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WEIGHED ESTIMATES FOR NONLINEAR ELLIPTIC PROBLEMS WITH ORLICZ DATA

SUN-SIG BYUN AND SEUNGJIN RYU

Abstract. We study the Dirichlet problem for a divergence structure elliptic equation of p-Laplacian type that is not necessarily of variational form. A global maximal regularity is obtained for such a problem by proving that the gradient of the weak solution is as globally integrable as the nonhomogeneous term in weighted Orlicz spaces under minimal conditions on the nonlinearity and the domain. We find not only reasonable conditions imposed on the nonlinearity and the domain but also a correct relationship between the associated weight and Young function for such a weighted Orlicz regularity.

1. Introduction and main result

In this paper we look at the following Dirichlet problem for a divergence structure elliptic equation of p-Laplacian type, $1 < p < \infty$:

$$\left\{ \begin{array}{rclcc} \operatorname{div} \mathbf{a}(Du, x) & = & \operatorname{div}(|F|^{p-2}F) & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega, \end{array} \right. \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with its nonsmooth boundary $\partial\Omega$, the nonhomogeneous term $F \in L^p(\Omega, \mathbb{R}^n)$ is given, as is the nonlinearity $\mathbf{a} = \mathbf{a}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. We hereafter assume

$$\left\{ \begin{array}{l} \gamma|\xi|^{p-2}|\eta|^2 \leq \langle D_\xi \mathbf{a}(\xi, x)\eta, \eta\rangle, \\ |\mathbf{a}(\xi, x)| + |\xi||D_\xi \mathbf{a}(\xi, x)| \leq \Lambda|\xi|^{p-1} \end{array} \right. \tag{1.2}$$

for all $\xi, \eta \in \mathbb{R}^n$, almost every $x \in \mathbb{R}^n$ and some positive constants γ, Λ. Then it is well known that there exists a unique weak solution $u \in W_0^{1,p}(\Omega)$, which means

$$\int_\Omega \mathbf{a}(Du, x) \cdot D\varphi \, dx = \int_\Omega |F|^{p-2}F \cdot D\varphi \, dx,$$

for all $\varphi \in W_0^{1,p}(\Omega)$, and we have the estimate

$$\||Du|^p\|_{L^1(\Omega)} \leq c \, \||F|^p\|_{L^1(\Omega)}, \tag{1.3}$$

the constant c is independent of u and F.

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The main purpose in this paper is to obtain a global Calderón-Zygmund-type estimate in weighed Orlicz spaces for the problem (1.1) under minimal assumptions on the given data. To this end, we first introduce the Muchenhoupt weight. Let $B_\rho(y)$ denote the open ball on $\mathbb{R}^n$ centered $y \in \mathbb{R}^n$ and radius $\rho > 0$. A weight w is a positive and locally integrable function on $\mathbb{R}^n$. Then we say that w is *of class* A_q for some $q \in (1, \infty)$, denoted by $w \in A_q$, if

$$[w]_q := \sup \left(\frac{1}{|B|} \int_B w(x)\, dx \right) \left(\frac{1}{|B|} \int_B w(x)^{\frac{-1}{q-1}}\, dx \right)^{q-1} < \infty,$$

where the supremum is taken over all balls $B = B_\rho(y)$. For example, $w_\alpha(x) = |x|^\alpha$ $(x \in \mathbb{R}^n)$ belongs to A_q class if $-n < \alpha < n(q-1)$. The classes A_q are increasing as q increases, more precisely, for $1 < \alpha_1 < \alpha_2 < \infty$, $A_{\alpha_1} \subset A_{\alpha_2}$ with the estimate $[w]_{\alpha_2} \leq [w]_{\alpha_1}$.

Given a weight $w \in A_q$ and a measurable set $E \subset \mathbb{R}^n$, we denote by

$$w(E) \;=\; \int_E w(x)\, dx$$

to mean the weighed Lebesgue measure of E. (On the other hand, $|E|$ denotes the Lebesgue measure of E.) A remarkable feature of the weighed Lebesgue measure for $w \in A_q$ is the following comparability with the Lebesgue measure:

$$\frac{1}{c_0}\left(\frac{|E|}{|B|}\right)^q \leq \frac{w(E)}{w(B)} \leq c_0 \left(\frac{|E|}{|B|}\right)^{\tau_0}, \qquad E \subset B \tag{1.4}$$

for some constants $c_0 > 1$ and $\tau_0 \in (0,1)$ depending only on n, q and $[w]_q$, but not on E and B.

We now turn to Orlicz spaces. The function $\Phi : [0, \infty) \to [0, \infty)$ is said to be a *Young function* if Φ is increasing, convex, and satisfies

$$\Phi(0) = 0,\ \Phi(\infty) = \lim_{\rho \to +\infty} \Phi(\rho) = +\infty,\ \lim_{\rho \to 0+} \frac{\Phi(\rho)}{\rho} = 0,\ \lim_{\rho \to +\infty} \frac{\Phi(\rho)}{\rho} = +\infty.$$

Throughout this paper, the Young function Φ is assumed to satisfy the following Δ_2 and ∇_2 conditions, denoted by $\Phi \in \Delta_2 \cap \nabla_2$,

