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ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF NONLOCAL p-LAPLACE EQUATIONS DEPENDING ON THE L^p NORM OF THE GRADIENT

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Abstract. In this paper we extend some results regarding the asymptotic behaviour of a class of nonlocal nonlinear parabolic problems, which have been previously considered in [7]. In particular, we obtain a local stability result for isolated local minima of the energy functional associated to this class of problems.

1. Introduction

In this paper we consider the asymptotic behaviour of the solution $u = u(x,t)$ of the following problem

$$\begin{cases} u_t - \nabla \cdot a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 1$ with Lipschitz boundary Γ. In what follows we assume a' is continuous and that there exist constants λ, Λ such that

$$0 < \lambda \leq a(\mu) \leq \Lambda \quad \forall \mu \in \mathbb{R}. \tag{1.2}$$

By $|\cdot|_p$ we denote the $L^p(\Omega)$-norm, $2 \leq p < +\infty$ and we assume

$$f = f(x) \in L^2(\Omega),\ u_0 \in W_0^{1,p}(\Omega),\ \frac{1}{p} + \frac{1}{q} = 1. \tag{1.3}$$

The motivation to study this type of problems can be found in [1], [3] –[8] and the references therein. This problem has been considered in our previous work [7], where the existence and uniqueness of a weak solution has been obtained and the question of the asymptotic behaviour has been addressed. In particular, we know that if the stationary problem has a unique solution, then the solution of problem (1.1) converges to this unique equilibrium.

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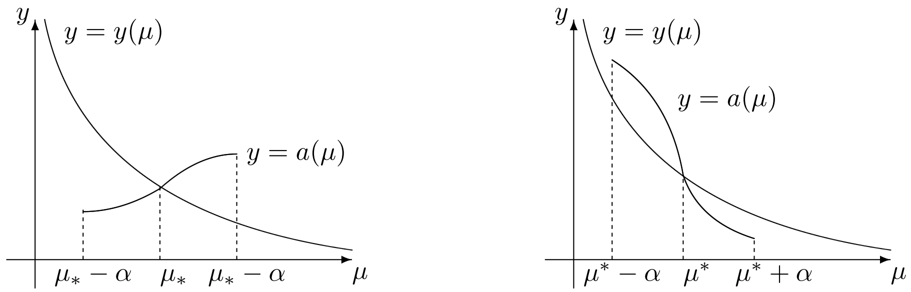
However, it has been shown that the corresponding stationary problem may have from one up to a continuum of solutions, which are also critical points of the energy functional:

$$E(u) = \frac{1}{p} A\left(\int_\Omega |\nabla u|^p dx\right) - \int_\Omega f u dx \tag{1.4}$$

with

$$A(z) = \int_0^z a(s)ds. \tag{1.5}$$

Furthermore, in [7] it was shown that the critical points can be either local minima or saddle points of the energy functional (1.4), depending on the function a (see Figure 1.1 and (4.3)). We already know [7] that an isolated global minimum of E is asymptotically stable. The



(a) Case of a local minimum

(b) Case of a saddle point

Figure 1.1

main result of this paper is the following theorem:

Theorem 1.1. *Under the assumptions above the isolated local minimizers of the energy E defined by* (1.4) *are asymptotically stable.*

The paper is organized as follows. In the next Section we formulate and prove some auxiliary lemmas, which are used throughout the paper. In Section 3 existence, uniqueness of a strong solution and its convergence to a stationary solution is shown. In the last Section we describe the proof of Theorem 1.1.

2. Some auxiliary lemmas

Lemma 2.1. *Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function with $g(x) > 0$ $\forall x > 0$ or such that*

$$\forall \alpha > 0 \text{ small}, \qquad \sup_{[\alpha, 2\alpha]} g = C_\alpha > 0. \tag{2.1}$$

