p = \left(\kappa^2 + |\nabla u|^2\right)^{\frac{(2-p)p}{4}} \frac{|\nabla v|^p}{(\kappa^2 + |\nabla u|^2)^{\frac{(2-p)p}{4}}} \qquad \text{a.e. in } \Omega\,.$$

From Hölder's inequality we infer that

$$\left(\int_\Omega |\nabla v|^p\, dx\right)^{\frac{2}{p}} \leq \left(\int_\Omega \left(\kappa^2 + |\nabla u|^2\right)^{\frac{p}{2}}\, dx\right)^{\frac{2-p}{p}} \int_\Omega \frac{|\nabla v|^2}{(\kappa^2 + |\nabla u|^2)^{\frac{2-p}{2}}}\, dx$$

and the assertion follows from (1.4).

Taking into account (1.5), the case $\kappa = 0$ and $p < 2$ can be proved in a similar way, while the case $p = 2$ is obvious. ☐

The next result is contained in [10, Proposition 3.1].

Proposition 3. *For every $u, v \in W_0^{1,p}(\Omega)$, the function*

$$\left\{(x,t) \mapsto (1-t)\underline{\Psi}''\big(\nabla u(x) + t(\nabla v(x) - \nabla u(x))\big)\left[\nabla v(x) - \nabla u(x)\right]^2\right\}$$

belongs to $L^1(\Omega\times]0,1[)$ and one has

$$\begin{aligned}\int_\Omega \Psi(\nabla v)\, dx - \int_\Omega \Psi(\nabla u)\, dx - \int_\Omega \nabla\Psi(\nabla u)\cdot(\nabla v - \nabla u)\, dx \\ = \int_0^1 (1-t)\left\{\int_\Omega \underline{\Psi}''\big(\nabla u(x) + t(\nabla v(x) - \nabla u(x))\big)\left[\nabla v(x) - \nabla u(x)\right]^2 dx\right\} dt\,.\end{aligned}$$

Theorem 3. *Let (u_k), (v_k) be two sequences in $W_0^{1,p}(\Omega)$ such that (u_k) is convergent to u in $W_0^{1,p}(\Omega)$, while (v_k) is weakly convergent to v in $W_0^{1,p}(\Omega)$.*

Then we have

$$\begin{aligned}\int_\Omega \underline{\Psi}''(\nabla u)[\nabla v]^2\, dx + \int_\Omega D_s g(x,u)v^2\, dx \\ \leq \liminf_k \left(\int_\Omega \underline{\Psi}''(\nabla u_k)[\nabla v_k]^2\, dx + \int_\Omega D_s g(x,u_k)v_k^2\, dx\right).\end{aligned}$$

Proof. We have

$$\begin{aligned}D_s g(x,u_k)v_k^2 - D_s g(x,u)v^2 \\ = \left(D_s g(x,u_k)v_k^2 - D_s g(x,u)v_k^2\right) + \left(D_s g(x,u)v_k^2 - D_s g(x,u)v^2\right).\end{aligned}$$

Since (Ψ_1) implies $p^* > 2$ in the case $p < N$, from (g) we infer that

$$\int_\Omega D_s g(x,u)v^2\, dx = \lim_k \int_\Omega D_s g(x,u_k)v_k^2\, dx\,.$$

Then the assertion follows from the Theorem in [17]. ☐

Proposition 4. *There exists a direct sum decomposition*

$$W_0^{1,p}(\Omega) = V \oplus W$$

such that:

(a) $\dim V = m^*(f, u_0) < +\infty$ *and* W *is closed in* $W_0^{1,p}(\Omega)$*;*
(b) *we have*

$$\begin{aligned}\int_\Omega \underline{\Psi}''(\nabla u_0)[\nabla(v+w)]^2\,dx &+ \int_\Omega D_s g(x,u_0)(v+w)^2\,dx \\ &= \int_\Omega \underline{\Psi}''(\nabla u_0)[\nabla v]^2\,dx + \int_\Omega D_s g(x,u_0)v^2\,dx \\ &\qquad + \int_\Omega \underline{\Psi}''(\nabla u_0)[\nabla w]^2\,dx + \int_\Omega D_s g(x,u_0)w^2\,dx \\ &\qquad\qquad \textit{for any } v\in V \textit{ and } w \in W\,,\end{aligned}$$

$$\int_\Omega \underline{\Psi}''(\nabla u_0)[\nabla v]^2\,dx + \int_\Omega D_s g(x,u_0)v^2\,dx \le 0 \qquad \textit{for any } v \in V\,,$$

$$\int_\Omega \underline{\Psi}''(\nabla u_0)[\nabla w]^2\,dx + \int_\Omega D_s g(x,u_0)w^2\,dx > 0 \qquad \textit{for any } w \in W\setminus\{0\}\,.$$

Proof. Let us treat the case $\kappa = 0$ and $p < 2$. The case $\kappa > 0$ is similar and even simpler. If $u_0 = 0$, we have

$$\int_\Omega \underline{\Psi}''(\nabla u_0)[\nabla v]^2\,dx = +\infty \qquad \text{for any } v \in W_0^{1,p}(\Omega)\setminus\{0\}$$

and the assertion is true with $V = \{0\}$ and $W = W_0^{1,p}(\Omega)$. Otherwise, let

$$X_{u_0} = \left\{ v \in W_0^{1,p}(\Omega):\quad \nabla v(x) = 0 \text{ a.e. in } Z_{u_0} \text{ and } \frac{|\nabla v|^2}{|\nabla u_0|^{2-p}} \in L^1(\Omega\setminus Z_{u_0})\right\}\,,$$

where Z_{u_0} is defined in (1.6). By Proposition 2 it follows that

$$(v|w)_{u_0} = \int_{\Omega\setminus Z_{u_0}} \Psi''(\nabla u_0)\,[\nabla v, \nabla w]\,dx$$

is a scalar product on X_{u_0} which makes X_{u_0} a Hilbert space continuously embedded in $W_0^{1,p}(\Omega)$, which is in turn compactly embedded in $L^2(\Omega)$.

Moreover, we have that $\{v \mapsto D_s g(x,u_0)v\}$ is a compact operator from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$, hence from X_{u_0} into X'_{u_0}.

It follows that there exist a finite-dimensional subspace V of X_{u_0} and a closed subspace $\widetilde{W}$ of $L^2(\Omega)$ such that $L^2(\Omega)$ is the orthogonal sum of V and $\widetilde{W}$ and

$$\forall v \in V, \forall w \in X_{u_0} \cap \widetilde{W} : \int_{\Omega\setminus Z_{u_0}} \Psi''(\nabla u_0)[\nabla v, \nabla w]\, dx + \int_\Omega D_s g(x, u_0) vw\, dx = 0\,,$$

$$\forall v \in V : \qquad \int_{\Omega\setminus Z_{u_0}} \Psi''(\nabla u_0)[\nabla v]^2\, dx + \int_\Omega D_s g(x, u_0) v^2\, dx \le 0\,,$$

$$\forall w \in X_{u_0} \cap \widetilde{W} : \qquad w \neq 0 \Longrightarrow \int_{\Omega\setminus Z_{u_0}} \Psi''(\nabla u_0)[\nabla w]^2\, dx + \int_\Omega D_s g(x, u_0) w^2\, dx > 0\,.$$

