JEPE Vol 1, 2015, p. 31-48

Orthogonal ⊥

EXISTENCE THEOREM FOR A NONLINEAR ELLIPTIC SHELL MODEL

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Abstract. In this paper we introduce a new nonlinear shell model with the following properties. First, we show that, if the middle surface of the undeformed shell is elliptic, then this new nonlinear shell model possesses solutions which are also elliptic surfaces. Second, we show that, if in addition the middle surface of the undeformed shell is a portion of a sphere, then the total energy of this nonlinear shell model coincides to within the first order, i.e., for "small enough" change of metric and change of curvature tensors, with the total energy of the well-known Koiter nonlinear shell model.

1. Introduction

A fundamental existence theorem that applies to a large class of models in three-dimensional nonlinear elasticity was established in a landmark paper [1] by J. Ball. By contrast, no existence theorem is as yet available for any nonlinear shell model that combines the "membrane effects" and the "flexural effects" that may classically arise in a deformed shell (except in the very special case of nonlinearly elastic "shallow shells"; cf., e.g., [5] and the references quoted therein).

This paper aims at achieving this objective in the particular case where the middle surface of the shell is *elliptic*. Specifically, our main result (Theorem 5) establishes the *existence of a solution* to a nonlinear shell model in this particular case. More specifically, we show that the unknown deformation $\boldsymbol{\psi} : \omega \to \mathbb{R}^3$ of the middle surface $S := \boldsymbol{\theta}(\overline{\omega})$ of the reference configuration of a shell is a minimizer, over a specific set $\boldsymbol{U}(\omega)$ of admissible deformations, of a functional $J : \boldsymbol{U}(\omega) \to \mathbb{R}$ of the form

$$J[\boldsymbol{\varphi}] := \int_\omega \big(\varepsilon W_M^\sharp[\boldsymbol{\varphi}] + \varepsilon^3 W_F^\sharp[\boldsymbol{\varphi}]\big)\sqrt{a}\,dy - L[\boldsymbol{\varphi}] \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{U}(\omega),$$

where $2\varepsilon > 0$ denotes the thickness of the shell, $\sqrt{a}\,dy$ denotes the area element along S, and L denotes a linear form that takes into account the applied forces.

The integrands $W_M^\sharp[\boldsymbol{\varphi}]$ and $W_F^\sharp[\boldsymbol{\varphi}]$ appearing in the above expression of $J[\boldsymbol{\varphi}]$ respectively model the "membrane effects" and the "flexural effects" that arise in the deformed shell (the middle surface of which is $\boldsymbol{\varphi}(\omega)$). These integrands are defined explicitly in terms of the fundamental forms of the surface $\boldsymbol{\varphi}(\omega)$ by means of specific stored energy functions W_M and W_F (see Theorems 2 and 3) that are *polyconvex* and *orientation-preserving* in a sense

2010 *Mathematics Subject Classification.* Primary: 74B20; Secondary: 74K25.
Key words and phrases. Nonlinear shell theory, Calculus of variations, Polyconvexity.
Received 23/10/2014, accepted 23/03/2015.

specified in Section 4. Besides, we prove in Theorem 4(b) that, if S is a portion of a sphere, then the integrand

$$\varepsilon W_M^\sharp[\boldsymbol{\varphi}] + \varepsilon^3 W_F^\sharp[\boldsymbol{\varphi}]$$

appearing in the above definition of $J[\boldsymbol{\varphi}]$ coincide for "small enough" change of metric and change of curvature tensors with the integrand

$$\frac{\varepsilon}{2} a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\varphi}) G_{\alpha\beta}(\boldsymbol{\varphi}) + \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\varphi}) R_{\alpha\beta}(\boldsymbol{\varphi})$$

appearing in the definition of the total energy of the well-known Koiter nonlinear shell model (see Section 3).

2. Preliminaries

This section gathers the notions about the differential geometry of surfaces in $\mathbb{R}^3$ that will be used throughout the paper. For more details on these notions, we refer the reader to, e.g., [3].

Greek indices and exponents range in the set $\{1, 2\}$, Latin indices and exponents range in the set $\{1, 2, 3\}$ (save when they are used for indexing sequences), and the summation convention with respect to repeated indices and exponents is used.

The Euclidean norm, the inner product, and the vector product, of vectors in $\mathbb{R}^3$ are respectively denoted $|\boldsymbol{a}|$, $\boldsymbol{a} \cdot \boldsymbol{b}$, and $\boldsymbol{a} \wedge \boldsymbol{b}$. The set of all 2×2 real positive-definite symmetric matrices is denoted $\mathbb{S}^2_>$.

A *domain* in $\mathbb{R}^2$ is a bounded, connected, open subset $\omega \subset \mathbb{R}^2$ with a Lipschitz-continuous boundary $\gamma := \partial\omega$, the set ω being locally on the same side of γ. A generic point in the set $\overline{\omega}$ is denoted $y = (y_\alpha)$ and partial derivatives, in the classical or distributional sense, are denoted $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$.

A mapping $\boldsymbol{\varphi} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ is an *immersion* if it satisfies $\partial_1\boldsymbol{\varphi}(y) \wedge \partial_2\boldsymbol{\varphi}(y) \neq \mathbf{0}$ at each point $y \in \overline{\omega}$.

Throughout this paper, the middle surface of the reference configuration of a nonlinearly elastic shell is denoted and defined by $S = \boldsymbol{\theta}(\overline{\omega})$, where $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ is a given immersion. The two vectors

$$\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y)$$

are then linearly independent at all points $y \in \overline{\omega}$ and span the tangent plane to the surface S, and the vector field

$$\boldsymbol{a}_3 := \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$$

is a unit normal vector field along the surface S. For each $y \in \overline{\omega}$, the three vectors $\boldsymbol{a}_i(y)$ form a basis in $\mathbb{R}^3$; its dual basis is denoted and defined by

$$\boldsymbol{a}^i(y) \cdot \boldsymbol{a}_j(y) = \delta^i_j,$$

where δ^i_j is the Kronecker symbol. The *area* element along S is $\sqrt{a}\, dy$, where

$$a := \det(a_{\alpha\beta}) = |\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|^2.$$

The covariant and contravariant components $a_{\alpha\beta}$ and $a^{\alpha\beta}$ of the *first fundamental form*, or *metric tensor*, of S, the covariant and mixed components $b_{\alpha\beta}$ and b^β_α of the *second* *fundamental form* of S, and the covariant components $c_{\alpha\beta}$ of the *third fundamental form* of S, are then defined by letting:

$$\begin{aligned} a_{\alpha\beta} &:= \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta, \\ b_{\alpha\beta} &:= \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = -\boldsymbol{a}_\beta \cdot \partial_\alpha \boldsymbol{a}_3, \quad b_\alpha^\beta := a^{\beta\sigma} b_{\sigma\alpha}, \\ c_{\alpha\beta} &:= b_\alpha^\sigma b_{\sigma\beta} = \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3. \end{aligned}$$

The notation $L^2(\omega;\mathbb{R}^3)$ denotes the space of vector fields $\boldsymbol{\xi} = (\xi_i) : \omega \to \mathbb{R}^3$ with components ξ_i in the usual Lebesgue space $L^2(\omega)$. It is equipped with the norm

$$\|\boldsymbol{\xi}\|_{L^2(\omega)} := \Big(\int_\omega |\boldsymbol{\xi}(y)|^2 dy \Big)^{1/2} \quad \text{for any } \boldsymbol{\xi} \in L^2(\omega;\mathbb{R}^3),$$

where $|\boldsymbol{\xi}(y)|$ denotes the Euclidean norm of the vector $\boldsymbol{\xi}(y) \in \mathbb{R}^3$ (as already mentioned before).

Likewise, the notation $H^1(\omega;\mathbb{R}^3)$ denotes the space of vector fields $\boldsymbol{\xi} = (\xi_i) : \omega \to \mathbb{R}^3$ with components ξ_i in the usual Sobolev space $H^1(\omega)$. It is equipped with the norm

$$\|\boldsymbol{\xi}\|_{H^1(\omega)} := \Big(\|\boldsymbol{\xi}\|^2_{L^2(\omega)} + \sum_{\alpha=1}^{2} \|\partial_\alpha \boldsymbol{\xi}\|^2_{L^2(\omega)} \Big)^{1/2} \quad \text{for any } \boldsymbol{\xi} \in H^1(\omega;\mathbb{R}^3).$$

Strong and weak convergences are respectively denoted $\to$ and $\rightharpoonup$.

3. Koiter's nonlinear shell model

We consider a shell made of a homogeneous and isotropic hyperelastic material, whose reference configuration is a natural state; hence the constituting material of the shell is characterized by its two *Lamé constants* $\lambda > 0$ and $\mu > 0$. The *reference configuration* of the shell is the set

$$\{\boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y);\ y \in \overline{\omega},\ -\varepsilon \le x_3 \le \varepsilon\},$$

defined in terms of a surface $S = \boldsymbol{\theta}(\overline{\omega}) \subset \mathbb{R}^3$ and a parameter $\varepsilon > 0$.

A *deformation* of the middle surface of the shell is a smooth enough mapping $\boldsymbol{\varphi} : \omega \to \mathbb{R}^3$.

Koiter's nonlinear shell model, introduced by Koiter [8] in 1966, is one of the most commonly used two-dimensional nonlinear shell models in computational mechanics. It states that the *unknown deformation* $\boldsymbol{\psi} : \omega \to \mathbb{R}^3$ of the middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ of the shell subjected to applied forces should minimize a functional, called the total energy of the deformed shell, over an appropriate set of admissible deformations, both of which we now define.