- $(\Phi \in \Delta_2)$ there exists $c_1 > 1$ such that $\Phi(2\rho) \leq c_1 \Phi(\rho)$,
- $(\Phi \in \nabla_2)$ there exists $c_2 > 1$ such that $2c_2 \Phi(\rho) \leq \Phi(c_2 \rho)$,

for all $\rho > 0$. This $\Delta_2 \cap \nabla_2$-condition is unavoidable for the type of regularity estimate under consideration, see [22]. In particular, the $\Delta_2 \cap \nabla_2$-condition ensures that the Young function Φ grows neither too fast nor too slowly. For instance, the functions such as

$$\Phi_1(\rho) = \exp(\rho^2) (\notin \Delta_2), \qquad \Phi_2(\rho) = \rho \log(1+\rho) (\notin \nabla_2)$$

cannot be considered here. Observe that in view of $\Phi \in \Delta_2 \cap \nabla_2$, one can find two constants q_1 and q_2 with $1 < q_1 \leq q_2 < \infty$ such that

$$\frac{1}{c_3} \min\{\lambda^{q_1}, \lambda^{q_2}\} \Phi(\rho) \leq \Phi(\lambda\rho) \leq c_3 \max\{\lambda^{q_1}, \lambda^{q_2}\} \Phi(\rho), \qquad \lambda, \rho \geq 0, \tag{1.5}$$

the constant c_3 being independent of λ and ρ.

We next define the lower index of a Young function Φ, denoted by $i(\Phi)$, by

$$i(\Phi) \;=\; \lim_{\lambda \to 0+} \frac{\log(h_\Phi(\lambda))}{\log \lambda} \;=\; \sup_{0<\lambda<1} \frac{\log(h_\Phi(\lambda))}{\log \lambda},$$

where

$$h_\Phi(\lambda) \;=\; \sup_{\rho>0} \frac{\Phi(\lambda\rho)}{\Phi(\rho)} \qquad (\lambda > 0).$$

Note that the index number $i(\Phi)$ is equal to the supremum of q_1 satisfying (1.5).

Given a weight w and a Young function Φ, the condition $w \in A_{i(\Phi)}$ is a crucial assumption in this work. Since $\Phi \in \Delta_2 \cap \nabla_2$, we see that $1 < i(\Phi) < \infty$; and so if a weight belongs to $A_{i(\Phi)}$, then it has a self-improving property, i.e., there exists a small positive constant ϵ_0 depending the index $i(\Phi)$ and the dimension n such that $w \in A_{i(\Phi)-\epsilon_0}$ with the estimate $[w]_{i(\Phi)-\epsilon_0} \le c_{n,i(\Phi)}[w]_{i(\Phi)}$. Consequently, we have

$$[w]_{i(\Phi)} \le [w]_{i(\Phi)-\epsilon_0} \le c[w]_{i(\Phi)}.$$

Then in view of (1.5), we discover that

$$\lambda^{i(\Phi)-\epsilon_0}\Phi(t) \le c\,\Phi(\lambda t), \qquad \lambda \ge 1,\ t \ge 0.$$

We refer to [14, 15] for a more in-depth discussion on the $A_{i(\Phi)}$ class.

We are now ready to introduce the weighted Orlicz space. For a Young function $\Phi \in \Delta \cap \nabla_2$ and a weight w satisfying $w \in A_{i(\Phi)}$, the weighted Orlicz space $L^\Phi_w(\Omega)$ consists of all measurable functions $g : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega \Phi(|g(x)|)w(x)dx \;<\; +\infty.$$

And $L^\Phi_w(\Omega)$ in fact is a Banach space equipped with the following weighted Luxemburg norm:

$$\|g\|_{L^\Phi_w(\Omega)} = \inf\left\{\kappa > 0 : \int_\Omega \Phi\left(\frac{|g(x)|}{\kappa}\right) w(x)\,dx \le 1\right\}.$$

Furthermore from the convexity of Φ and (1.5) we see that

$$\begin{aligned} \frac{1}{c_3}\min\left\{\|g\|^{q_1}_{L^\Phi_w(\Omega)}, \|g\|^{q_2}_{L^\Phi_w(\Omega)}\right\} \;&\le\; \int_\Omega \Phi(|g(x)|)w(x)dx \\ &\le\; c_3 \max\left\{\|g\|^{q_1}_{L^\Phi_w(\Omega)}, \|g\|^{q_2}_{L^\Phi_w(\Omega)}\right\}. \end{aligned} \tag{1.6}$$

Here the constants c_3, q_1 and q_2 as in (1.5). The Young function $\Phi(\rho) = \rho^q$ $(1 < q < \infty)$ satisfies the $\Delta_2 \cap \nabla_2$ condition and we deduce that $i(\Phi) = q$. Therefore the weighted Lebesgue space L^q_w for $w \in A_q$ is a special case of weighted Orlicz spaces L^Φ_w for $w \in A_{i(\Phi)}$. We refer the reader to [14, 15] for details concerning weighted Orlicz spaces.

We next introduce the regularity requirement on the nonlinearity $\mathbf{a} = \mathbf{a}(\xi, x)$ and the geometric assumption on the domain Ω considered here. For a ball $B_\rho(y)$, we define a function $\Theta(\mathbf{a}; B_\rho(y))$ on $B_\rho(y)$ by

$$\Theta(\mathbf{a}; B_\rho(y))(x) = \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{\left|\mathbf{a}(\xi, x) - \overline{\mathbf{a}}_{B_\rho(y)}(\xi)\right|}{|\xi|^{p-1}},$$

where

$$\overline{\mathbf{a}}_{B_\rho(y)}(\xi) = \fint_{B_\rho(y)} \mathbf{a}(\xi, x)\,dx = \frac{1}{|B_\rho(y)|}\int_{B_\rho(y)} \mathbf{a}(\xi, x)\,dx.$$

The function $\Theta(\mathbf{a}; B_\rho(y))$ provides the measurement of the oscillation of $\frac{\mathbf{a}(\xi,x)}{|\xi|^{p-1}}$ in the variable x over $B_\rho(y)$, uniformly in ξ. It follows from (1.2) that the function Θ is bounded by 2Λ.