In particular, if we set $W = W_0^{1,p}(\Omega) \cap \widetilde{W}$, we have $W_0^{1,p}(\Omega) = V \oplus W$ with W closed in $W_0^{1,p}(\Omega)$. Moreover, $\dim V$ is the supremum of the dimensions of the linear subspaces of X_{u_0} on which the quadratic form

$$\left\{ u \mapsto \int_{\Omega\setminus Z_{u_0}} \Psi''(\nabla u_0)[\nabla u]^2\, dx + \int_\Omega D_s g(x, u_0) u^2\, dx \right\}$$

is negative semidefinite. Since

$$\int_\Omega \underline{\Psi}''(\nabla u_0)[\nabla u]^2\, dx + \int_\Omega D_s g(x, u_0) u^2\, dx = +\infty \qquad \text{for any } u \in W_0^{1,p}(\Omega) \setminus X_{u_0}\,,$$

the assertion follows. ☐

In the following, we consider a direct sum decomposition as in the previous Proposition. We also set, for any $r > 0$,

$$B_r = \left\{ u \in W_0^{1,p}(\Omega) :\ \|\nabla u\|_p < r \right\}\,,$$
$$D_r = \left\{ u \in W_0^{1,p}(\Omega) :\ \|\nabla u\|_p \le r \right\}\,.$$

Lemma 1. *There exist $r, \delta > 0$ such that, for every $u \in (u_0 + D_r)$ and every $w \in W$, one has*

$$\int_\Omega \underline{\Psi}''(\nabla u)[\nabla w]^2\, dx + \int_\Omega D_s g(x, u) w^2\, dx \ge \delta \|\nabla w\|_p^2\,.$$

Proof. Assume, for a contradiction, that there exist a sequence (v_k) strongly convergent to u_0 in $W_0^{1,p}(\Omega)$ and a sequence (w_k) in W such that

$$\int_\Omega \underline{\Psi}''(\nabla v_k)[\nabla w_k]^2\, dx + \int_\Omega D_s g(x, v_k) w_k^2\, dx < \frac{1}{k}\, \|\nabla w_k\|_p^2\,. \tag{2.1}$$

Without loss of generality, we may assume that $\|\nabla w_k\|_p = 1$. Then, up to a subsequence, (w_k) is weakly convergent to some w in $W_0^{1,p}(\Omega)$. In particular, $w \in W$. From Theorem 3 we infer that

$$\int_\Omega \underline{\Psi}''(\nabla u_0)[\nabla w]^2\, dx + \int_\Omega D_s g(x, u_0) w^2\, dx \le 0\,,$$

whence $w = 0$.

Coming back to (2.1), now we deduce that

$$\lim_k \int_\Omega \underline{\Psi}''(\nabla v_k)[\nabla w_k]^2\, dx = 0\,.$$

By Proposition 2 we infer that $\nabla w_k \to 0$ in $L^p(\Omega)$. Since $\|\nabla w_k\|_p = 1$, a contradiction follows. $\square$

Theorem 4. *There exists $r > 0$ such that:*

(*a*) *the map f' is of class $(S)_+$ on $u_0 + D_{2r}$;*

(*b*) *for every $v \in V \cap D_r$, the C^1-functional*

$$\begin{array}{ccc} W & \to & \mathbb{R} \\ w & \mapsto & f(u_0 + v + w) \end{array}$$

is strictly convex on $W \cap D_r$.

Proof. By [3, Theorem 3.4] there exists $r > 0$ such that the map f' is of class $(S)_+$ on $u_0 + D_{2r}$. By decreasing r we infer by Lemma 1 that

$$\int_\Omega \underline{\Psi}''(\nabla(u_0 + u))[\nabla w]^2\, dx + \int_\Omega D_s g(x, u_0 + u) w^2\, dx \geq \delta \|\nabla w\|_p^2 \tag{2.2}$$

for every $u \in D_{2r}$ and every $w \in W$.