Given an arbitrary deformation $\boldsymbol{\varphi} : \omega \to \mathbb{R}^3$ of the surface $S = \boldsymbol{\theta}(\overline{\omega})$ with smooth enough components, the functions

$$a_{\alpha\beta}(\boldsymbol{\varphi}) := \boldsymbol{a}_\alpha(\boldsymbol{\varphi}) \cdot \boldsymbol{a}_\beta(\boldsymbol{\varphi}), \text{ where } \boldsymbol{a}_\alpha(\boldsymbol{\varphi}) := \partial_\alpha \boldsymbol{\varphi},$$

denote the covariant components of the *first fundamental form* of the *deformed surface* $\boldsymbol{\varphi}(\omega)$, and the functions

$$G_{\alpha\beta}(\boldsymbol{\varphi}) := \frac{1}{2}(a_{\alpha\beta}(\boldsymbol{\varphi}) - a_{\alpha\beta})$$

denote the covariant components of the *change of metric* tensor field associated with the deformation $\boldsymbol{\varphi}$ of S. The *area* element along the surface $\boldsymbol{\varphi}(\omega) \subset \mathbb{R}^3$ is $\sqrt{a(\boldsymbol{\varphi})}\, dy$, where

$$a(\boldsymbol{\varphi}) := \det(a_{\alpha\beta}(\boldsymbol{\varphi})) = |\boldsymbol{a}_1(\boldsymbol{\varphi}) \wedge \boldsymbol{a}_2(\boldsymbol{\varphi})|^2.$$

If the two vectors $\boldsymbol{a}_\alpha(\boldsymbol{\varphi})$ are linearly independent at all points of ω, then the vector field

$$\boldsymbol{a}_3(\boldsymbol{\varphi}) := \frac{\boldsymbol{a}_1(\boldsymbol{\varphi}) \wedge \boldsymbol{a}_2(\boldsymbol{\varphi})}{|\boldsymbol{a}_1(\boldsymbol{\varphi}) \wedge \boldsymbol{a}_2(\boldsymbol{\varphi})|}$$

is well-defined and defines a *unit normal vector field* to the deformed surface $\boldsymbol{\varphi}(\omega)$, the functions $a^{\alpha\beta}(\boldsymbol{\varphi})$ defined by

$$(a^{\alpha\beta}(\boldsymbol{\varphi})) := (a_{\alpha\beta}(\boldsymbol{\varphi}))^{-1}$$

denote the contravariant components of the *first fundamental form* of the deformed surface $\boldsymbol{\varphi}(\omega)$, the functions

$$b_{\alpha\beta}(\boldsymbol{\varphi}) := \partial_{\alpha\beta}\boldsymbol{\varphi} \cdot \boldsymbol{a}_3(\boldsymbol{\varphi}) = -\partial_\alpha\boldsymbol{\varphi} \cdot \partial_\beta\boldsymbol{a}_3(\boldsymbol{\varphi})$$

denote the covariant components of the *second fundamental form* of the deformed surface $\boldsymbol{\varphi}(\omega)$, the functions

$$R_{\alpha\beta}(\boldsymbol{\varphi}) := b_{\alpha\beta}(\boldsymbol{\varphi}) - b_{\alpha\beta}$$

denote the covariant components of the *change of curvature* tensor field associated with the deformation $\boldsymbol{\varphi}$ of S, the functions

$$c_{\alpha\beta}(\boldsymbol{\varphi}) := \partial_\alpha\boldsymbol{a}_3(\boldsymbol{\varphi}) \cdot \partial_\beta\boldsymbol{a}_3(\boldsymbol{\varphi}) = b_{\alpha\sigma}(\boldsymbol{\varphi})a^{\sigma\tau}(\boldsymbol{\varphi})b_{\tau\beta}(\boldsymbol{\varphi})$$

denote the covariant components of the *third fundamental form* tensor field of the deformed surface $\boldsymbol{\varphi}(\omega)$, and the functions

$$P_{\alpha\beta}(\boldsymbol{\varphi}) := \frac{1}{2}(c_{\alpha\beta}(\boldsymbol{\varphi}) - c_{\alpha\beta})$$

denote the covariant components of the *change of third fundamental form* associated with the deformation $\boldsymbol{\varphi}$ of S. The *area* element along the surface $(\boldsymbol{a}_3(\boldsymbol{\varphi}))(\omega) \subset \mathbb{R}^3$ is $\sqrt{c(\boldsymbol{\varphi})}\,dy$, where

$$c(\boldsymbol{\varphi}) := \det(c_{\alpha\beta}(\boldsymbol{\varphi})).$$

Note that

$$c(\boldsymbol{\varphi}) = K(\boldsymbol{\varphi})^2 a(\boldsymbol{\varphi}),$$

where

$$K(\boldsymbol{\varphi}) := \det(b_{\alpha\beta}(\boldsymbol{\varphi}))/\det(a_{\alpha\beta}(\boldsymbol{\varphi}))$$

denotes the total curvature of the surface $\boldsymbol{\varphi}(\omega)$, and that

$$\begin{aligned}\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi}) \wedge \partial_2\boldsymbol{a}_3(\boldsymbol{\varphi}) &= K(\boldsymbol{\varphi})(\boldsymbol{a}_1(\boldsymbol{\varphi}) \wedge \boldsymbol{a}_2(\boldsymbol{\varphi})) \\ &= K(\boldsymbol{\varphi})\sqrt{a(\boldsymbol{\varphi})}\,\boldsymbol{a}_3(\boldsymbol{\varphi}).\end{aligned}$$

The last relation implies in particular that

$$(\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi}) \wedge \partial_2\boldsymbol{a}_3(\boldsymbol{\varphi})) \cdot \boldsymbol{a}_3(\boldsymbol{\varphi}) = K(\boldsymbol{\varphi})\sqrt{a(\boldsymbol{\varphi})}.$$

The unknown $\boldsymbol{\psi} : \omega \to \mathbb{R}^3$ appearing in Koiter's nonlinear shell model represents the *position vector field of the unknown deformed middle surface* $\boldsymbol{\psi}(\omega)$ of the shell, and as such is assumed to satisfy a *boundary condition* of the form

$$\boldsymbol{\psi} = \boldsymbol{\theta} \text{ and } \boldsymbol{a}_3(\boldsymbol{\psi}) = \boldsymbol{a}_3 \text{ on } \gamma_0,$$

where γ_0 is a non-empty relatively open subset of $\gamma := \partial\omega$, which means that the shell is assumed to be clamped on $\boldsymbol{\theta}(\gamma_0)$. In addition, the unknown $\boldsymbol{\psi}$ is subjected to the *constraint*

$$\partial_1\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi} \neq \boldsymbol{0} \text{ in } \omega,$$

so as to insure that the tangent plane is well defined at each point of the deformed surface.

Taking appropriate *a priori* assumptions into account, W.T. Koiter concludes that the unknown deformation $\boldsymbol{\psi}$ of the middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ of the shell should be a *minimizer*, over a set of smooth enough vector fields $\boldsymbol{\varphi} : \omega \to \mathbb{R}^3$ satisfying the boundary conditions

$$\boldsymbol{\varphi} = \boldsymbol{\theta} \text{ and } \boldsymbol{a}_3(\boldsymbol{\varphi}) = \boldsymbol{a}_3 \text{ on } \gamma_0,$$

of the *total energy* of the deformed surface $\boldsymbol{\varphi}(\omega)$, denoted and defined by

$$J_K[\boldsymbol{\varphi}] := \int_\omega \Big\{ \frac{\varepsilon}{2} a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\varphi}) G_{\alpha\beta}(\boldsymbol{\varphi}) + \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\varphi}) R_{\alpha\beta}(\boldsymbol{\varphi}) \Big\} \sqrt{a}\, dy - L[\boldsymbol{\varphi}],$$

where the functions

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma})$$

are the contravariant components of the (uniformly positive-definite) elasticity tensor of the shell, $\lambda > 0$ and $\mu > 0$ are the Lamé constants of the constitutive material, and L is a linear functional that takes into account the applied forces.

The integral $\frac{\varepsilon}{2} \int_\omega a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\varphi}) G_{\alpha\beta}(\boldsymbol{\varphi}) \sqrt{a}\, dy$ is called the *membrane part* of Koiter's energy, while the integral $\frac{\varepsilon^3}{6} \int_\omega a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\varphi}) R_{\alpha\beta}(\boldsymbol{\varphi}) \sqrt{a}\, dy$ is called the *flexural part* of Koiter's energy.

4. Polyconvex and orientation-preserving stored energy functions defined on a surface

The notion of polyconvexity has been introduced by Ball [1] in three-dimensional elasticity in order to establish an existence theorem for the minimization problem of nonlinear elasticity. It has been subsequently generalized to a class of more general functionals in [2], and has been adapted in [4] to "orientation-preserving" functionals, whose argument is a pair of vector fields defined on a surface, representing the deformation of the middle surface of a shell and the rotated unit normal vector field along the deformed middle surface. In this paper we adapt the definition of polyconvexity on a surface of [4] to a class of functionals that are "orientation-preserving" and whose argument is a single vector field defined on a surface (like Koiter's energy defined in Section 3), representing the deformation of the middle surface of a shell.