Let y, h be nonnegative functions, y continuous such that

$$\int_0^{+\infty} y(s)ds, \int_0^{+\infty} h(s)ds < +\infty, \tag{2.2}$$

$$y(t) - y(s) \le \int_s^t (g(y(\xi)) + h(\xi))d\xi, \ \forall s < t.$$

Then it holds that

$$\lim_{t\to+\infty} y(t) = 0.$$

Proof. From the condition $\int_0^{+\infty} y(s)ds$ we have that $\liminf_{t\to+\infty} y(t) = 0$.
Suppose that $\limsup_{t\to+\infty} y(t) > 0$ and choose α such that $\limsup_{t\to+\infty} y(t) > 2\alpha$. By the mean value theorem one can find a sequence of disjoint intervals (t_n, t_n'), $t_n \to +\infty$ such that

$$y(t_n) = \alpha \leq y(t) \leq 2\alpha = y(t_n') \quad \forall t \in (t_n, t_n').$$

Then from the last inequality of (2.2) and (2.1) it holds that

$$\alpha = y(t_n') - y(t_n) \leq \int_{t_n}^{t_n'} g(y(s))ds + \int_{t_n}^{t_n'} h(s)ds \leq C_\alpha(t_n' - t_n) + \int_{t_n}^{t_n'} h(s)ds.$$

For $n \geq n_0$ large enough, by (2.2), $\int_{t_n}^{t_n'} h(s)ds \leq \frac{\alpha}{2}$ and from above we get

$$t_n' - t_n \geq \frac{\alpha}{2C_\alpha}.$$

It follows that

$$\int_{t_{n_0}}^{+\infty} y(s)ds \geq \sum_{n\geq n_0} \int_{t_n}^{t_n'} y(s)ds \geq \sum_{n\geq n_0} \frac{\alpha^2}{2C_\alpha} = +\infty$$

and a contradiction. □

Lemma 2.2. *Let $p \geq 2$, $a, b \in \mathbb{R}$. Then*

$$\int_0^1 (1-s)|a+sb|^{p-2}|b|^2 ds \geq \frac{1}{8(18)^{\frac{p}{2}}}|b|^p.$$

Proof.

(i) Let us first assume that $|a| \geq |b|$. Then we have that

$$|a+sb| \geq |a| - s|b| \geq |b| - s|b| = (1-s)|b|, \quad s \in [0,1].$$

Consider now

$$\int_0^1 (1-s)|a+sb|^{p-2}|b|^2 ds \geq \int_0^1 (1-s)^{p-1}|b|^p ds = \frac{|b|^p}{p}$$

and the statement of the lemma holds.

(ii) Let now $|a| < |b|$. Then we see

$$|a+sb| \leq |a| + s|b| < (1+s)|b| \leq 2|b| \quad s \in [0,1].$$

Hence,

$$\int_0^1 (1-s)|a+sb|^{p-2}|b|^2 ds = \int_0^1 (1-s)\frac{|a+sb|^p}{|a+sb|^2}|b|^2 ds \geq \frac{1}{4}\int_0^1 (1-s)|a+sb|^p ds.$$

Since $\int_0^1 2(1-s)ds = 1$ and $X \to X^{\frac{p}{2}}$ is convex by Jensens's inequality we get

$$\int_0^1 (1-s)(|a+sb|^2)^{\frac{p}{2}} ds \geq \frac{1}{2}\Big(\int_0^1 2(1-s)(|a|^2 + 2sab + s^2|b|^2)ds\Big)^{\frac{p}{2}}$$
$$= \frac{1}{2}\Big(|a|^2 + \frac{2}{3}ab + \frac{1}{6}|b|^2\Big)^{\frac{p}{2}} \geq \frac{1}{2}\Big(|a|^2 - \frac{2}{3}|a||b| + \frac{1}{6}|b|^2\Big)^{\frac{p}{2}}.$$

Using the Young inequality $ab \leq \dfrac{3a^2}{2} + \dfrac{b^2}{6}$ and combining the two inequalities above we obtain the statement of the lemma.

□

3. Asymptotic behaviour and regularity

Theorem 3.1. *Let the assumptions above hold. Then for any $T > 0$ there exists a unique L^2-strong solution u of* (1.1) *such that*

$$u \in C([0,T]; W_0^{1,p}(\Omega)), \quad u_t,\ \nabla \cdot |\nabla u|^{p-2}\nabla u \in L^2(0,T;L^2(\Omega)). \tag{3.1}$$

Moreover, $u(t) = u(\cdot, t)$ converges to a stationary point in $W_0^{1,p}(\Omega)$ when t goes to infinity.