If $v \in V \cap D_r$, $t \in [0,1]$ and $w_0, w_1 \in W \cap D_r$, by Proposition 3 and (2.2) we deduce that

$$\begin{aligned} &(1-t) f(u_0 + v + w_0) + t f(u_0 + v + w_1) \\ &\qquad \geq f(u_0 + v + (1-t) w_0 + t w_1) + \frac{\delta}{2} t(1-t) \|\nabla w_1 - \nabla w_0\|_p^2 . \end{aligned}$$

Therefore $\{w \mapsto f(u_0 + v + w)\}$ is strictly convex on $W \cap D_r$. $\square$

Theorem 5. *There exist $r > 0$ and $\varrho \in]0, r]$ such that, for every $v \in V \cap D_\varrho$, there exists one and only one $\overline{w} \in W \cap D_r$ such that*

$$f(u_0 + v + \overline{w}) \leq f(u_0 + v + w) \qquad \text{for any } w \in W \cap D_r .$$

Moreover, $\overline{w} \in B_r$ and $\overline{w}$ is the unique critical point of $\{w \mapsto f(u_0 + v + w)\}$ in $W \cap D_r$.

Finally, if we set $\psi(v) = \overline{w}$, the map ψ is continuous from $V \cap D_\varrho$ into $W_0^{1,p}(\Omega)$ with $\psi(0) = 0$.

Proof. Let $r > 0$ be as in Theorem 4. Since u_0 is a critical point of f and the functional $\{w \mapsto f(u_0 + w)\}$ is strictly convex on $W \cap D_r$, we have $f(u_0) < f(u_0 + w)$ for every $w \in W \cap D_r$ with $w \neq 0$.

We claim that there exists $\varrho \in]0, r]$ such that

$$f(u_0 + v) < f(u_0 + v + w) \qquad \text{for any } v \in V \cap D_\varrho \text{ and any } w \in W \text{ with } \|\nabla w\|_p = r.$$

By contradiction, let (v_k) be a sequence in V with $v_k \to 0$ and let (w_k) be a sequence in W with $\|\nabla w_k\|_p = r$ and $f(u_0 + v_k) \geq f(u_0 + v_k + w_k)$. Up to a subsequence, (w_k) is weakly convergent to some $w \in W \cap D_r$. Then $(u_0 + v_k + w_k)$ is weakly convergent to $u_0 + w$ with

$$\limsup_k f(u_0 + v_k + w_k) \leq \lim_k f(u_0 + v_k) = f(u_0) \leq f(u_0 + w) .$$

Combining Proposition 1 with Theorem 4, we deduce that $(u_0 + v_k + w_k)$ is strongly convergent to $u_0 + w$, whence $f(u_0 + w) = f(u_0)$ with $\|\nabla w\|_p = r$, and a contradiction follows.

Again by Theorem 4, for any $v \in V \cap D_\varrho$ there exists one and only one minimum point $\overline{w} \in W \cap D_r$ and in fact $\overline{w} \in B_r$. In particular, we have

$$\langle f'(u_0 + v + \overline{w}), w \rangle = 0 \qquad \text{for any } w \in W .$$

If $v = 0$, then $\overline{w} = 0$.

Finally, we set $\psi(v) = \overline{w}$. If (v_k) is convergent to v in $V \cap D_\varrho$, up to a subsequence $(\psi(v_k))$ is weakly convergent to some w in $W \cap D_r$. Since

$$f(u_0 + v_k + \psi(v_k)) \leq f(u_0 + v_k + z) \qquad \text{for any } z \in W \cap D_r \,,$$

from Proposition 1 we infer that

$$f(u_0 + v + w) \leq f(u_0 + v + z) \qquad \text{for any } z \in W \cap D_r \,,$$

whence $w = \psi(v)$. Then the choice $z = \psi(v)$ implies that

$$\limsup_k f(u_0 + v_k + \psi(v_k)) \leq f(u_0 + v + \psi(v)) \,,$$

whence $(v_k + \psi(v_k)) \to (v + \psi(v))$ strongly in $W_0^{1,p}(\Omega)$. Therefore the map ψ is continuous from $V \cap D_\varrho$ into W endowed with the topology of $W_0^{1,p}(\Omega)$. $\square$