Let ω be a domain in $\mathbb{R}^2$ and let

$$\begin{aligned}
&\mathbb{E}_+ := \{(\boldsymbol{q}, \boldsymbol{u}_\alpha) \in \mathbb{R}^3 \times (\mathbb{R}^3)^2;\ (\boldsymbol{u}_1 \wedge \boldsymbol{u}_2) \cdot \boldsymbol{q} > 0\},\\
&\mathbb{D}_+ := \{(\boldsymbol{q}, \boldsymbol{u}_\alpha, \boldsymbol{e}) \in \mathbb{R}^3 \times (\mathbb{R}^3)^2 \times \mathbb{R}^3;\ \boldsymbol{e} \cdot \boldsymbol{q} > 0\},\\
&\mathbb{D}_+(\boldsymbol{q}) := \{(\boldsymbol{u}_\alpha, \boldsymbol{e}) \in (\mathbb{R}^3)^2 \times \mathbb{R}^3;\ \boldsymbol{e} \cdot \boldsymbol{q} > 0\} \text{ for each } \boldsymbol{q} \in \mathbb{R}^3 \text{ with } \boldsymbol{q} \neq \boldsymbol{0}.
\end{aligned}$$

Note that the set $\mathbb{D}_+(\boldsymbol{q})$ is *convex* for each $\boldsymbol{q} \in \mathbb{R}^3$ with $\boldsymbol{q} \neq \boldsymbol{0}$.

A stored energy function $W : \omega \times \mathbb{E}_+ \to \mathbb{R}$ is said to be **orientation-preserving** if, for almost all $y \in \omega$,

$$W(y, \boldsymbol{q}, \boldsymbol{u}_\alpha) \to \infty \text{ if } (\boldsymbol{q}, \boldsymbol{u}_\alpha) \in \mathbb{E}_+ \text{ satisfies } (\boldsymbol{u}_1 \wedge \boldsymbol{u}_2) \cdot \boldsymbol{q} \to 0^+.$$

An orientation-preserving stored energy function $W : \omega \times \mathbb{E}_+ \to \mathbb{R}$ is said to be **polyconvex** if there exists a function $\mathbb{W} : \omega \times \mathbb{D}_+ \to \mathbb{R}$ with the following properties:

$$W(y, \boldsymbol{q}, \boldsymbol{u}_\alpha) = \mathbb{W}(y, \boldsymbol{q}, \boldsymbol{u}_\alpha, \boldsymbol{u}_1 \wedge \boldsymbol{u}_2) \text{ for a.e. } y \in \omega \text{ and for all } (\boldsymbol{q}, \boldsymbol{u}_\alpha) \in \mathbb{E}_+,$$
$$\mathbb{W}(y, \boldsymbol{q}, \cdot) : \mathbb{D}_+(\boldsymbol{q}) \to \mathbb{R} \text{ is } \textit{convex} \text{ for a.e. } y \in \omega \text{ and for all } \boldsymbol{q} \in \mathbb{R}^3 \text{ with } \boldsymbol{q} \neq \boldsymbol{0},$$
$$\mathbb{W}(y, \cdot) : \mathbb{D}_+ \to \mathbb{R} \text{ is } \textit{continuous} \text{ for a.e. } y \in \omega,$$
$$\mathbb{W}(\cdot, \boldsymbol{q}, \boldsymbol{u}_\alpha, \boldsymbol{e}) : \omega \to \mathbb{R} \text{ is } \textit{measurable} \text{ for all } (\boldsymbol{q}, \boldsymbol{u}_\alpha, \boldsymbol{e}) \in \mathbb{D}_+.$$

These notions of polyconvexity and orientation-preserving will be used in Section 8 to prove the existence of solution to a new nonlinear shell model, defined in Section 7 by replacing in Koiter's model the stored energy function. To do this, we will need in addition a theorem due to Ball, Currie & Olver [2, Theorem 5.4], recorded here with our notation for reader's convenience.

Theorem 1. *Let $\mathbb{W} : \omega \times \mathbb{R}^3 \times \mathbb{R}^9 \to \mathbb{R} \cup \{+\infty\}$ satisfy the following properties:*

$$\mathbb{W}(\cdot, \boldsymbol{q}, \boldsymbol{F}) : \omega \to \mathbb{R} \cup \{+\infty\} \textit{ is measurable for every } (\boldsymbol{q}, \boldsymbol{F}) \in \mathbb{R}^3 \times \mathbb{R}^9,$$
$$\mathbb{W}(y, \cdot, \cdot) : \mathbb{R}^3 \times \mathbb{R}^9 \to \mathbb{R} \cup \{+\infty\} \textit{ is continuous for almost all } y \in \omega,$$
$$\mathbb{W}(y, \boldsymbol{q}, \cdot) : \mathbb{R}^9 \to \mathbb{R} \cup \{+\infty\} \textit{ is convex for almost all } y \in \omega \textit{ and all } \boldsymbol{q} \in \mathbb{R}^3.$$

Let $\boldsymbol{\eta}_n : \omega \to \mathbb{R}^3$, $n \in \mathbb{N}$, and $\boldsymbol{\eta} : \omega \to \mathbb{R}^3$ be measurable functions such that $\boldsymbol{\eta}_n \to \boldsymbol{\eta}$ a.e. in ω as $n \to \infty$, and let $\boldsymbol{F}_n \in L^1(\omega; \mathbb{R}^9)$ and $\boldsymbol{F} \in L^1(\omega; \mathbb{R}^9)$ such that $\boldsymbol{F}_n \rightharpoonup \boldsymbol{F}$ in $L^1(\omega; \mathbb{R}^9)$ as $n \to \infty$. Suppose further that there exists a function $g \in L^1(\omega)$ such that

$$\mathbb{W}(y, \boldsymbol{\eta}_n(y), \boldsymbol{F}_n(y)) \geq g(y) \textit{ and } \mathbb{W}(y, \boldsymbol{\eta}(y), \boldsymbol{F}(y)) \geq g(y)$$

for all $n \in \mathbb{N}$ and almost all $y \in \Omega$. Then

$$\int_\omega \mathbb{W}(y, \boldsymbol{\eta}(y), \boldsymbol{F}(y))dy \leq \liminf_{n\to\infty} \int_\omega \mathbb{W}(y, \boldsymbol{\eta}_n(y), \boldsymbol{F}_n(y))dy.$$

5. A polyconvex and orientation-preserving stored energy function of membrane type

In this section we define a polyconvex and orientation-preserving stored energy function $W_M^\sharp$ that coincides to within the first order with the membrane part of Koiter's energy. To begin with, we need to establish the following preliminary result:

Lemma 1. *Given any mapping $\boldsymbol{\varphi} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$, the following relations hold in ω:*

$$a^{\alpha\beta} a_{\alpha\beta}(\boldsymbol{\varphi}) = 2 + 2a^{\alpha\beta} G_{\alpha\beta}(\boldsymbol{\varphi}),$$
$$a^{\alpha\sigma} a^{\beta\tau} a_{\sigma\tau}(\boldsymbol{\varphi}) a_{\alpha\beta}(\boldsymbol{\varphi}) = 2 + 4a^{\alpha\beta} G_{\alpha\beta}(\boldsymbol{\varphi}) + 4a^{\alpha\sigma} a^{\beta\tau} G_{\alpha\beta}(\boldsymbol{\varphi}) G_{\sigma\tau}(\boldsymbol{\varphi}),$$

and

$$a(\boldsymbol{\varphi})/a = 1 + 2\, a^{\alpha\beta} G_{\alpha\beta}(\boldsymbol{\varphi}) + 2(a^{\alpha\beta} G_{\alpha\beta}(\boldsymbol{\varphi}))^2 - 2\, a^{\alpha\beta} a^{\sigma\tau} G_{\alpha\sigma}(\boldsymbol{\varphi}) G_{\beta\tau}(\boldsymbol{\varphi}),$$
$$\log\big(a(\boldsymbol{\varphi})/a\big) = 2a^{\alpha\beta} G_{\alpha\beta}(\boldsymbol{\varphi}) - 2a^{\alpha\sigma} a^{\beta\tau} G_{\alpha\beta}(\boldsymbol{\varphi}) G_{\sigma\tau}(\boldsymbol{\varphi}) + o(\|\boldsymbol{G}(\boldsymbol{\varphi})\|^2),$$

where

$$\|\boldsymbol{G}(\boldsymbol{\varphi})\|^2 := a^{\alpha\beta} a^{\sigma\tau} G_{\alpha\sigma}(\boldsymbol{\varphi}) G_{\beta\tau}(\boldsymbol{\varphi}).$$

Proof. The first two relations are easily deduced from the definition of the functions $G_{\alpha\beta}(\boldsymbol{\varphi})$, which implies that

$$a_{\alpha\beta}(\boldsymbol{\varphi}) = a_{\alpha\beta} + 2G_{\alpha\beta}(\boldsymbol{\varphi}) \text{ in } \omega.$$

The third relation is deduced from the identity

$$\begin{aligned} a(\boldsymbol{\varphi}) &= \det\left(a_{\alpha\beta}(\boldsymbol{\varphi})\right) = \det\left(a_{\alpha\beta} + 2G_{\alpha\beta}(\boldsymbol{\varphi})\right) \\ &= a\Big[1 + 2a^{\alpha\beta}G_{\alpha\beta}(\boldsymbol{\varphi}) + 4\det\left(a^{\alpha\sigma}G_{\sigma\beta}(\boldsymbol{\varphi})\right)\Big], \end{aligned}$$

and from the Cayley-Hamilton theorem applied to the matrix field with components $G^{\alpha}_{\beta}(\boldsymbol{\varphi}) := a^{\alpha\sigma}G_{\sigma\beta}(\boldsymbol{\varphi})$, which shows that

$$G^{\alpha}_{\beta}(\boldsymbol{\varphi})G^{\beta}_{\alpha}(\boldsymbol{\varphi}) - (G^{\alpha}_{\alpha}(\boldsymbol{\varphi}))^2 + 2\det\left(G^{\alpha}_{\beta}(\boldsymbol{\varphi})\right) = 0,$$

or equivalently, that

$$2\det\left(G^{\alpha}_{\beta}(\boldsymbol{\varphi})\right) = (a^{\alpha\beta}G_{\alpha\beta}(\boldsymbol{\varphi}))^2 - a^{\alpha\beta}a^{\sigma\tau}G_{\alpha\sigma}(\boldsymbol{\varphi})G_{\beta\tau}(\boldsymbol{\varphi}).$$

□

Theorem 2. *Given any immersion $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$ and any two constants $\lambda_* > 0$ and $\mu_* > 0$, define the function*

$$W^{\sharp}_{M}[\boldsymbol{\varphi}] := \mu_*\Big[a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi}) - 2\Big] + \lambda_*\Big[\frac{a(\boldsymbol{\varphi})}{a} - 1\Big] - (\lambda_* + \mu_*)\log\Big(\frac{a(\boldsymbol{\varphi})}{a}\Big)$$

for all $\boldsymbol{\varphi} \in H^1(\omega;\mathbb{R}^3)$ that satisfy $\partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi} \neq 0$ a.e. in ω.