Proof. Consider $\varphi_1, \dots, \varphi_n, \dots$ a basis in $W^{2,2(p-1)}(\Omega)$ such that

$$\varphi_i \in H_0^s(\Omega),\ (\varphi_i, v)_{H_0^s(\Omega)} = \mu_i(\varphi_i, v)_{L^2(\Omega)}\ \forall v \in H_0^s(\Omega), \tag{3.2}$$

where s is chosen in such a way that $H_0^s(\Omega) \subset W^{2,2(p-1)}(\Omega)$ (see [11]). We will suppose that φ_i are orthonormal in $L^2(\Omega)$ ($W^{2,2(p-1)}(\Omega) \subset L^2(\Omega)$, since $p \geq 2$). If $u_0 = \sum_i \beta_i\varphi_i$ consider

$$u_n(t) = \sum_{i=1}^n \gamma_i(t)\varphi_i$$

solution to

$$\begin{cases} \displaystyle\int_\Omega u_n' v dx + a(\|\nabla u_n\|_p^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx = \int_\Omega f v dx \\ \forall v \in [\varphi_1, \dots, \varphi_n], \\ u_n(0) = \displaystyle\sum_{i=1}^n \beta_i\varphi_i, \end{cases} \tag{3.3}$$

where $[\varphi_1,\dots,\varphi_n]$ is the space spanned by $\varphi_1,\dots,\varphi_n$. Taking $v=\varphi_j$ and using the fact that the φ_i's are orthonormal we see that (3.3) is equivalent to the Cauchy problem

$$\tag{3.4}\begin{cases}\gamma_j'(t)=-a\Big(\Big\|\sum_{i=1}^n\gamma_i(t)\nabla\varphi_i\Big\|_p^p\Big)\int_\Omega\Big|\sum_{i=1}^n\gamma_i(t)\nabla\varphi_i\Big|^{p-2}\sum_{i=1}^n\gamma_i(t)\nabla\varphi_i\nabla\varphi_j dx\\ \qquad\qquad\qquad +\int_\Omega f\varphi_j dx,\quad \forall j=1,\dots n,\\ \gamma_j(0)=\beta_j,\quad \forall j=1,\dots n.\end{cases}$$

By the existence theorem for ordinary differential equations this Cauchy problem possesses a solution $\gamma_j\in C^2([0,\delta))$, $\delta>0$. Taking $v=u_n'$ in (3.3) we get

$$\tag{3.5}\int_\Omega|u_n'|^2dx+\frac{d}{dt}E(u_n(t))=0,$$

where E is defined by (1.4). By integration we obtain

$$\tag{3.6}\int_0^t\int_\Omega|u_n'|^2dxdt=E(u_n(0))-E(u_n(t))\le C,$$

since E is uniformly bounded from below. Indeed, from (1.4) using (1.2), Hölder's and Young's inequalities and the fact that $W_0^{1,p}(\Omega)\subset L^2(\Omega)$ for $p\ge 2$ we have that

$$E(u_n)\ge\frac{\lambda}{p}\|\nabla u_n\|_p^p-|f|_2|u_n|_2\ge\frac{\lambda}{p}\|\nabla u_n\|_p^p-C|f|_2\|\nabla u_n\|_p\ge-\frac{(C|f|_2)^q}{q\lambda^{q-1}}$$

Hence, (3.6) implies that

$$\tag{3.7}u_n'\in L^2(0,T;L^2(\Omega))=L^2(Q_T),\quad Q_T=(0,T)\times\Omega.$$

We can now differentiate (3.3) with respect to t and since

$$\frac{d}{dt}|\nabla u_n|^p=\frac{d}{dt}\big(|\nabla u_n|^2\big)^{\frac{p}{2}}=\frac{p}{2}\big(|\nabla u_n|^2\big)^{\frac{p}{2}-1}\frac{d}{dt}|\nabla u_n|^2$$
$$=\frac{p}{2}|\nabla u_n|^{p-2}2\nabla u_n\nabla u_n'=p|\nabla u_n|^{p-2}\nabla u_n\nabla u_n'$$

we get

$$\int_\Omega u_n''vdx+pa'(\|\nabla u_n\|_p^p)\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla u_n'dx\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla vdx$$
$$+a(\|\nabla u_n\|_p^p)\int_\Omega(p-2)|\nabla u_n|^{p-4}\nabla u_n\nabla u_n'\nabla u_n\nabla v+|\nabla u_n|^{p-2}\nabla u_n'\nabla vdx=0.$$