Definition 1. A vector field $\mathbf{a}$ is said to be (δ, R)-vanishing if

$$\sup_{0<\rho\le R}\ \sup_{y\in\mathbb{R}^n} \fint_{B_\rho(y)} \Theta(\mathbf{a}; B_\rho(y))(x)\ dx \le \delta. \tag{1.7}$$

The assumption (1.7) asserts the BMO semi-norm of $\frac{\mathbf{a}(\xi,\cdot)}{|\xi|^{p-1}}$ has small oscillation with respect to x and uniformly in ξ, less than δ from being averaged over each ball. The Calderón-Zygmund-type theory of nonlinear PDEs and systems with discontinuous nonlinearities has been extensively studied under this type of small BMO condition (or VMO condition) in the literature, see [1, 2, 3, 4, 6, 7, 9, 11, 13, 16, 17].

To measure the deviation of $\partial\Omega$ from being a hyperplane at each scale $\rho > 0$, we use the following the so-called "Reifenberg flatness" condition.

Definition 2. A bounded domain Ω is said to be (δ, R)-Reifenberg flat if for every $x \in \partial\Omega$ and every $r \in (0, R]$, there exists a coordinate system $\{y_1, \dots, y_n\}$, which can depend on r and x such that $x = 0$ in this coordinate system and that

$$B_r(0) \cap \{y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\delta r\}\,. \tag{1.8}$$

Remark 1. *We have a few comments about Definition 1 and 2. Knowing that the problem (1.1) has a scaling invariance property, the constant R can be taken as 1 or any other constant larger than one. On the other hand, the constant δ is still invariant under such scaling. (See Lemma 1.) Furthermore we note that this Reifenberg flatness (1.8) is meaningful only for small values of δ and in fact δ is to be chosen so small in the range $0 < \delta < \frac{1}{2^{n+1}}$ that the regularity theory under consideration is available, as we will see in Main Theorem below. With such small δ, this flatness condition means that the deviation of $\partial\Omega$ from being an $(n-1)$-dimensional affine space is small enough at each scale $r > 0$. Furthermore from (1.8), we see that $\mathbb{R}^n \setminus \Omega$ enjoys the following measure density condition is obtained.*

$$|\Omega \cap B_r(y)| \ge \left(\frac{1-\delta}{2}\right)^n |B_r(y)| \ge \left(\frac{7}{16}\right)^n |B_r(y)|$$

for all $y \in \Omega$ and $r \in (0, R)$. We refer the reader to [5, 18, 21] *for details.*

We now state the main result of this paper.

Main Theorem. *Given a Young function $\Phi \in \Delta_2 \cap \nabla_2$, let $w \in A_{i(\Phi)}$. Suppose that $|F|^p \in L^\Phi_w(\Omega)$. Then there exists a small positive constant $\delta = \delta(\gamma, \Lambda, n, \Phi, w)$ such that if $\mathbf{a}$ is (δ, R)-vanishing and Ω is (δ, R)-Reifenberg flat, then the weak solution u of (1.1) satisfies $|Du|^p \in L^\Phi_w(\Omega)$ and we have*

$$\||Du|^p\|_{L^\Phi_w(\Omega)} \ \le\ c\,\||F|^p\|_{L^\Phi_w(\Omega)}, \tag{1.9}$$

the constant c depending on $\gamma, \Lambda, n, \Phi, w,$ and Ω.

Remark 2. *The regularity result above is a natural extension of the previous work in* [9]. *More precisely, we generalize a global weighed L^q $(q > p)$ estimate of the gradient of the weak solution for elliptic equations of p-Laplacian type, obtained in* [9], *to the setting of weighed Orlicz spaces. Our approach here for the estimate (1.9) is based on the Hardy-Littlewood maximal function and the Calderón-Zygmund type covering lemma. This approach, introduced in* [12] *and later developed in* [3, 7, 8], *is not new but it is very effective for a global estimate of the elliptic and parabolic problems being invariant under scaling and normalization* *like (1.1), see Lemma 1 below. We clearly point out that the so-called "Harmonic-analysis-free" method, which was first introduced in* [1] *and later employed for the same problem (1.1) in* [9] *is quite useful when the problem encounters a scaling deficit difficulty, as the parabolic equations/systems of $p(x,t)$-Laplacian type (see, for instance,* [1, 2, 6, 13]*. Needless to say, one can instead follow this influential approach from* [1] *to obtain the same result in Main Theorem.*

Remark 3. *We note that the existence and uniqueness of a weak solution to the main problem (1.1) is guaranteed by the assumption $|F|^p \in L^{\Phi}_w(\Omega)$. In fact, if $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i(\Phi)}$, and $g \in L^{\Phi}_w(\Omega)$, then $g \in L^1(\Omega)$ and we have the estimate*

$$\int_\Omega |g(x)|\,dx \le c\left[\left(\int_\Omega \Phi(|g(x)|)w(x)dx\right)^{\frac{1}{q_1}} + \left(\int_\Omega \Phi(|g(x)|)w(x)dx\right)^{\frac{1}{q_2}}\right], \tag{1.10}$$

where q_1 and q_2 are defined as in (1.5), see [20].