Then there exists a polyconvex and orientation-preserving function $W_M : \omega \times \mathbb{E}_+ \to \mathbb{R}$ (see Section 4) *such that*

$$W^{\sharp}_{M}[\boldsymbol{\varphi}] = W_M(\cdot, \boldsymbol{a}_3(\boldsymbol{\varphi}), \partial_\alpha\boldsymbol{\varphi}) \text{ a.e. in } \omega,$$

for all $\boldsymbol{\varphi} \in H^1(\omega;\mathbb{R}^3)$ that satisfy $\partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi} \neq \boldsymbol{0}$ a.e. in ω.

Besides, for each $\boldsymbol{\varphi} \in \mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$,

$$W^{\sharp}_{M}[\boldsymbol{\varphi}] = a^{\alpha\beta\sigma\tau}_{*}G_{\sigma\tau}(\boldsymbol{\varphi})G_{\alpha\beta}(\boldsymbol{\varphi}) + o(\|\boldsymbol{G}(\boldsymbol{\varphi})\|^2) \text{ in } \omega,$$

where

$$a^{\alpha\beta\sigma\tau}_{*} := 2\lambda_* a^{\alpha\beta}a^{\sigma\tau} + \mu_*(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}).$$

Proof. Let (cf. Section 4)

$$\begin{aligned} \mathbb{E}_+ &:= \{(\boldsymbol{q}, \boldsymbol{u}_\alpha) \in (\mathbb{R}^3)^3;\ (\boldsymbol{u}_1 \wedge \boldsymbol{u}_2)\cdot\boldsymbol{q} > 0\}, \\ \mathbb{D}_+ &:= \{(\boldsymbol{q}, \boldsymbol{u}_\alpha, \boldsymbol{e}) \in (\mathbb{R}^3)^4;\ \boldsymbol{e}\cdot\boldsymbol{q} > 0\}, \\ \mathbb{D}_+(\boldsymbol{q}) &:= \{(\boldsymbol{u}_\alpha, \boldsymbol{e}) \in (\mathbb{R}^3)^3;\ \boldsymbol{e}\cdot\boldsymbol{q} > 0\} \text{ for each } \boldsymbol{q} \in \mathbb{R}^3 \text{ with } \boldsymbol{q} \neq \boldsymbol{0}. \end{aligned}$$

Define the function $\mathbb{W}_M : \omega \times \mathbb{D}_+ \to \mathbb{R}$ by letting, for each $y \in \omega$ and each $(\boldsymbol{q}, \boldsymbol{u}_\alpha, \boldsymbol{e}) \in \mathbb{D}_+$,

$$\begin{aligned} \mathbb{W}_M(y, \boldsymbol{q}, \boldsymbol{u}_\alpha, \boldsymbol{e}) := \mu_*\Big[a^{\alpha\beta}(y)\boldsymbol{u}_\alpha\cdot\boldsymbol{u}_\beta - 2\Big] + \lambda_*\Big[\frac{(\boldsymbol{e}\cdot\boldsymbol{q})^2}{a(y)} - 1\Big] \\ - (\lambda_* + \mu_*)\log\Big(\frac{(\boldsymbol{e}\cdot\boldsymbol{q})^2}{a(y)}\Big), \end{aligned}$$

and the function $W_M : \omega \times \mathbb{E}_+ \to \mathbb{R}$ by letting, for each $y \in \omega$ and each $(\boldsymbol{q}, \boldsymbol{u}_\alpha) \in \mathbb{E}_+$,

$$W_M(y, \boldsymbol{q}, \boldsymbol{u}_\alpha) := \mathbb{W}_M(y, \boldsymbol{q}, \boldsymbol{u}_\alpha, \boldsymbol{u}_1 \wedge \boldsymbol{u}_2).$$

Since the matrix $(a^{\alpha\beta}(y))$ is symmetric and positive-definite at each $y \in \overline{\omega}$, the mapping

$$(\boldsymbol{u}_\alpha) \in \mathbb{R}^3 \times \mathbb{R}^3 \to \big(a^{\alpha\beta}(y)\boldsymbol{u}_\alpha \cdot \boldsymbol{u}_\beta\big)^{1/2} \in \mathbb{R}$$

is a norm over the space $\mathbb{R}^3 \times \mathbb{R}^3$. Together with the convexity of the functions $t \in \mathbb{R} \to t^2 \in \mathbb{R}$ and $t \in (0,\infty) \to -\log t \in \mathbb{R}$, this property implies that, for each $y \in \omega$ and each $\boldsymbol{q} \neq \boldsymbol{0}$, the mapping $\mathbb{W}_M(y, \boldsymbol{q}, \cdot) : \mathbb{D}_+(\boldsymbol{q}) \to \mathbb{R}$ is convex. Besides, the mapping $\mathbb{W}_M : \omega \times \mathbb{D}_+ \to \mathbb{R}$ is continuous and satisfies

$$\mathbb{W}_M(y, \boldsymbol{q}, \boldsymbol{u}_\alpha, \boldsymbol{e}) \to \infty \text{ if } (y, \boldsymbol{q}, \boldsymbol{u}_\alpha, \boldsymbol{e}) \in \omega \times \mathbb{D}_+ \text{ with } \boldsymbol{e} \cdot \boldsymbol{q} \to 0^+.$$

Hence the function $W_M : \omega \times \mathbb{E}_+ \to \mathbb{R}$ is polyconvex and orientation-preserving in the sense of Section 4.

The relations $\boldsymbol{a}_\alpha(\boldsymbol{\varphi}) \cdot \boldsymbol{a}_\beta(\boldsymbol{\varphi}) = a_{\alpha\beta}(\boldsymbol{\varphi})$ and $(\boldsymbol{a}_1(\boldsymbol{\varphi}) \wedge \boldsymbol{a}_2(\boldsymbol{\varphi})) \cdot \boldsymbol{a}_3(\boldsymbol{\varphi}) = \sqrt{a(\boldsymbol{\varphi})}$ show that, for each $\boldsymbol{\varphi} \in H^1(\omega; \mathbb{R}^3)$ such that $\partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi} \neq \boldsymbol{0}$ a.e. in ω,

$$W_M^\sharp[\boldsymbol{\varphi}] = W_M(\cdot, \boldsymbol{a}_3(\boldsymbol{\varphi}), \boldsymbol{a}_\alpha(\boldsymbol{\varphi})) = \mathbb{W}_M(\cdot, \boldsymbol{a}_3(\boldsymbol{\varphi}), \boldsymbol{a}_\alpha(\boldsymbol{\varphi}), \boldsymbol{a}_1(\boldsymbol{\varphi}) \wedge \boldsymbol{a}_2(\boldsymbol{\varphi})) \text{ a.e. in } \omega.$$

This completes the proof of the first part of the theorem.

It remains to prove that $W_M^\sharp[\boldsymbol{\varphi}]$ depends on $\boldsymbol{\varphi}$ only by means of $G_{\alpha\beta}(\boldsymbol{\varphi})$, then to identify the first order term of $W_M^\sharp[\boldsymbol{\varphi}]$ with respect to $G_{\alpha\beta}(\boldsymbol{\varphi})$. Let for brevity

$$G_{\alpha\beta} := G_{\alpha\beta}(\boldsymbol{\varphi}) \text{ and } \|\boldsymbol{G}\|^2 := a^{\alpha\beta}a^{\sigma\tau}G_{\alpha\sigma}G_{\beta\tau}.$$

Using the Taylor expansions of Lemma 1 in the definition of $W_M^\sharp[\boldsymbol{\varphi}]$, we deduce that

$$\begin{aligned}
W_M^\sharp[\boldsymbol{\varphi}] &= \mu_*\Big[a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi}) - 2\Big] + \lambda_*\Big[\frac{a(\boldsymbol{\varphi})}{a} - 1\Big] - (\lambda_* + \mu_*)\log\big(\frac{a(\boldsymbol{\varphi})}{a}\big) \\
&= 2\mu_*\, a^{\alpha\beta}G_{\alpha\beta} + 2\lambda_*\Big[a^{\alpha\beta}G_{\alpha\beta} + (a^{\alpha\beta}G_{\alpha\beta})^2 - a^{\alpha\beta}a^{\sigma\tau}G_{\alpha\sigma}G_{\beta\tau}\Big] \\
&\quad - 2(\lambda_* + \mu_*)\Big[a^{\alpha\beta}G_{\alpha\beta} - a^{\alpha\beta}a^{\sigma\tau}G_{\alpha\sigma}G_{\beta\tau}\Big] + o(\|\boldsymbol{G}\|^2) \\
&= 2\lambda_*(a^{\alpha\beta}G_{\alpha\beta})^2 + 2\mu_*\, a^{\alpha\beta}a^{\sigma\tau}G_{\alpha\sigma}G_{\beta\tau} + o(\|\boldsymbol{G}\|^2) \\
&= a_*^{\alpha\beta\sigma\tau}G_{\sigma\tau}G_{\alpha\beta} + o(\|\boldsymbol{G}\|^2).
\end{aligned}$$

□

Remark. The definition of the function $W_M^\sharp$ can be replaced in Theorem 2 by the following more general expression:

$$\begin{aligned}
W_M^\sharp[\boldsymbol{\varphi}] := C_1\Big[a^{\alpha\sigma}a^{\beta\tau}a_{\sigma\tau}(\boldsymbol{\varphi})a_{\alpha\beta}(\boldsymbol{\varphi})\Big]^{p_1} + C_2\Big[a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi})\Big]^{p_2} \\
+ C_3\Big[\frac{a(\boldsymbol{\varphi})}{a}\Big]^{p_3} - C_4\log\Big[\frac{a(\boldsymbol{\varphi})}{a}\Big] - C_5,
\end{aligned}$$

for appropriate choices (of which there exist infinitely many) of the constants $C_k > 0$ and $p_k > 1$.