Taking $v=u_n'$ and noting that the last term is nonnegative we get

$$\tag{3.8}\frac{1}{2}\frac{d}{dt}\int_\Omega|u_n'|^2dx\le-pa'(\|\nabla u_n\|_p^p)\Big(\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla u_n'dx\Big)^2.$$

From the first equation in (3.3) written with $v=u_n'$ we have

$$a(\|\nabla u_n\|_p^p)\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla u_n'dx=\int_\Omega fu_n'dx-\int_\Omega|u_n'|^2dx$$

and from (3.8) follows

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_n'|^2dx \le p\frac{|a'(\|\nabla u_n\|_p^p)|}{a^2(\|\nabla u_n\|_p^p)}\left(\int_\Omega f u_n' dx - \int_\Omega |u_n'|^2dx\right)^2.$$

Since $E(u_n)$ is uniformly bounded so is $\|\nabla u_n\|_p^p$. Due to the fact that $a\in C^1$ from Hölder's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_n'|^2dx \le C\left(\int_\Omega |f|^2dx + \int_\Omega |u_n'|^2dx\right)\int_\Omega |u_n'|^2dx. \tag{3.9}$$

Denote by $y_n(t)=|u_n'(t)|_2^2$. Integrating (3.9) we get

$$y_n(t)-y_n(s)\le 2C\int_s^t \big(|f|_2^2+y_n(\xi)\big)y_n(\xi)d\xi.$$

Passing to the limit in (3.6) as $t\to+\infty$ we obtain that

$$\int_0^{+\infty} y_n(s)ds<+\infty.$$

Hence, since $g(x)=2C(|f|_2^2x+x^2)>0$ on $x>0$ from Lemma 2.1 we derive

$$y_n(t)\to 0 \text{ as } t\to+\infty. \tag{3.10}$$

Thus y_n remains bounded in time. Remark that

$$\nabla\cdot|\nabla u|^{p-2}\nabla u = |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}\sum_{i,j=1}^n u_{x_i}u_{x_j}u_{x_ix_j}.$$

Applying twice the Cauchy-Schwarz inequality we get

$$\int_\Omega |\nabla\cdot|\nabla u|^{p-2}\nabla u|^2dx \le \frac{1}{2}\Big(\int_\Omega |\nabla u|^{2p-4}|\Delta u|^2dx + (p-2)^2\int_\Omega |\nabla u|^{2p-4}\sum_{i,j=1}^n u_{x_ix_j}^2dx\Big). \tag{3.11}$$

From Hölder's inequality with the exponents $\frac{p-1}{p-2}$, $p-1$ we get that

$$\int_\Omega |\nabla u|^{2p-4}|\Delta u|^2dx \le \left(\int_\Omega |\nabla u|^{2(p-1)}dx\right)^{\frac{p-2}{p-1}}\left(\int_\Omega |\Delta u|^{2(p-1)}dx\right)^{\frac{1}{p-1}}.$$

We can estimate the second term in (3.11) in a similar way. Thus, since $\varphi_j\in W^{2,2(p-1)}(\Omega)$, we can multiply the first equation in (3.4) by $\varphi_j\nabla\cdot|\nabla u_n|^{p-2}\nabla u_n$, then integrating over Ω and summing in j we get

$$\int_\Omega \nabla\cdot|\nabla u_n|^{p-2}\nabla u_n u_n' dx = a(\|\nabla u_n\|_p^p)\sum_{j=1}^n\left(\int_\Omega \nabla\cdot|\nabla u_n|^{p-2}\nabla u_n\varphi_j dx\right)^2 + \sum_{j=1}^n\int_\Omega f\varphi_j dx\int_\Omega \nabla\cdot|\nabla u_n|^{p-2}\nabla u_n\varphi_j dx.$$

Since $\varphi_1, \dots, \varphi_n$ are orthonormal in $L^2(\Omega)$ the equality above can be written as