2. Auxiliary results

We begin this section with the following invariance property under normalization and scaling. The proof follows by direct computations (for further details, see [9, 10]).

Lemma 1. *Let u be the weak solution to the problem (1.1). Assume that the nonlinearity $\mathbf{a}(\xi,x)$ satisfies (1.2) and is (δ,R)-vanishing. For each $\lambda > 1$ and $0 < r < 1$, define the rescaled maps*

$$\tilde{\mathbf{a}}(\xi,x) = \frac{\mathbf{a}(\lambda\xi, rx)}{\lambda^{p-1}}, \quad \tilde\Omega = \left\{\frac{1}{r}x : x \in \Omega\right\}, \quad \tilde u(x) = \frac{u(rx)}{\lambda r}, \quad \tilde F(x) = \frac{F(rx)}{\lambda}.$$

Then

(1) *$\tilde u \in W^{1,p}_0(\tilde\Omega)$ is the weak solution of*

$$\text{div}\, \tilde{\mathbf{a}}(D\tilde u, x) \;=\; \text{div}\,(|\tilde F|^{p-2}\tilde F) \qquad \text{in} \quad \tilde\Omega,$$

(2) *$\tilde{\mathbf{a}}(\xi,x)$ satisfies the structural assumption (1.2) with the same constants γ and Λ,*

(3) *$\tilde{\mathbf{a}}$ is $(\delta,\frac{R}{r})$-vanishing and $\tilde\Omega$ is $(\delta,\frac{R}{r})$-Reifenberg flat.*

We now recall the Hardy-Littlewood maximal function and its basic properties. Let g be a locally integrable function on $\mathbb{R}^n$. Then the Hardy-Littlewood maximal function of g is given by

$$(\mathcal{M}g)(x) = \sup_{\rho>0} ⨍_{B_\rho(x)} |g(y)|\,dy = \sup_{\rho>0}\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)} |g(y)|\,dy.$$

If g is defined only on a bounded domain U, we define its restricted maximal function as

$$\mathcal{M}_U g = \mathcal{M}(g\chi_U),$$

where χ_U is the standard characteristic function on U. The maximal function operator satisfies the so-called *weak (1,1) inequality.* More specifically, there exists a positive constant $c = c(n)$ such that

$$\big|\{x \in \mathbb{R}^n : (\mathcal{M}g)(x) > \lambda\}\big| \le \frac{c}{\lambda}\int_{\mathbb{R}^n} |g(x)|\,dx \tag{2.1}$$

for any $\lambda > 0$. As the well-known Muchenhoupt characterization of the A_q-weight, the Hardy-Littlewood maximal operator is bounded from the weighted Lebesgue space $L^q_w(\mathbb{R}^n)$ to itself and the $A_{i(\Phi)}$-weight can be classified as follows. Given a Young function $\Phi \in \Delta_2 \cap \nabla_2$, the weight w belongs to the $A_{i(\Phi)}$ class if and only if there exists $c = c(n, \Phi, w)$ such that

$$\int_{\mathbb{R}^n} \Phi\Big(\mathcal{M}g(x)\Big)w(x)dx \leq c\int_{\mathbb{R}^n} \Phi\Big(|g(x)|\Big)w(x)dx, \tag{2.2}$$

for all $g \in L^\Phi_w(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$. We refer to [14, 15] and the references therein.

We will use the following simple result.

Lemma 2. *Given a Young function $\Phi \in \Delta_2 \cap \nabla_2$, let $w \in A_{i(\Phi)}$. Assume that g is a nonnegative and measurable function defined on a bounded domain Ω in $\mathbb{R}^n$. Let $\theta > 0$ and $\lambda > 1$ be constants. Then*

$$g \in L^\Phi_w(\Omega) \quad \Longleftrightarrow \quad S := \sum_{k\geq 1} \Phi\big(\lambda^k\big)\, w\Big(\big\{x \in \Omega : g(x) > \theta\lambda^k\big\}\Big) < \infty$$

and

$$\frac{1}{c}S \leq \int_\Omega \Phi\Big(g(x)\Big)w(x)\,dx \leq c(w(\Omega) + S), \tag{2.3}$$

the positive constant c depending only on θ, λ, Φ, and w.

The following version of the Calderón-Zygmund type covering lemma is used to prove the main theorem. The proof can be found in [4, Lemma 5.4] or [19, Lemma 3.4] with slight modifications.

Lemma 3. *Given a Young function $\Phi \in \Delta_2 \cap \nabla_2$, let $w \in A_{i(\Phi)}$. Let Ω be a bounded $(\delta, 1)$-Reifenberg flat domain for some small $\delta > 0$ and let C and D be measurable sets with $C \subset D \subset \Omega$. Suppose that there exists small $\epsilon > 0$ such that*

1. *for any $y \in \Omega$, $w(C \cap B_1(y)) < \epsilon w(B_1(y))$,*
2. *for each $y \in \Omega$ and $r \in (0, 1)$,*

$$\textit{if} \;\; w(C \cap B_r(y)) \geq \epsilon w(B_r(y)), \quad \textit{then} \;\; B_r(y) \cap \Omega \subset D.$$

Then

$$w(C) \leq c_4\, \epsilon\, w(D),$$

the constant c_4 depending only on n, Φ, and w.