6. A polyconvex and orientation-preserving stored energy function of flexural type

We define in this section a polyconvex and orientation-preserving stored energy function "of flexural type" in the particular case where the middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ of the shell is *elliptic*, in contrast with the previous section where no such restriction was needed. This assumption is needed here to ensure that the third fundamental form of S is positive-definite at each point of $\overline{\omega}$.

More specifically, we assume in this section that the immersion $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ satisfies

$$(b_{\alpha\beta}(y)) \in \mathbb{S}^2_> \text{ for every } y \in \overline{\omega},$$

where $b_{\alpha\beta} := \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \boldsymbol{a}_3$ denote the covariant components of the second fundamental form of $S = \boldsymbol{\theta}(\overline{\omega})$ (see Section 2). This assumption implies that the covariant components $c_{\alpha\beta} := \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3 \in \mathcal{C}^1(\overline{\omega})$ of the third fundamental form of S also satisfy

$$(c_{\alpha\beta}(y)) \in \mathbb{S}^2_> \text{ for every } y \in \overline{\omega}.$$

For each $y \in \overline{\omega}$, let $(\hat{b}^{\alpha\beta}(y)) \in \mathbb{S}^2_>$, resp. $(\hat{c}^{\alpha\beta}(y)) \in \mathbb{S}^2_>$, denote the inverse matrix of the matrix $(b_{\alpha\beta}(y)) \in \mathbb{S}^2_>$, resp. $(c_{\alpha\beta}(y)) \in \mathbb{S}^2_>$. Note that $\hat{c}^{\alpha\beta} := \hat{b}^{\alpha\sigma} a_{\sigma\tau} \hat{b}^{\tau\beta}$ and that the functions $\hat{b}^{\alpha\beta}$ and $\hat{c}^{\alpha\beta}$ are distinct from the contravariant components

$$b^{\alpha\beta} = a^{\alpha\sigma} b_{\sigma\tau} a^{\tau\beta} \text{ and } c^{\alpha\beta} = a^{\alpha\sigma} c_{\sigma\tau} a^{\tau\beta}$$

of the second and third fundamental forms of $S = \boldsymbol{\theta}(\overline{\omega})$.

To begin with, we show how the tensor field $P_{\alpha\beta}(\boldsymbol{\varphi})$ can be expressed in terms of the tensor fields $G_{\alpha\beta}(\boldsymbol{\varphi})$ and $R_{\alpha\beta}(\boldsymbol{\varphi})$ (these tensor fields are defined in Section 3).

Lemma 2. *Let*

$$\boldsymbol{U}^\flat(\overline{\omega}) := \{\boldsymbol{\varphi} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3);\ \partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi} \neq \boldsymbol{0} \text{ in } \overline{\omega},\ \boldsymbol{a}_3(\boldsymbol{\varphi}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)\}.$$

Then the following relation holds in ω for every $\boldsymbol{\varphi} \in \boldsymbol{U}^\flat(\overline{\omega})$ that is sufficiently close in the $\mathcal{C}^1(\overline{\omega})$-norm to the immersion $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$:

$$\begin{aligned} P_{\alpha\beta}(\boldsymbol{\varphi}) = & -b^\sigma_\alpha b^\tau_\beta G_{\sigma\tau}(\boldsymbol{\varphi}) + \frac{1}{2}\Big[b^\sigma_\alpha R_{\sigma\beta}(\boldsymbol{\varphi}) + b^\sigma_\beta R_{\alpha\sigma}(\boldsymbol{\varphi}) + a^{\sigma\tau} R_{\alpha\sigma}(\boldsymbol{\varphi}) R_{\tau\beta}(\boldsymbol{\varphi})\Big] \\ & + 2 b^\gamma_\alpha b^\delta_\beta a^{\sigma\tau} G_{\gamma\sigma}(\boldsymbol{\varphi}) G_{\delta\tau}(\boldsymbol{\varphi}) - a^{\sigma\tau} G_{\gamma\sigma}(\boldsymbol{\varphi}) \Big[b^\gamma_\alpha R_{\beta\tau}(\boldsymbol{\varphi}) + b^\gamma_\beta R_{\alpha\tau}(\boldsymbol{\varphi})\Big] \\ & + o(\|\boldsymbol{G}(\boldsymbol{\varphi})\|^2 + \|\boldsymbol{R}(\boldsymbol{\varphi})\|^2). \end{aligned}$$

Proof. Using the power series expansion of the inverse of a matrix of the form $(I + A)^{-1}$, where I denotes the identity matrix and $\|A\| < 1$ for some subordinate matrix norm (this is where the assumption that $\boldsymbol{\varphi}$ be sufficiently close to $\boldsymbol{\theta}$ in the $\mathcal{C}^1(\overline{\omega})$-norm is used), in the identity

$$a_{\alpha\beta}(\boldsymbol{\varphi}) = a_{\alpha\sigma}(\delta^\sigma_\beta + 2a^{\sigma\tau} G_{\tau\beta}(\boldsymbol{\varphi})),$$

we first deduce that

$$a^{\alpha\beta}(\boldsymbol{\varphi}) = a^{\alpha\beta} - 2\hat{G}^{\alpha\beta}(\boldsymbol{\varphi}),$$

where

$$\hat{G}^{\alpha\beta}(\boldsymbol{\varphi}) = a^{\alpha\sigma} a^{\beta\tau} G_{\sigma\tau}(\boldsymbol{\varphi}) - 2a^{\alpha\gamma} a^{\beta\nu} a^{\sigma\tau} G_{\gamma\sigma}(\boldsymbol{\varphi}) G_{\nu\tau}(\boldsymbol{\varphi}) + o(\|\boldsymbol{G}(\boldsymbol{\varphi})\|^2).$$

Using these relations and the relation

$$b_{\alpha\beta}(\boldsymbol{\varphi}) = b_{\alpha\beta} + R_{\alpha\beta}(\boldsymbol{\varphi})$$

in the right-hand side of the relation

$$P_{\alpha\beta}(\boldsymbol{\varphi}) := \frac{1}{2}(c_{\alpha\beta}(\boldsymbol{\varphi}) - c_{\alpha\beta}) = \frac{1}{2}(a^{\sigma\tau}(\boldsymbol{\varphi})b_{\alpha\sigma}(\boldsymbol{\varphi})b_{\tau\beta}(\boldsymbol{\varphi}) - a^{\sigma\tau}b_{\alpha\sigma}b_{\tau\beta}),$$

we then deduce, after a series of straightforward calculations, that (for simplicity we omit the dependence on $\boldsymbol{\varphi}$ of the functions $P_{\alpha\beta}, G_{\alpha\beta}$, and $R_{\alpha\beta}$, appearing in the formula below)

$$\begin{aligned} P_{\alpha\beta} &= -b_{\alpha\sigma}a^{\sigma\tau}G_{\tau\gamma}a^{\gamma\delta}b_{\delta\beta} + \frac{1}{2}(b_{\alpha\sigma}a^{\sigma\tau}R_{\tau\beta} + R_{\alpha\sigma}a^{\sigma\tau}b_{\tau\beta} + R_{\alpha\sigma}a^{\sigma\tau}R_{\tau\beta}) \\ &\quad + 2b_{\alpha\sigma}a^{\sigma\tau}G_{\tau\gamma}a^{\gamma\delta}G_{\delta\mu}a^{\mu\nu}b_{\nu\beta} - R_{\alpha\sigma}a^{\sigma\tau}G_{\tau\gamma}a^{\gamma\delta}b_{\delta\beta} - b_{\alpha\sigma}a^{\sigma\tau}G_{\tau\gamma}a^{\gamma\delta}R_{\delta\beta} \\ &\quad + o(\|\boldsymbol{G}\|^2 + \|\boldsymbol{R}\|^2). \end{aligned}$$