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u_n' dx + a(\|\nabla u_n\|_p^p)|P_n(\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n)|_2^2 = -\int_\Omega P_n f \nabla \cdot |\nabla u_n|^{p-2}\nabla u_n dx,$$

where P_n denotes a projection operator from $L^2(\Omega)$ onto $[\varphi_1, \dots, \varphi_n]$. Then from (1.2), Hölder's and Young's inequalities we get

$$\frac{1}{p}\frac{d}{dt}\|\nabla u_n\|_p^p + \lambda |P_n(\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n)|_2^2 \le |(f, P_n(\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n))| \le |f|_2 |P_n(\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n)|_2 \le \frac{|f|_2^2}{2\lambda} + \frac{\lambda |P_n(\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n)|_2^2}{2}.$$

Therefore, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\nabla u_n\|_p^p + \frac{\lambda}{2}|P_n(\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n)|_2^2 \le \frac{|f|_2^2}{2\lambda}$$

And after integration in time

$$\frac{1}{p}\|\nabla u_n\|_p^p + \frac{\lambda}{2}\int_0^t |P_n(\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n)|_2^2 dt \le \frac{1}{p}\|\nabla u_0\|_p^p + \frac{|f|_2^2 T}{2\lambda}. \tag{3.12}$$

From (3.7), (3.12) follow that we can find a subsequence of n such that

$$u_n' \rightharpoonup u' \text{ in } L^2(Q_T),$$
$$P_n(\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n) \rightharpoonup \chi \quad \text{in} \quad L^2(Q_T).$$

One can prove (see a proof of existence for a weak solution [7]) that

$$\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n \rightharpoonup \nabla \cdot |\nabla u|^{p-2}\nabla u \quad \text{in} \quad L^q(0,T; W^{-1,q}(\Omega)) \subset L^q(0,T; H^{-s}(\Omega)),$$
$$\|\nabla u_n\|_p^p \to \|\nabla u\|_p^p \quad \text{a.e. } t.$$

Let $w \in L^2(\Omega)$, then $P_n w \in [\varphi_1, \dots, \varphi_n]$. Taking now in (3.3) $v = P_n w$ and passing to the limit ($P_n w \to w$ in $L^2(\Omega)$, see [12]), we obtain

$$\int_\Omega u' w dx - a(\|\nabla u\|_p^p)\int_\Omega \chi w dx = \int_\Omega f w dx \quad \forall w \in L^2(\Omega) \quad \text{in} \quad D'(0,T).$$

Remark that for $w \in H_0^s(\Omega)$ it holds that $P_n w \to w$ in $H_0^s(\Omega)$. Indeed,

$$w = \sum_{j=1}^\infty (\varphi_j, w)\varphi_j$$

and due to (3.2) we get that

$$\|w\|_{H_0^s(\Omega)}^2 = \sum_{j=1}^\infty |(\varphi_j, w)|^2 \mu_j < +\infty.$$

Then

$$\|P_n w - w\|_{H_0^s(\Omega)}^2 = \left\| \sum_{j=n+1}^\infty (\varphi_j, w)\varphi_j \right\|_{H_0^s(\Omega)}^2 = \sum_{j=n+1}^\infty |(\varphi_j, w)|^2 \mu_j \to 0.$$

Therefore for $w \in H_0^s(\Omega)$, $\varphi \in D(0,T)$ we obtain

$$\int_0^T \int_\Omega \chi w \varphi dxdt = \lim_{n\to+\infty} \int_0^T \int_\Omega P_n(\nabla \cdot |\nabla u_n|^{p-2}\nabla u_n) w\varphi dxdt$$
$$= \lim_{n\to+\infty} \int_0^T \int_\Omega \nabla \cdot |\nabla u_n|^{p-2}\nabla u_n P_n w\varphi dxdt = \int_0^T \int_\Omega \nabla \cdot |\nabla u|^{p-2}\nabla u w\varphi dxdt.$$

Hence, $\chi = \nabla \cdot |\nabla u|^{p-2}\nabla u$ and u is a solution to (1.1) and

$$u_t - a(\|\nabla u\|_p^p)\nabla \cdot |\nabla u|^{p-2}\nabla u = f \text{ in } L^2(\Omega).$$