We remark that the constant c_4 is depending on $\frac{1}{1-\delta}$. Since δ is to be selected less than $\frac{1}{2^{n+1}}$, c_4 is bounded by some universal constant being independent of δ.

From now on, for simplicity and clearance, the symbol c denotes a universal constant that can be explicitly calculated in terms of known quantities. This constant may vary in different occurrences. The relevant connections with known quantities will be specified, if necessary.

3. Global Weighted Orlicz Estimates

In this section, we will complete the proof of Main Theorem. The following lemma is a crucial ingredient for its proof.

Lemma 4. *Let $u \in W_0^{1,p}(\Omega)$ be the weak solution of (1.1). Then there exists a constant $N = N(\gamma, \Lambda, n, p) > 1$ such that for each $0 < \epsilon < 1$ fixed, one can select small $\delta = \delta(\epsilon, \gamma, \Lambda, n, p, \Phi, w) \in (0, \frac{1}{8})$ such that if $\mathbf{a}$ is $(\delta, 42)$-vanishing, Ω is $(\delta, 42)$-Reifenberg flat, and if for $0 < r < 1$ and $y \in \Omega$, $B_r(y)$ satisfies*

$$w\left(\{x \in \Omega : \mathcal{M}(|Du|^p) > N^p\} \cap B_r(y)\right) \geq \epsilon w(B_r(y)), \tag{3.1}$$

then we have

$$B_r(y) \cap \Omega \subset \{x \in \Omega : \mathcal{M}(|Du|^p) > 1\} \cup \{x \in \Omega : \mathcal{M}(|F|^p) > \delta^p\}. \tag{3.2}$$

For the proof, we need the following comparison estimates. We start with interior estimates in $B_6 \subset\subset \Omega$.

Lemma 5. *(See [9, 11].) Let $u \in W^{1,p}(B_6)$ be a weak solution of*

$$div\, \mathbf{a}(Du, x) \;=\; div\, (|F|^{p-2}F) \quad in \quad B_6$$

with

$$\fint_{B_6} |Du|^p \, dx \leq 1.$$

Then there exists a constant $n_1 = n_1(\gamma, \Lambda, n, p) > 1$ so that for any $\epsilon \in (0, 1)$ fixed, one can find a small positive constant $\delta = \delta(\epsilon, \gamma, \Lambda, n, p)$ such that if

$$\fint_{B_6} \Theta\Big(\mathbf{a}; B_6\Big) \, dx \leq \delta \quad and \quad \fint_{B_6} |F|^p \, dx \leq \delta^p$$

hold for such small δ, then there exists a weak solution $v \in W^{1,p}(B_4)$ of

$$div\, \overline{\mathbf{a}}_{B_4}(Dv) \;=\; 0 \quad in \quad B_4$$

such that

$$\fint_{B_2} |D(u - v)|^p \, dx \;\leq\; \epsilon^p \quad and \quad \|Dv\|_{L^\infty(B_3)} \leq n_1.$$

The next lemma is a boundary version of Lemma 5. We use the following notation.

$$B_r^+ = B_r \cap \{x_n > 0\}, \quad \Omega_r = B_r \cap \Omega, \quad T_r = B_r \cap \{x_n = 0\}, \quad \partial_w \Omega_r = \partial\Omega \cap B_r.$$

Lemma 6. *(See [9].) Let u be a weak solution of*

$$\begin{cases} div\, \mathbf{a}(Du, x) &= div\, (|F|^{p-2}F) \quad in \quad \Omega_6, \\ u &= 0 \quad on \quad \partial_w \Omega_6, \end{cases}$$

with

$$\fint_{\Omega_6} |Du|^p \, dx \leq 1.$$

Then, there exists a constant $n_2 = n_2(\gamma, \Lambda, n, p) > 1$ so that for any $\epsilon \in (0, 1)$ fixed, one can find a small positive constant $\delta = \delta(\epsilon, \gamma, \Lambda, n, p)$ such that if

$$B_6^+ \subset \Omega_6 \subset B_6 \cap \{x_n > -12\delta r\},$$

$$\fint_{B_6^+} \Theta\Big(\mathbf{a}; B_6^+\Big)\, dx \le \delta \quad \textit{and} \quad \fint_{\Omega_6} |F|^p\, dx \le \delta^p$$

hold for such small δ, *then there exists a weak solution* v *of*

$$\begin{cases} div\, \overline{\mathbf{a}}_{B_4^+}(Dv) &= 0 \quad in \quad B_4^+, \\ v &= 0 \quad on \quad T_4, \end{cases}$$

such that

$$\fint_{\Omega_2} |D(u-\overline{v})|^p\, dx \ \le\ \epsilon^p \quad \textit{and} \quad \|D\overline{v}\|_{L^\infty(\Omega_3)} \le n_2$$

where $\overline{v}$ *is the zero extension of* v *from* B_4^+ *to* B_4.

Now, we are ready to prove Lemma 4.