□

Theorem 3. *Given any immersion $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$ such that the surface $S = \boldsymbol{\theta}(\overline{\omega})$ is* elliptic *and given any two constants $\hat{\lambda}_* > 0$ and $\hat{\mu}_* > 0$, let*

$$W_F^\sharp[\boldsymbol{\varphi}] := K\Big[\hat{\mu}_*\Big(\hat{c}^{\alpha\beta}c_{\alpha\beta}(\boldsymbol{\varphi}) - 2\Big) + \hat{\lambda}_*\Big(\frac{c(\boldsymbol{\varphi})}{c} - 1\Big) - (\hat{\lambda}_* + \hat{\mu}_*)\log\Big(\frac{c(\boldsymbol{\varphi})}{c}\Big)\Big]$$

for each $\boldsymbol{\varphi} \in \boldsymbol{U}^\sharp(\omega)$, where

$$\begin{aligned} \boldsymbol{U}^\sharp(\omega) := \{\boldsymbol{\varphi} \in H^1(\omega;\mathbb{R}^3);\ &\partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi} \neq \boldsymbol{0} \text{ a.e. in } \omega, \\ &\boldsymbol{a}_3(\boldsymbol{\varphi}) \in H^1(\omega;\mathbb{R}^3),\ K(\boldsymbol{\varphi}) > 0 \text{ a.e. in } \omega\}, \end{aligned}$$

and

$$K(\boldsymbol{\varphi}) := \frac{1}{|\partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi}|}\,(\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi}) \wedge \partial_2\boldsymbol{a}_3(\boldsymbol{\varphi})) \cdot \boldsymbol{a}_3(\boldsymbol{\varphi}) \quad \text{and} \quad K := K(\boldsymbol{\theta}).$$

(a) *Then there exists a polyconvex and orientation-preserving function $W_F : \omega \times \mathbb{E}_+ \to \mathbb{R}$* (see Section 4) *such that, for each $\boldsymbol{\varphi} \in \boldsymbol{U}^\sharp(\omega)$,*

$$W_F^\sharp[\boldsymbol{\varphi}] = W_F(\cdot, \boldsymbol{a}_3(\boldsymbol{\varphi}), \partial_\alpha\boldsymbol{a}_3(\boldsymbol{\varphi})) \text{ a.e. in } \omega.$$

Besides, for each $\boldsymbol{\varphi} \in \boldsymbol{U}^\flat(\overline{\omega})$ (the set $\boldsymbol{U}^\flat(\overline{\omega})$ is defined in Lemma 2),

$$\begin{aligned} W_F^\sharp[\boldsymbol{\varphi}] &= \hat{c}_*^{\alpha\beta\sigma\tau}P_{\sigma\tau}(\boldsymbol{\varphi})P_{\alpha\beta}(\boldsymbol{\varphi}) + o(\|\boldsymbol{P}(\boldsymbol{\varphi})\|^2) \\ &= \hat{b}_*^{\alpha\beta\sigma\tau}R_{\sigma\tau}(\boldsymbol{\varphi})R_{\alpha\beta}(\boldsymbol{\varphi}) + o(\|\boldsymbol{R}(\boldsymbol{\varphi})\|^2) + \mathcal{O}(\|\boldsymbol{G}(\boldsymbol{\varphi})\|(\|\boldsymbol{G}(\boldsymbol{\varphi})\| + \|\boldsymbol{R}(\boldsymbol{\varphi})\|)), \end{aligned}$$

where

$$\begin{aligned} \hat{b}_*^{\alpha\beta\sigma\tau} &:= K\Big[2\hat{\lambda}_*\hat{b}^{\alpha\beta}\hat{b}^{\sigma\tau} + \frac{\hat{\mu}_*}{2}(\hat{b}^{\alpha\sigma}\hat{b}^{\beta\tau} + \hat{b}^{\alpha\tau}\hat{b}^{\beta\sigma}) + \frac{\hat{\mu}_*}{4}(\hat{c}^{\alpha\sigma}a^{\beta\tau} + \hat{c}^{\alpha\tau}a^{\beta\sigma} + a^{\alpha\sigma}\hat{c}^{\beta\tau} + a^{\alpha\tau}\hat{c}^{\beta\sigma})\Big], \\ \hat{c}_*^{\alpha\beta\sigma\tau} &:= K\Big[2\hat{\lambda}_*\hat{c}^{\alpha\beta}\hat{c}^{\sigma\tau} + \hat{\mu}_*(\hat{c}^{\alpha\sigma}\hat{c}^{\beta\tau} + \hat{c}^{\alpha\tau}\hat{c}^{\beta\sigma})\Big]. \end{aligned}$$

(b) *If the surface $S = \boldsymbol{\theta}(\overline{\omega})$ is a portion of a sphere, then, for each $\boldsymbol{\varphi} \in \boldsymbol{U}^\flat(\overline{\omega})$,*

$$W_F^\sharp[\boldsymbol{\varphi}] = \hat{a}_*^{\alpha\beta\sigma\tau}R_{\sigma\tau}(\boldsymbol{\varphi})R_{\alpha\beta}(\boldsymbol{\varphi}) + o(\|\boldsymbol{R}(\boldsymbol{\varphi})\|^2) + \mathcal{O}(\|\boldsymbol{G}(\boldsymbol{\varphi})\|(\|\boldsymbol{G}(\boldsymbol{\varphi})\| + \|\boldsymbol{R}(\boldsymbol{\varphi})\|)),$$

where

$$\hat{a}_*^{\alpha\beta\sigma\tau} := 2\hat{\lambda}_*a^{\alpha\beta}a^{\sigma\tau} + \hat{\mu}_*(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}).$$

Proof. Define the function $\mathbb{W}_F : \omega \times \mathbb{D}_+ \to \mathbb{R}$ by letting, for each $y \in \omega$ and each $(\boldsymbol{q}, \boldsymbol{v}_\alpha, \boldsymbol{f}) \in \mathbb{D}_+$,

$$\mathbb{W}_F(y, \boldsymbol{q}, \boldsymbol{v}_\alpha, \boldsymbol{f}) := K\Big[\hat{\mu}_*\Big(\hat{c}^{\alpha\beta}(y)\boldsymbol{v}_\alpha \cdot \boldsymbol{v}_\beta - 2\Big) + \hat{\lambda}_*\Big(\frac{(\boldsymbol{f} \cdot \boldsymbol{q})^2}{c(y)} - 1\Big) - (\hat{\lambda}_* + \hat{\mu}_*) \log\Big(\frac{(\boldsymbol{f} \cdot \boldsymbol{q})^2}{c(y)}\Big)\Big],$$

and define the function $W_F : \omega \times \mathbb{E}_+ \to \mathbb{R}$ by letting, for each $y \in \omega$ and each $(\boldsymbol{q}, \boldsymbol{v}_\alpha) \in \mathbb{E}_+$,

$$W_F(y, \boldsymbol{q}, \boldsymbol{v}_\alpha) := \mathbb{W}_F(y, \boldsymbol{q}, \boldsymbol{v}_\alpha, \boldsymbol{v}_1 \wedge \boldsymbol{v}_2).$$

Since the functions $c_{\alpha\beta}$ coincide with the covariant components of the first fundamental form of the surface $\boldsymbol{a}_3(\overline{\omega}) \subset \mathbb{R}^3$ ($\boldsymbol{a}_3 : \overline{\omega} \to \mathbb{R}^3$ is an immersion since the surface $S = \boldsymbol{\theta}(\overline{\omega})$ is elliptic), and since $K > 0$ in $\overline{\omega}$, Theorem 2 shows that W_F is polyconvex and orientation-preserving (these notions have been defined in Section 4), that

$$W_F^\sharp[\boldsymbol{\varphi}] = W_F(\cdot, \boldsymbol{a}_3(\boldsymbol{\varphi}), \partial_\alpha \boldsymbol{a}_3(\boldsymbol{\varphi})) = \mathbb{W}_F(\cdot, \boldsymbol{a}_3(\boldsymbol{\varphi}), \partial_\alpha \boldsymbol{a}_3(\boldsymbol{\varphi}), \partial_1 \boldsymbol{a}_3(\boldsymbol{\varphi}) \wedge \partial_2 \boldsymbol{a}_3(\boldsymbol{\varphi}))$$

for all $\boldsymbol{\varphi} \in \boldsymbol{U}^\sharp(\omega)$, and that

$$W_F^\sharp[\boldsymbol{\varphi}] = \hat{c}_*^{\alpha\beta\sigma\tau} P_{\sigma\tau}(\boldsymbol{\varphi}) P_{\alpha\beta}(\boldsymbol{\varphi}) + o(\|\boldsymbol{P}(\boldsymbol{\varphi})\|^2)$$

for all $\boldsymbol{\varphi} \in \boldsymbol{U}^\flat(\overline{\omega})$.

It remains to identify the first order term of $W_F^\sharp[\boldsymbol{\varphi}]$ with respect to $G_{\alpha\beta}(\boldsymbol{\varphi})$ and $R_{\alpha\beta}(\boldsymbol{\varphi})$. By replacing the functions $P_{\sigma\tau}(\boldsymbol{\varphi})$ and $P_{\alpha\beta}(\boldsymbol{\varphi})$ appearing in the above formula by their expressions given in Lemma 2, viz. (for simplicity, we omit in this proof the dependence on $\boldsymbol{\varphi}$ of the functions $P_{\alpha\beta}, G_{\alpha\beta}$, and $R_{\alpha\beta}$),

$$\begin{aligned} P_{\alpha\beta} &= -b_\alpha^\gamma b_\beta^\delta G_{\gamma\delta} + \frac{1}{2}\Big[b_\alpha^\gamma R_{\gamma\beta} + b_\beta^\gamma R_{\alpha\gamma} + a^{\gamma\delta} R_{\alpha\gamma} R_{\delta\beta}\Big] \\ &\quad + 2b_\alpha^\gamma b_\beta^\delta a^{\lambda\mu} G_{\gamma\lambda} G_{\delta\mu} - a^{\lambda\mu} G_{\gamma\lambda}\Big[b_\alpha^\gamma R_{\beta\mu} + b_\beta^\gamma R_{\alpha\mu}\Big] + o(\|\boldsymbol{G}\|^2 + \|\boldsymbol{R}\|^2), \\ P_{\sigma\tau} &= -b_\sigma^\lambda b_\tau^\mu G_{\lambda\mu} + \frac{1}{2}\Big[b_\sigma^\lambda R_{\lambda\tau} + b_\tau^\lambda R_{\sigma\lambda} + a^{\lambda\mu} R_{\sigma\lambda} R_{\mu\tau}\Big] \\ &\quad + 2b_\sigma^\gamma b_\tau^\delta a^{\rho\nu} G_{\gamma\rho} G_{\delta\nu} - a^{\rho\nu} G_{\gamma\rho}\Big[b_\sigma^\gamma R_{\tau\nu} + b_\tau^\gamma R_{\sigma\nu}\Big] + o(\|\boldsymbol{G}\|^2 + \|\boldsymbol{R}\|^2), \end{aligned}$$