It remains to show that $u \in C([0,T]; W_0^{1,p}(\Omega))$. By rescaling the time in the following way, setting

$$\alpha(t) = \int_0^t a(\|\nabla u(\cdot,s)\|_p^p)ds, \tag{3.13}$$

we reduce solving the problem (1.1) to solving the problem (see [9], [7]):

$$\begin{cases} w_t - \nabla \cdot |\nabla w|^{p-2}\nabla w = \dfrac{f}{a(\|\nabla w\|_p^p)} & \text{in } \Omega \times (0, \alpha(T)), \\ w = 0 & \text{on } \Gamma \times (0,\alpha(T)), \\ w(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.14}$$

where $w(x,\alpha(t)) = u(x,t)$. Then (we keep denoting the solution by u) multiplying the first equation in (3.14) by u_t and integrating over Ω we get

$$\int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^{p-2}\nabla u \nabla u_t dx = \int_\Omega \frac{f u_t}{a(\|\nabla u(\cdot,t)\|_p^p)} dx.$$

Using (1.2) and Hölder's and Young's inequalities we obtain

$$|u_t|_2^2 + \frac{1}{p}\frac{d}{dt}\|\nabla u\|_p^p \le \frac{1}{\lambda}|f|_2|u_t|_2 \le \frac{|f|_2^2}{2\lambda^2} + \frac{|u_t|_2^2}{2}.$$

Therefore, it holds that

$$\frac{d}{dt}\|\nabla u\|_p^p \le C|f|_2^2.$$

Integrating from t_0 to t we deduce

$$\|\nabla u(t)\|_p^p \le \|\nabla u(t_0)\|_p^p + C|f|_2^2(t-t_0).$$

Hence, letting $t \to t_0$ we get

$$\limsup_{t\to t_0} \|\nabla u(t)\|_p \le \|\nabla u(t_0)\|_p. \tag{3.15}$$

Recall that

$$\|\nabla u(t)\|_p \le C \quad \forall t \ge 0,$$

thus for a subsequence

$$\nabla u(t_k) \rightharpoonup \tilde{u} \text{ in } (L^p(\Omega))^n \text{ as } t_k \to t_0.$$

Note, that since $u \in C([0,T]; L^2(\Omega))$ we have $u(t) \to u(t_0)$ in $L^2(\Omega)$. Then for $\varphi \in (D(\Omega))^n$ we see

$$\int_\Omega \nabla u(t_k)\varphi dx = -\int_\Omega u(t_k)\nabla\varphi dx \to -\int_\Omega u(t_0)\nabla \varphi dx = \int_\Omega \nabla u(t_0)\varphi dx.$$

Thus we get that $\tilde{u} = \nabla u(t_0)$. Then by the weak lower semicontinuity of the norm we know that

$$\|\nabla u(t_0)\|_p \le \liminf_{t_k \to t_0} \|\nabla u(t_k)\|_p.$$

Therefore, by (3.15) we see

$$\|\nabla u(t_k)\|_p^p \to \|\nabla u(t_0)\|_p^p \text{ as } t_k \to t_0,\ t_0 \ge 0. \tag{3.16}$$

Combining (3.16) and the fact that $\nabla u(t_k) \rightharpoonup \nabla u(t_0)$ in $(L^p(\Omega))^n$ we get that

$$\|\nabla (u(t_k) - u(t_0))\|_p^p \to 0 \text{ as } t_k \to t_0,\ t_0 \ge 0.$$

Since the limit is unique and this holds for every subsequence, hence, we get the result. Uniqueness follows by the uniqueness result for a weak solution.

It is known [7, Lemma 6.1] that there exists a subsequence $t_k \to +\infty$ such that $u(t_k)$ convergenes to a stationary point in $W_0^{1,p}(\Omega)$. The last statement of the theorem can be obtained as in [7] using (3.10). □

4. Local convergence results

Let us recall some results [7] on the associated stationary problem to the problem (1.1), that is the following problem

$$\begin{cases} -\nabla \cdot a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{4.1}$$

Let φ be the unique solution to

$$\begin{cases} -\nabla \cdot |\nabla \varphi|^{p-2}\nabla \varphi = f & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma. \end{cases} \tag{4.2}$$

Then the stationary points are determined by the solutions to

$$a(\mu) = \|\nabla \varphi\|_p^{p-1} \mu^{\frac{1}{p}-1} := y(\mu). \tag{4.3}$$

Hence, if (4.3) admits a unique solution, so does the stationary problem, then for any initial data u_0 the solution $u(t)$ converges to this solution of the stationary problem.