Proof of Lemma 4. We argue by contradiction. Suppose that $B_r(y)$ satisfies (3.1) but the claim (3.2) is false. Then there exists a point $y_1 \in B_r(y)\cap\Omega$ such that for every $\rho > 0$, one has

$$\frac{1}{|B_\rho(y_1)|}\int_{\Omega\cap B_\rho(y_1)} |Du|^p\, dx \le 1 \quad \text{and} \quad \frac{1}{|B_\rho(y_1)|}\int_{\Omega\cap B_\rho(y_1)} |F|^p\, dx \le \delta^p. \tag{3.3}$$

Case 1: $B_{6r}(y) \subset\subset \Omega$.
Without loss of generality, we may assume that $y = 0$. Since $B_{6r} \subset B_{7r}(y_1)\cap\Omega$, it follows from (3.3) that

$$\begin{aligned} \fint_{B_{6r}} |Du|^p\, dx &\le \frac{1}{|B_{6r}|}\int_{\Omega_{7r}(y_1)} |Du|^p\, dx \\ &\le \frac{|B_{7r}(y_1)|}{|B_{6r}|}\frac{1}{|B_{7r}(y_1)|}\int_{\Omega_{7r}(y_1)} |Du|^p\, dx \ \le\ 2^n. \end{aligned} \tag{3.4}$$

Similarly, we have

$$\fint_{B_{6r}} |F|^p\, dx \le 2^n\delta^p. \tag{3.5}$$

Consider the rescaled maps

$$\tilde{\mathbf{a}}(\xi, x) = \frac{\mathbf{a}(2^{\frac{n}{p}}\xi, rx)}{2^{\frac{n(p-1)}{p}}}, \quad \tilde{u}(x) = \frac{u(rx)}{2^{\frac{n}{p}}r}, \quad \tilde{F}(x) = \frac{F(rx)}{2^{\frac{n}{p}}} \tag{3.6}$$

for $x \in B_6 \subset \tilde{\Omega}$ and $\xi \in \mathbb{R}^n$. In light of Lemma 1 and (3.4)-(3.6), we are under the hypotheses of Lemma 5, which implies that after scaling back, there exists $v \in W^{1,p}(B_{4r})$ such that

$$\|Dv\|_{L^\infty(B_{3r})} \le n_1 \tag{3.7}$$

for some $n_1 = n_1(\gamma, \Lambda, n, p) > 1$ and

$$\fint_{B_{2r}} |D(u-v)|^p\, dx \le \eta, \tag{3.8}$$

where $\eta \in (0,1)$ is to be determined as below.

Now we let $N_1 = \max\Big\{2n_1, 2^{\frac{n}{p}}\Big\}$. Then by (3.7), we discover that

$$\Big\{x \in B_r : \mathcal{M}(|Du|^p) > N_1^p\Big\} \subset \Big\{x \in B_r : \mathcal{M}_{B_{4r}}(|D(u-v)|^p) > n_1^p\Big\}. \tag{3.9}$$

Using (3.9), *weak (1-1) estimate* (2.1) and (3.8), we have

$$\begin{aligned} \frac{1}{|B_r|}\big|\{x \in B_r : \mathcal{M}(|Du|^p) > N_1^p\}\big| &\le \frac{1}{|B_r|}\big|\{x \in B_r : \mathcal{M}_{B_{4r}}(|D(u-v)|^p) > n_1^p\}\big| \\ &\le c\fint_{B_{4r}} |D(u-v)|^p\,dx \\ &\le c\eta. \end{aligned}$$

Consequently, we conclude

$$\big|\{x \in \Omega : \mathcal{M}(|Du|^p) > N_1^p\} \cap B_r\big| \le c\eta|B_r|$$

for some $c = c(\gamma, \Lambda, n, p) > 0$. We next recall (1.4) to find

$$\begin{aligned} \frac{w\big(\{x \in \Omega : \mathcal{M}(|Du|^p) > N_1^p\} \cap B_r\big)}{w\big(B_r\big)} &\le c_0 \left(\frac{\big|\{x \in \Omega : \mathcal{M}(|Du|^p) > N_1^p\} \cap B_r\big|}{|B_r|}\right)^{\tau_0} \\ &\le c\eta^{\tau_0}, \end{aligned}$$

for some $c = c(\lambda, \Lambda, n, p, i(\Phi), [w]_{i(\Phi)}) > 0$. And so

$$w\big(\{x \in \Omega : \mathcal{M}(|Du|^p) > N_1^p\} \cap B_r\big) \le c\eta^{\tau_0} w\big(B_r\big) < \epsilon w\big(B_r\big),$$

by taking $\eta > 0$ so small that the last inequality holds. This contradicts (3.1).

Case 2: $B_{6r}(y) \not\subseteq \Omega$.
In view of the $(\delta, 42)$-*Reifenberg flatness* of the domain Ω and $(\delta, 42)$-*vanishing* property of the nonlinearity $\mathbf{a}$, there exists a point $y_0 \in \partial\Omega \cap B_{6r}(y)$ and a new coordinate system, depending on the point y_0 and the scale $r \in (0, 1)$, whose variables we denote by $z = (z_1, \dots, z_{n-1}, z_n)$, such that in this new coordinate system the origin is $y_0 + \delta_0\overrightarrow{n_0}$ for some small $\delta_0 > 0$ and some inward unit vector $\overrightarrow{n_0}$ at y_0, $y = z_0$, $y_1 = z_1$,

$$B_{42r}^+ \subset \Omega_{42r} (= \Omega \cap B_{42r}) \subset \{z \in B_{42r} : z_n > -84r\delta\}, \tag{3.10}$$

and

$$\fint_{B_{42r}^+} \Theta\big(\mathbf{a}, B_{42r}^+\big)(z)\,dz \le \delta. \tag{3.11}$$

Note that $\Omega \cap B_{42r} \subset \Omega \cap B_{49r}(z_1)$. Then by (3.3), we have

$$\fint_{\Omega_{42r}} |Du|^p\,dz \le 2\left(\frac{49}{42}\right)^n \fint_{\Omega \cap B_{49r}(z_1)} |Du|^p\,dz < 2^{n+1}. \tag{3.12}$$