we first deduce that

$$\begin{aligned} W_F^\sharp[\boldsymbol{\varphi}] &= \hat{c}_*^{\alpha\beta\sigma\tau} b_\sigma^\lambda b_\tau^\mu G_{\lambda\mu} b_\alpha^\gamma b_\beta^\delta G_{\gamma\delta} \\ &\quad - \frac{1}{2}\hat{c}_*^{\alpha\beta\sigma\tau} b_\sigma^\lambda b_\tau^\mu G_{\lambda\mu}\Big[b_\alpha^\gamma R_{\gamma\beta} + b_\beta^\gamma R_{\alpha\gamma}\Big] - \frac{1}{2}\hat{c}_*^{\alpha\beta\sigma\tau}\Big[b_\sigma^\lambda R_{\lambda\tau} + b_\tau^\lambda R_{\sigma\lambda}\Big] b_\alpha^\gamma b_\beta^\delta G_{\gamma\delta} \\ &\quad + \frac{1}{4}\hat{c}_*^{\alpha\beta\sigma\tau}\Big[b_\sigma^\lambda R_{\lambda\tau} + b_\tau^\lambda R_{\sigma\lambda}\Big]\Big[b_\alpha^\gamma R_{\gamma\beta} + b_\beta^\gamma R_{\alpha\gamma}\Big] + o(\|\boldsymbol{G}\|^2 + \|\boldsymbol{R}\|^2). \end{aligned}$$

This in turn implies that

$$\begin{aligned} W_F^\sharp[\boldsymbol{\varphi}] &= \frac{1}{4}\hat{c}_*^{\alpha\beta\sigma\tau}\Big[b_\sigma^\lambda R_{\lambda\tau} + b_\tau^\lambda R_{\sigma\lambda}\Big]\Big[b_\alpha^\gamma R_{\gamma\beta} + b_\beta^\gamma R_{\alpha\gamma}\Big] \\ &\quad + \mathcal{O}(\|\boldsymbol{G}\|^2 + \|\boldsymbol{G}\|\|\boldsymbol{R}\|) + o(\|\boldsymbol{R}\|^2) \\ &= \frac{1}{4}\Big[\hat{c}_*^{\gamma\beta\lambda\tau} b_\lambda^\sigma b_\gamma^\alpha + \hat{c}_*^{\alpha\gamma\lambda\tau} b_\lambda^\sigma b_\gamma^\beta + \hat{c}_*^{\gamma\beta\sigma\lambda} b_\gamma^\alpha b_\lambda^\tau + \hat{c}_*^{\alpha\gamma\sigma\lambda} b_\gamma^\beta b_\lambda^\tau\Big] R_{\sigma\tau} R_{\alpha\beta} \\ &\quad + \mathcal{O}(\|\boldsymbol{G}\|^2 + \|\boldsymbol{G}\|\|\boldsymbol{R}\|) + o(\|\boldsymbol{R}\|^2). \end{aligned}$$

Then the conclusion follows by noting that

$$\frac{1}{4}\Big[\hat{c}_*^{\gamma\beta\lambda\tau} b_\lambda^\sigma b_\gamma^\alpha + \hat{c}_*^{\alpha\gamma\lambda\tau} b_\lambda^\sigma b_\gamma^\beta + \hat{c}_*^{\gamma\beta\sigma\lambda} b_\gamma^\alpha b_\lambda^\tau + \hat{c}_*^{\alpha\gamma\sigma\lambda} b_\gamma^\beta b_\lambda^\tau\Big] = \hat{b}_*^{\alpha\beta\sigma\tau}.$$

In the particular case where $S = \boldsymbol{\theta}(\overline{\omega})$ is a *portion of a sphere*, the mean curvature H and the total curvature K of the surface S are related by $K = H^2$. Consequently, the Cayley-Hamilton theorem implies that

$$(b_{\alpha\sigma} - Ha_{\alpha\sigma})a^{\sigma\tau}(b_{\tau\beta} - Ha_{\tau\beta}) = b_{\alpha\sigma}a^{\sigma\tau}b_{\tau\beta} - 2Hb_{\alpha\beta} + Ka_{\alpha\beta} = 0,$$

which in turn implies that

$$b_{\alpha\beta} = Ha_{\alpha\beta},\ c_{\alpha\beta} = Ka_{\alpha\beta},\ \hat{b}^{\alpha\beta} = (1/H)a^{\alpha\beta},\ \hat{c}^{\alpha\beta} = (1/K)a^{\alpha\beta}.$$

Since $K = H^2$, it follows that

$$\hat{b}_*^{\alpha\beta\sigma\tau} = \hat{a}_*^{\alpha\beta\sigma\tau}.$$

This completes the proof. □

7. A polyconvex and orientation-preserving stored energy function of Koiter's type

By combining the results of the previous two sections, we are now able to define a polyconvex and orientation-preserving stored energy function $W_K^\sharp[\boldsymbol{\varphi}]$ that has in addition the property that, when the middle surface of the undeformed shell is a portion of a sphere, its leading term with respect to the change of metric and change of curvature tensors is precisely Koiter's stored energy function.

Theorem 4. (a) *Given any immersion $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$ such that the surface $S = \boldsymbol{\theta}(\overline{\omega})$ is elliptic, define for each $\varepsilon > 0$ the stored energy function*

$$W_K^\sharp[\boldsymbol{\varphi}] := \varepsilon W_M^\sharp[\boldsymbol{\varphi}] + \varepsilon^3 W_F^\sharp[\boldsymbol{\varphi}] \ \text{ for all } \boldsymbol{\varphi} \in \boldsymbol{U}^\sharp(\omega),$$

where (see Theorems 2 and 3)

$$\begin{aligned}
W_M^\sharp[\boldsymbol{\varphi}] &:= \mu_*\Big[a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi}) - 2\Big] + \lambda_*\Big[\frac{a(\boldsymbol{\varphi})}{a} - 1\Big] - (\lambda_* + \mu_*)\log\Big(\frac{a(\boldsymbol{\varphi})}{a}\Big),\\
W_F^\sharp[\boldsymbol{\varphi}] &:= K\Big[\hat{\mu}_*\Big(\hat{c}^{\alpha\beta}c_{\alpha\beta}(\boldsymbol{\varphi}) - 2\Big) + \hat{\lambda}_*\Big(\frac{c(\boldsymbol{\varphi})}{c} - 1\Big) - (\hat{\lambda}_* + \hat{\mu}_*)\log\Big(\frac{c(\boldsymbol{\varphi})}{c}\Big)\Big],\\
\boldsymbol{U}^\sharp(\omega) &:= \{\boldsymbol{\varphi} \in H^1(\omega;\mathbb{R}^3);\ \partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi} \neq \boldsymbol{0} \text{ a.e. in } \omega,\\
&\qquad \boldsymbol{a}_3(\boldsymbol{\varphi}) \in H^1(\omega;\mathbb{R}^3),\ K(\boldsymbol{\varphi}) > 0 \text{ a.e. in } \omega\},
\end{aligned}$$

with $K(\boldsymbol{\varphi}) := |\partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi}|^{-1}\,(\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi}) \wedge \partial_2\boldsymbol{a}_3(\boldsymbol{\varphi})) \cdot \boldsymbol{a}_3(\boldsymbol{\varphi})$.

Then, for each $\boldsymbol{\varphi} \in \boldsymbol{U}^\flat(\overline{\omega})$,

$$\begin{aligned}
W_K^\sharp[\boldsymbol{\varphi}] &= \varepsilon a_*^{\alpha\beta\sigma\tau}G_{\sigma\tau}(\boldsymbol{\varphi})G_{\alpha\beta}(\boldsymbol{\varphi}) + \varepsilon^3\hat{b}_*^{\alpha\beta\sigma\tau}R_{\sigma\tau}(\boldsymbol{\varphi})R_{\alpha\beta}(\boldsymbol{\varphi})\\
&\quad + \mathcal{O}(\varepsilon^2\|\boldsymbol{G}(\boldsymbol{\varphi})\|^2 + \varepsilon^4\|\boldsymbol{R}(\boldsymbol{\varphi})\|^2) + o(\varepsilon\|\boldsymbol{G}(\boldsymbol{\varphi})\|^2 + \varepsilon^3\|\boldsymbol{R}(\boldsymbol{\varphi})\|^2).
\end{aligned}$$

(b) *In the particular case where $S = \boldsymbol{\theta}(\overline{\omega})$ is a portion of a sphere and the constants appearing in the definition of $W_M^\sharp[\boldsymbol{\varphi}]$ and $W_F^\sharp[\boldsymbol{\varphi}]$ are defined by*

$$\lambda_* := \frac{\lambda\mu}{\lambda + 2\mu},\ \mu_* := \mu,\ \hat{\lambda}_* := \frac{\lambda\mu}{3(\lambda + 2\mu)},\ \hat{\mu}_* := \frac{\mu}{3},$$

where $\lambda > 0$ *and* $\mu > 0$ *denote the Lamé constants appearing in the definition of the two-dimensional elasticity tensor* $a^{\alpha\beta\sigma\tau}$ *used to define Koiter's energy* J_K (see Section 3), *then*

$$W_K^\sharp[\boldsymbol{\varphi}] := \frac{\varepsilon}{2} a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\varphi}) G_{\alpha\beta}(\boldsymbol{\varphi}) + \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\varphi}) R_{\alpha\beta}(\boldsymbol{\varphi}) \\ + \mathcal{O}(\varepsilon^2 \|\boldsymbol{G}(\boldsymbol{\varphi})\|^2 + \varepsilon^4 \|\boldsymbol{R}(\boldsymbol{\varphi})\|^2) + o(\varepsilon \|\boldsymbol{G}(\boldsymbol{\varphi})\|^2 + \varepsilon^3 \|\boldsymbol{R}(\boldsymbol{\varphi})\|^2).$$

Proof. The assertions of the theorem are simple consequences of Theorems 2 and 3. ☐

8. Existence theorem for a nonlinear shell model of Koiter's type

We are now in a position to establish an existence theorem for a nonlinear shell model whose total energy coincides to within the first order with Koiter's energy when the middle surface of the undeformed shell is a portion of a sphere; cf. Theorem 4(b).