Theorem 4.1. *Let $p \ge 2$, u_* be an isolated solution to the problem* (4.1), *corresponding to the solution μ_* of the equation* (4.3). *Assume that the function a' is continuous and*

$$\frac{p}{p-1} a'(\mu_*)\mu_* + a(\mu_*) = \delta > 0. \tag{4.4}$$

Then there exists $\varepsilon > 0$ such that if the initial value $u_0 \in \mathcal{N}_\varepsilon(u_*)$, where

$$\mathcal{N}_\varepsilon(u_*) := \left\{ u \in W_0^{1,p}(\Omega) : \|\nabla(u - u_*)\|_p < \varepsilon,\ E(u) < E(u_*) + \frac{\delta \varepsilon^p}{16(18)^{\frac{p}{2}}} \right\} \tag{4.5}$$

then

$$u(t) \to u_* \textit{ in } W_0^{1,p}(\Omega). \tag{4.6}$$

Proof. Set $\mathcal{E}(s) = E(u_* + s(u - u_*))$. Then one has

$$E(u) - E(u_*) = \mathcal{E}(1) - \mathcal{E}(0) = \int_0^1 \mathcal{E}'(s)ds = \mathcal{E}'(0) + \int_0^1 (1-s)\mathcal{E}''(s)ds = \int_0^1 (1-s)\mathcal{E}''(s)ds, \tag{4.7}$$

since $\mathcal{E}'(0) = 0$ due to the fact that u_* is a stationary point.

Denote by $w = u - u_*$. After a simple computation we see that

$$\begin{aligned} \mathcal{E}''(s) = pa'(\|\nabla(u_* + sw)\|_p^p)\left(\int_\Omega |\nabla(u_* + sw)|^{p-2}\nabla(u_* + sw)\nabla w dx\right)^2 \\ + a(\|\nabla(u_* + sw)\|_p^p)\Big(\int_\Omega (p-2)|\nabla(u_* + sw)|^{p-4}(\nabla(u_* + sw)\nabla w)^2 \\ + |\nabla(u_* + sw)|^{p-2}|\nabla w|^2 dx\Big). \end{aligned} \tag{4.8}$$

If $a'(\|\nabla(u_* + sw)\|_p^p) \geq 0$ since $p \geq 2$ one has

$$\mathcal{E}''(s) \geq a(\|\nabla(u_* + sw)\|_p^p)\int_\Omega |\nabla(u_* + sw)|^{p-2}|\nabla w|^2 dx. \tag{4.9}$$

Remark that by the Hölder and the Cauchy-Schwarz inequalities we have that

$$\begin{aligned} &\left(\int_\Omega |\nabla(u_* + sw)|^{p-2}\nabla(u_* + sw)\nabla w dx\right)^2 \\ &\leq \frac{p-2+1}{p-1}\int_\Omega |\nabla(u_* + sw)|^{p-4}(\nabla(u_* + sw)\nabla w)^2 dx \int_\Omega |\nabla(u_* + sw)|^p dx \\ &\leq \frac{1}{p-1}\Big((p-2)\int_\Omega |\nabla(u_* + sw)|^{p-4}(\nabla(u_* + sw)\nabla w)^2 dx \\ &\quad + \int_\Omega |\nabla(u_* + sw)|^{p-2}|\nabla w|^2 dx\Big)\int_\Omega |\nabla(u_* + sw)|^p dx. \end{aligned}$$

Therefore, if $a'(\|\nabla(u_* + sw)\|_p^p) < 0$ we get that

$$\begin{aligned} \mathcal{E}''(s) \geq \left(\frac{p}{p-1}a'(\|\nabla(u_* + sw)\|_p^p)\|\nabla(u_* + sw)\|_p^p + a(\|\nabla(u_* + sw)\|_p^p)\right) \\ \times \Big(\int_\Omega (p-2)|\nabla(u_* + sw)|^{p-4}(\nabla(u_* + sw)\nabla w)^2 \\ + |\nabla(u_* + sw)|^{p-2}|\nabla w|^2 dx\Big). \end{aligned} \tag{4.10}$$