3. The finite dimensional reduction

Let u_0 still denote a critical point of the functional f defined in (1.3). We also keep the notations of Theorem 5 and define the reduced functional $\varphi : V \cap B_\varrho \to \mathbb{R}$ as

$$\varphi(v) = f(u_0 + v + \psi(v)) = \min\left\{f(u_0 + v + w) : \; w \in W \cap D_r\right\} .$$

Theorem 6. *The functional φ is of class C^1 and*

$$\langle \varphi'(z), v\rangle = \langle f'(u_0 + z + \psi(z)), v\rangle \qquad \textit{for any } z \in V \cap B_\varrho \textit{ and } v \in V \,. \tag{3.1}$$

In particular, 0 is a critical point of φ. Moreover, we have

$$C_m(\varphi, 0) \approx C_m(f, u_0) \qquad \textit{for any } m \geq 0 \,.$$

Finally, 0 is an isolated critical point of φ if and only if u_0 is an isolated critical point of f.

Proof. For any $v_0, v_1 \in V \cap B_\varrho$, we have

$$\begin{aligned}
\varphi(v_1) &= f(u_0 + v_1 + \psi(v_1)) \\
&= f(u_0 + v_0 + \psi(v_1)) + \langle f'(u_0 + v_0 + t(v_1 - v_0) + \psi(v_1)), v_1 - v_0\rangle \\
&\geq f(u_0 + v_0 + \psi(v_0)) + \langle f'(u_0 + v_0 + t(v_1 - v_0) + \psi(v_1)), v_1 - v_0\rangle \\
&= \varphi(v_0) + \langle f'(u_0 + v_0 + t(v_1 - v_0) + \psi(v_1)), v_1 - v_0\rangle
\end{aligned}$$

for some $t \in]0,1[$. Since ψ is continuous from $V \cap B_\varrho$ into $W_0^{1,p}(\Omega)$, it follows that

$$\liminf_{\substack{(v_0,v_1)\to(z,z)\\ v_0\neq v_1}} \frac{\varphi(v_1) - \varphi(v_0) - \langle f'(u_0 + z + \psi(z)), v_1 - v_0\rangle}{\|v_1 - v_0\|} \geq 0 \,.$$

We also have

$$\begin{aligned}
\varphi(v_1) &= f(u_0 + v_1 + \psi(v_1)) \leq f(u_0 + v_1 + \psi(v_0)) \\
&= f(u_0 + v_0 + \psi(v_0)) + \langle f'(u_0 + v_0 + t(v_1 - v_0) + \psi(v_0)), v_1 - v_0\rangle \\
&= \varphi(v_0) + \langle f'(u_0 + v_0 + t(v_1 - v_0) + \psi(v_0)), v_1 - v_0\rangle
\end{aligned}$$

for some $t \in]0,1[$, whence

$$\limsup_{\substack{(v_0,v_1)\to(z,z)\\ v_0\neq v_1}} \frac{\varphi(v_1) - \varphi(v_0) - \langle f'(u_0 + z + \psi(z)), v_1 - v_0\rangle}{\|v_1 - v_0\|} \leq 0 \,.$$

Therefore φ is of class C^1 with

$$\langle \varphi'(z), v\rangle = \langle f'(u_0 + z + \psi(z)), v\rangle \,.$$

Since $\psi(0) = 0$, we also have $\varphi'(0) = 0$.