Similarly,

$$\fint_{\Omega_{42r}} |F|^p\,dz \le 2^{n+1}\delta^p. \tag{3.13}$$

As for the *Case 1*, we apply Lemma 1 with the scale $7r$ and $\lambda = 2^{\frac{n+1}{p}}$. Utilizing (3.10)-(3.13), we see that we are under the hypotheses of Lemma 6, and thus there exists a function $\overline{v} \in W^{1,p}(\Omega_{28r})$ satisfying

$$\fint_{\Omega_{14r}} |D(u - \overline{v})|^p\,dz \le \eta_*\epsilon,$$

where η_* is to be selected as below, and

$$\|D\overline{v}\|_{L^\infty(\Omega_{21r})} \le n_2,$$

where n_2 is a universal constant depending on γ, Λ, n, and p.

Writing $N_2 = \max\left\{2n_2, 2^{\frac{n+1}{p}}\right\}$, we conclude, as in the *Case 1*, that

$$\frac{\left|\left\{z \in \Omega : \mathcal{M}(|Du|^p) > N_2^p\right\} \cap B_{7r}\right|}{|B_{7r}|} \;\leq\; c\eta_*\epsilon,$$

which implies that after scaling back,

$$\left|\left\{x \in \Omega : \mathcal{M}(|Du|^p) > N_2^p\right\} \cap B_r(y)\right| \leq c\eta_*\epsilon|B_r(y)|$$

The rest of proof is very similar to that of the *Case 1*, and so we omit it here. Finally, we take $N = \max\{N_1, N_2\}$ to complete the proof of Lemma 4. □

We are now all set to prove the main theorem. In the proof, Lemma 3 and 4 will play an important role.

Proof of Main Theorem. We first fix N, ϵ, and the corresponding δ given by Lemma 4. We first claim that

$$S = \sum_{k\geq 1} \Phi\big(N^{pk}\big)\, w\left(\left\{x \in \Omega : \mathcal{M}(|Du|^p) > N^{pk}\right\}\right) < \infty, \tag{3.14}$$

under the assumption

$$\||F|^p\|_{L^\Phi_w(\Omega)} \leq \delta^p. \tag{3.15}$$

To employ Lemma 3, we write

$$C = \{x \in \Omega : \mathcal{M}(|Du|^p) > N^p\}$$

and

$$D = \{x \in \Omega : \mathcal{M}(|Du|^p) > 1\} \cup \{x \in \Omega : \mathcal{M}(|F|^p) > \delta^p\}.$$

Then clearly $C \subset D \subset \Omega$. We next fix any $y \in \Omega$. Then it follows from (1.4), *weak (1-1) estimate* (2.1) and *standard L^p estimate with zero boundary data* (1.3) that

$$\begin{aligned}
\frac{w(C \cap B_1(y))}{w(B_1(y))} \leq c_0\left(\frac{|C \cap B_1(y)|}{|B_1(y)|}\right)^{\tau_0} &\leq c\,|C|^{\tau_0} \\
&\leq c\left(\int_\Omega |Du|^p\,dx\right)^{\tau_0} \leq c\left(\int_\Omega |F|^p\,dx\right)^{\tau_0}.
\end{aligned}$$

But then by (1.10), (1.6) and (3.15), we have

$$\begin{aligned}
\int_\Omega |F|^p\,dx &\leq c\left[\left(\int_\Omega \Phi(|F|^p)w(x)dx\right)^{\frac{1}{q_1}} + \left(\int_\Omega \Phi(|F|^p)w(x)dx\right)^{\frac{1}{q_2}}\right] \\
&\leq c\left(\||F|^p\|_{L^\Phi_w(\Omega)}\right)^{\frac{q_1}{q_2}} \leq c\delta^{\frac{pq_1}{q_2}}.
\end{aligned}$$

Consequently, we conclude

$$w(C \cap B_1(y)) \leq c\delta^{\frac{pq_1\tau_0}{q_2}}\, w(B_1(y)).$$

And so the first condition of Lemma 3 holds by taking δ further small enough. On the other hand, the second condition of Lemma 3 follows from Lemma 4. As a consequence, we have

$$
\begin{aligned}
&w\Big(\{x \in \Omega : \mathcal{M}(|Du|^p) > N^{pk}\}\Big) \\
&\qquad \le c_4\, \epsilon\, w\Big(\{x \in \Omega : \mathcal{M}(|Du|^p) > 1\}\Big) \\
&\qquad\qquad + \ c_4\, \epsilon\, w\Big(\big\{x \in \Omega : \mathcal{M}(|F|^p) > \delta^2 N^{p(k-i)}\big\}\Big).
\end{aligned} \tag{3.16}
$$

Thanks to Lemma 1, we apply the same estimate (3.16) can be obtained for $(\frac{u}{N}, \frac{F}{N})$, $(\frac{u}{N^2}, \frac{F}{N^2})$, $(\frac{u}{N^3}, \frac{F}{N^3})$, …, inductively, and so deduce the following power decay estimate.