For a, b non negative numbers we have that

$$|a^p - b^p| \leq p|a - b|\{a + b\}^{p-1}.$$

Then using the Hölder inequality for $s \in (0,1)$ we see that

$$\begin{aligned} \big|\|\nabla(u_* + sw)\|_p^p - \|\nabla u_*\|_p^p\big| &\leq ps\int_\Omega \big(|\nabla(u_* + sw)| + |\nabla u_*|\big)^{p-1}|\nabla w| dx \\ &\leq p\big||\nabla(u_* + sw)| + |\nabla u_*|\big|_p^{p-1}\|\nabla w\|_p. \end{aligned}$$

Hence, by the continuity of a' and due to the assumption (4.4) from (4.9) and (4.10) we can deduce that there exists $\eta > 0$ such that

$$\|\nabla w\|_p \le \eta \quad \Rightarrow \quad \mathcal{E}''(s) \ge \frac{\delta}{2}\int_\Omega |\nabla(u_* + sw)|^{p-2}|\nabla w|^2 dx, \tag{4.11}$$

i.e. by (4.7) and Lemma 2.2

$$E(u) - E(u_*) \ge \frac{\delta}{2}\int_0^1 (1-s)\int_\Omega |\nabla(u_* + sw)|^{p-2}|\nabla w|^2 dxds \ge \frac{\delta}{16(18)^{\frac{p}{2}}}\|\nabla w\|_p^p. \tag{4.12}$$

We choose $\varepsilon < \eta$ such that u_* is the unique stationary point in

$$B_\varepsilon = \{u : \|\nabla(u - u_*)\|_p < \varepsilon\}$$

(we can do this since u_* is an isolated stationary point) and $u_0 \in \mathcal{N}_\varepsilon(u_*)$. We introduce the set A defined by

$$A = \{t \in [0, +\infty) \mid u(t) \in \mathcal{N}_\varepsilon(u_*)\}.$$

Since $u \in C([0,T]; W_0^{1,p}(\Omega))$ it is clear that A contains a neighbourhood of 0 and is open. Denote by t_∞ the point such that $t_\infty = Sup\{t \mid [0,t) \subset A\}$. Let t_n be a sequence in A such that $t_n \to t_\infty, t_n < t_\infty$. Since $u \in C([0,T]; W_0^{1,p}(\Omega))$ one has

$$\|\nabla(u(t_\infty) - u_*)\|_p \le \varepsilon < \eta.$$

Hence using the fact that E is decreasing along the trajectories and (4.11), (4.12) we deduce that

$$\frac{\delta}{16(18)^{\frac{p}{2}}}\|\nabla(u(t_\infty) - u_*)\|_p^p \le E(u(t_\infty)) - E(u_*) < \frac{\delta}{16(18)^{\frac{p}{2}}}\varepsilon^p,$$

i.e. $t_\infty \in A$ and since A is open we get a contradiction with the definition of t_∞. Thus t_∞ is not finite and $A = [0, \infty)$. So $u(t) \in \mathcal{N}_\varepsilon(u_*)$ for all t. From Theorem 3.1 we know that $u(t)$ converges to a stationary point. Since u_* is the only stationary point in B_ε then the result follows. □

Remark 4.1. The assumption (4.4) is equivalent to

$$a'(\mu_*) > \frac{(1-p)a(\mu_*)}{p\mu_*} = y'(\mu_*).$$

Therefore,

$$\lim_{\mu \to \mu_*} \frac{a(\mu) - a(\mu_*) + y(\mu_*) - y(\mu)}{\mu - \mu_*} > 0$$

and it holds that there exists $\alpha > 0$ such that

$$(a(\mu) - y(\mu))(\mu - \mu_*) > 0 \quad \forall \mu \in (\mu_* - \alpha, \mu_* + \alpha), \ \mu \ne \mu_*,$$

that is we are in the case of Figure 1.1a.

Thus from Remark 4.1 we see that the stationary point u_* corresponds to an isolated local minimum of the energy E (see [7]). Therefore, Theorem 4.1 can be reformulated in Theorem 1.1.

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