Now consider

$$Y = \{u_0 + z + \psi(z) : \; z \in V \cap B_\rho\}$$

endowed with the $W_0^{1,p}(\Omega)$-topology. Since $\{z \mapsto u_0 + z + \psi(z)\}$ is a homeomorphism from $V \cap B_\rho$ onto Y which sends 0 into u_0, it is clear that

$$C_m(\varphi, 0) \approx C_m(f\big|_Y, u_0) \qquad \text{for any } m \geq 0\,.$$

Now set

$$U = u_0 + (V \cap B_\varrho) + (W \cap D_r)\,.$$

Since

$$\{w \mapsto f(u_0 + z + w)\}$$

is convex on $W \cap D_r$ for any $z \in V \cap B_\varrho$, we have that

$$\mathcal{H}(u_0 + z + w, t) = u_0 + z + (1-t)w + t\psi(z)$$

defines a strong deformation retraction of

$$\Big(f^c \cap U, (f^c \setminus \{u_0\}) \cap U\Big)$$

onto

$$\Big(f^c \cap Y \cap U, (f^c \setminus \{u_0\}) \cap Y \cap U\Big)\,.$$

It follows

$$H^m(f^c, f^c \setminus \{u_0\}) \approx H^m(f^c \cap Y, (f^c \cap Y) \setminus \{u_0\})\,,$$

whence

$$C_m(\varphi, 0) \approx C_m(f\big|_Y, u_0) \approx C_m(f, u_0) \qquad \text{for any } m \geq 0\,.$$

Since any critical point u of f in $u_0+(V\cap B_\varrho)+(W\cap D_r)$ must be of the form $u = u_0+z+\psi(z)$ with $z \in V \cap B_\varrho$, from (3.1) we infer that 0 is isolated for φ if and only if u_0 is isolated for f. $\square$

4. Proof of the main results

Proof of Theorems 2 and 1.
By Proposition 4, we have $m^*(f, u_0) = \dim V < +\infty$. From Theorem 6 we also know that

$$C_m(f, u_0) \approx C_m(\varphi, 0) \qquad \text{for any } m \geq 0\,.$$

Since the critical groups are defined using Alexander-Spanier cohomology, it is clear that $C_m(\varphi, 0) = \{0\}$ whenever $m > \dim V$, both in the case $\kappa > 0$ and in the case $\kappa = 0$ with $p < 2$.

Now assume that $\kappa > 0$. Let V_- be a linear subspace of $C_c^1(\Omega)$ of dimension $m(f, u_0)$ such that

$$\int_\Omega \Psi''(\nabla u_0)[\nabla v]^2\, dx + \int_\Omega D_s g(x, u_0) v^2\, dx < 0 \quad \text{for any } v \in V_- \setminus \{0\}\,.$$

By (1.4) we have

$$\Psi''(\eta)[\xi]^2 \leq \frac{C}{(\kappa^2 + |\eta|^2)^{\frac{2-p}{2}}}\,|\xi|^2 \leq \frac{C}{\kappa^{2-p}}\,|\xi|^2 \qquad \text{for any } \eta, \xi \in \mathbb{R}^N\,.$$

Then, it is easily seen that, for any $u \in W_0^{1,p}(\Omega)$, the function

$$f_u(v) = f(u+v)$$

is of class C^2 on V_- with

$$(f_u)''(z)[v]^2 = \int_\Omega \Psi''(\nabla u + \nabla z)[\nabla v]^2\,dx + \int_\Omega D_s g(x, u+z)v^2\,dx$$

and, for any $z, v \in V_-$, the function $\left\{u \mapsto (f_u)''(z)[v]^2\right\}$ is continuous on $W_0^{1,p}(\Omega)$. Moreover, we have

$$(f_{u_0})''(0)[v]^2 < 0 \quad \text{for any } v \in V_- \setminus \{0\}\,.$$

From [18, Theorem 3.1] it follows that $C_m(f, u_0) = \{0\}$ whenever $m < \dim V_- = m(f, u_0)$. □

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Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via Orabona 4, 70125 Bari, Italy

E-mail address: `silvia.cingolani@poliba.it`

Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, Via dei Musei 41, 25121 Brescia, Italy

E-mail address: `marco.degiovanni@unicatt.it`

Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via Orabona 4, 70125 Bari, Italy

E-mail address: `giuseppina.vannella@poliba.it`