$$
\begin{aligned}
&w\Big(\{x \in \Omega : \mathcal{M}(|Du|^p) > N^{pk}\}\Big) \\
&\qquad \le \epsilon_1^k w\Big(\{x \in \Omega : \mathcal{M}(|Du|^p) > 1\}\Big) \\
&\qquad\qquad + \sum_{i=1}^{k} \epsilon_1^i w\Big(\big\{x \in \Omega : \mathcal{M}(|F|^p) > \delta^p N^{p(k-i)}\big\}\Big),
\end{aligned}
$$

for each $k = 1, 2, \ldots$, where $\epsilon_1 = c_4 \epsilon$. Thus

$$
\begin{aligned}
S &= \sum_{k\ge1} \Phi(N^{pk})\, w\left(\big\{x \in \Omega : \mathcal{M}(|Du|^p) > N^{pk}\big\}\right) \\
&\le \sum_{k\ge1} \Phi(N^{pk}) \epsilon_1^k w\left(\{x \in \Omega : \mathcal{M}(|Du|^p) > 1\}\right) \\
&\qquad + \sum_{k\ge1} \Phi(N^{pk}) \sum_{i=1}^{k} \epsilon_1^i w\left(\Big\{x \in \Omega : \mathcal{M}(|F|^p) > \delta^2 N^{p(k-i)}\Big\}\right) \\
&=: \ S_1 \ + \ S_2.
\end{aligned}
$$

Since $\Phi \in \Delta_2$, there exists a constant μ_1, depending only on p, N, and c_1, such that $\Phi(N^p) \le \mu_1 \Phi(1)$, and so

$$
\Phi(N^{pk}) \le \mu_1^k \Phi(1) \qquad (k = 1, 2, 3, \cdots),
$$

thereby estimate S_1 as follow.

$$
S_1 \le \sum_{k\ge1} \Big(\Phi(1) \mu_1^k \epsilon_1^k w(\Omega)\Big) \le c \sum_{k\ge1} (\mu_1 \epsilon_1)^k .
$$

On the other hand,

$$
\begin{aligned}
S_2 &= \sum_{k\ge1} \Phi\Big(N^{p(k-i)} N^{pi}\Big) \sum_{i=1}^{k} \epsilon_1^i w\left(\Big\{x \in \Omega : \mathcal{M}(|F|^p) > \delta^p N^{p(k-i)}\Big\}\right) \\
&\le \sum_{i\ge1} \sum_{k\ge i} \Phi\Big(N^{p(k-i)}\Big) \mu_1^i \epsilon_1^i w\left(\Big\{x \in \Omega : \mathcal{M}(|F|^p) > \delta^p N^{p(k-i)}\Big\}\right) \\
&= \sum_{i\ge1} (\mu_1 \epsilon_1)^i \sum_{k\ge i} \Phi\Big(N^{p(k-i)}\Big) w\left(\Big\{x \in \Omega : \mathcal{M}(|F|^p) > \delta^p N^{p(k-i)}\Big\}\right) \\
&= \sum_{i\ge1} (\mu_1 \epsilon_1)^i \sum_{j\ge0} \Phi\Big(N^{pj}\Big) w\left(\left\{x \in \Omega : \mathcal{M}\left(\left|\frac{F}{\delta}\right|^p\right) > N^{pj}\right\}\right).
\end{aligned}
$$

We now recall (2.3), (2.2)-(1.6) and (3.15) to find

$$\begin{aligned} S_2 &\leq c\sum_{i\geq 1}(\mu_1\epsilon_1)^i \int_\Omega \Phi\left(\mathcal{M}\left(\left|\frac{F}{\delta}\right|^p\right)\right) w(x)dx \\ &\leq c\sum_{i\geq 1}(\mu_1\epsilon_1)^i \left\|\frac{|F|^p}{\delta^p}\right\|^{q_1}_{L^\Phi_w(\Omega)} \leq c\sum_{i\geq 1}(\mu_1\epsilon_1)^i. \end{aligned}$$

Therefore, we derive

$$S \leq c\sum_{k\geq 1}(\mu_1\epsilon_1)^k$$

for $\epsilon_1 = c_4\epsilon$, as in Lemma 3.

We now select a sufficiently small $\epsilon > 0$, in order to get

$$\mu_1\epsilon_1 < 1.$$

We then can find accordingly a small $\delta = \delta(\gamma, \Lambda, n, p, \Phi, w) > 0$ from Lemma 4. Therefore (3.14) is now claimed under the smallness assumption (3.15).

We next recall Lemma 2 and (1.6) to conclude

$$\||Du|^p\|_{L^\Phi_w(\Omega)} \leq c$$

under the assumption (3.15).

Now we need to drop the assumption (3.15). To this end, we consider

$$u_1 = \frac{\delta u}{\sqrt[p]{\||F|^p\|_{L^\Phi_w(\Omega)} + \sigma}} \qquad \text{and} \qquad F_1 = \frac{\delta F}{\sqrt[p]{\||F|^p\|_{L^\Phi_w(\Omega)} + \sigma}}$$

in place of u and F, respectively.

Clearly, we have $\||F_1|^p\|_{L^\Phi_w(\Omega)} \leq \delta^p$, and so $\||Du_1|^p\|_{L^\Phi_w(\Omega)} \leq c$. We then let $\sigma \to 0$ to find the required estimate

$$\||Du|^p\|_{L^\Phi_w(\Omega)} \leq c\,\||F|^p\|_{L^\Phi_w(\Omega)}.$$

This completes the proof.

□

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Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 151-747, Republic of Korea
E-mail address: `byun@snu.ac.kr`

Department of Mathematics, University of Seoul, Seoul 130-743, Republic of Korea
E-mail address: `seungjinryu@uos.ac.kr`