Theorem 5. *Given any immersion* $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ *such that the surface* $S = \boldsymbol{\theta}(\overline{\omega})$ *is elliptic, any* $\varepsilon > 0$, *and any non-empty relatively open subset* γ_0 *of* $\gamma := \partial\omega$, *define the functional* $J : \boldsymbol{U}(\omega) \to \mathbb{R} \cup \{+\infty\}$ *by letting*

$$\boldsymbol{U}(\omega) := \{\boldsymbol{\varphi} \in H^1(\omega; \mathbb{R}^3);\ a(\boldsymbol{\varphi}) \in L^1(\omega),\ \partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi} \neq \boldsymbol{0} \text{ a.e. in } \omega, \\ \boldsymbol{a}_3(\boldsymbol{\varphi}) \in H^1(\omega; \mathbb{R}^3),\ c(\boldsymbol{\varphi}) \in L^1(\omega),\ K(\boldsymbol{\varphi}) > 0 \text{ a.e. in } \omega, \\ \boldsymbol{\varphi} = \boldsymbol{\theta} \text{ and } \boldsymbol{a}_3(\boldsymbol{\varphi}) = \boldsymbol{a}_3\ d\gamma\text{-a.e. in } \gamma_0\},$$

and

$$J[\boldsymbol{\varphi}] := \int_\omega W_K^\sharp[\boldsymbol{\varphi}] \sqrt{a}\, dy - L[\boldsymbol{\varphi}] \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{U}(\omega),$$

where $W_K^\sharp := \varepsilon W_M^\sharp + \varepsilon^3 W_F^\sharp$ *denotes the stored energy function defined in* Theorem 4, *and* $L : H^1(\omega; \mathbb{R}^3) \to \mathbb{R}$ *is a linear and continuous functional.*

Then there exists a vector field $\boldsymbol{\psi} \in \boldsymbol{U}(\omega)$ *such that*

$$J[\boldsymbol{\psi}] = \inf_{\boldsymbol{\varphi} \in \boldsymbol{U}(\omega)} J[\boldsymbol{\varphi}].$$

Proof. (i) Since ω is bounded and, for each $\boldsymbol{\varphi} \in \boldsymbol{U}(\omega)$,

$$a_{\alpha\beta}(\boldsymbol{\varphi}) \in L^1(\omega),\ a(\boldsymbol{\varphi}) \in L^1(\omega), \text{ and } -\log\Big(\frac{a(\boldsymbol{\varphi})}{a}\Big) \geq 1 - \frac{a(\boldsymbol{\varphi})}{a},$$

the integrand $W_M^\sharp[\boldsymbol{\varphi}]\sqrt{a}$ is bounded from below by a function in $L^1(\omega)$. Hence, for each $\boldsymbol{\varphi} \in \boldsymbol{U}(\omega)$, the integral

$$\int_\omega \varepsilon W_M^\sharp[\boldsymbol{\varphi}] \sqrt{a}\, dy$$

is well defined, either as a real number or as $+\infty$. Applying the same argument to the function $W_F^\sharp[\boldsymbol{\varphi}]\sqrt{a}$ shows that, for each $\boldsymbol{\varphi} \in \boldsymbol{U}(\omega)$, the integral

$$\int_\omega \varepsilon^3 W_F^\sharp[\boldsymbol{\varphi}] \sqrt{a}\, dy$$

is likewise well defined, either as a real number or as $+\infty$. Hence the functional $J : \boldsymbol{U}(\omega) \to \mathbb{R} \cup \{+\infty\}$ appearing in the statement of the theorem is well-defined.

Using in particular the inequality

$$2\sqrt{a(\boldsymbol{\varphi})/a} \leq a^{\alpha\beta} a_{\alpha\beta}(\boldsymbol{\varphi}),$$

we next deduce that

$$\begin{aligned} W_M^\sharp[\boldsymbol{\varphi}] &= \lambda_* \Big[\frac{a(\boldsymbol{\varphi})}{a} - \log\Big(\frac{a(\boldsymbol{\varphi})}{a}\Big) - 1\Big] + \mu_*\Big[a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi}) - \log\Big(\frac{a(\boldsymbol{\varphi})}{a}\Big) - 2\Big] \\ &\geq \frac{\lambda_*}{2}\Big[\frac{a(\boldsymbol{\varphi})}{a} - 2\log 2\Big] + \mu_*\Big[a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi}) - 2\log\Big(\frac{a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi})}{2}\Big) - 2\Big] \\ &\geq \frac{\lambda_*}{2}\Big[\frac{a(\boldsymbol{\varphi})}{a} - 2\log 2\Big] + \frac{\mu_*}{2}\Big[a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi}) - 4\log 2\Big] \end{aligned}$$

and

$$a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi}) \geq \frac{1}{\sup_{y\in\overline{\omega}} \|(a_{\alpha\beta}(y))\|}\sum_\alpha |\partial_\alpha\boldsymbol{\varphi}|^2,$$

where $\|(a_{\alpha\beta}(y))\|$ denotes the spectral norm of the matrix $(a_{\alpha\beta}(y)) \in \mathbb{S}^2_>$.

The above two inequalities combined with two analogous inequalities for $W_F^\sharp[\boldsymbol{\varphi}]$, and with Poincaré's inequality, show that the functional $J : \boldsymbol{U}(\omega) \to \mathbb{R}\cup\{+\infty\}$ is coercive in the following sense: If a sequence $(\boldsymbol{\varphi}_n) \subset \boldsymbol{U}(\omega)$ satisfies

$$\sup_n J[\boldsymbol{\varphi}_n] < \infty,$$

then the sequences

$$(\boldsymbol{\varphi}_n) \text{ and } (\boldsymbol{a}_3(\boldsymbol{\varphi}_n)) \text{ are bounded in } H^1(\omega;\mathbb{R}^3),$$

and the sequences $(a(\boldsymbol{\varphi}_n))$ and $(c(\boldsymbol{\varphi}_n))$ are bounded in $L^1(\omega)$. Since $a(\boldsymbol{\varphi}_n) = |\partial_1\boldsymbol{\varphi}_n\wedge\partial_2\boldsymbol{\varphi}_n|^2$ and $c(\boldsymbol{\varphi}_n) = |\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\wedge\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi}_n)|^2$ a.e. in ω (see Section 3), the sequences

$$(\partial_1\boldsymbol{\varphi}_n\wedge\partial_2\boldsymbol{\varphi}_n) \text{ and } (\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\wedge\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi}_n)) \text{ are bounded in } L^2(\omega;\mathbb{R}^3).$$

(ii) Let $(\boldsymbol{\varphi}_n) \subset \boldsymbol{U}(\omega)$ denote an infimizing sequence of the functional J over the set $\boldsymbol{U}(\omega)$. Since then $\sup_n J[\boldsymbol{\varphi}_n] < \infty$ (note that $\inf_{\boldsymbol{\varphi}\in\boldsymbol{U}(\omega)} J[\boldsymbol{\varphi}] < \infty$ since $\boldsymbol{U}(\omega)$ contains at least one element, namely $\boldsymbol{\theta}$), the above coerciveness property of J implies that there exists an infimizing subsequence, still denoted $(\boldsymbol{\varphi}_n) \subset \boldsymbol{U}(\omega)$, of the functional J and there exist vector fields $\boldsymbol{\psi} \in H^1(\omega;\mathbb{R}^3)$, $\boldsymbol{\eta} \in H^1(\omega;\mathbb{R}^3)$, $\boldsymbol{\zeta} \in L^2(\omega)$, and $\boldsymbol{\xi} \in L^2(\omega)$, such that, as $n\to\infty$,

$$\begin{aligned} &\boldsymbol{\varphi}_n \rightharpoonup \boldsymbol{\psi} \text{ and } \boldsymbol{a}_3(\boldsymbol{\varphi}_n) \rightharpoonup \boldsymbol{\eta} \text{ in } H^1(\omega;\mathbb{R}^3), \\ &\boldsymbol{\varphi}_n \to \boldsymbol{\psi} \text{ and } \boldsymbol{a}_3(\boldsymbol{\varphi}_n) \to \boldsymbol{\eta} \text{ in } L^2(\omega;\mathbb{R}^3) \text{ and a.e. in } \omega, \\ &\partial_1\boldsymbol{\varphi}_n\wedge\partial_2\boldsymbol{\varphi}_n \rightharpoonup \boldsymbol{\zeta} \text{ and } \partial_1\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\wedge\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi}_n) \rightharpoonup \boldsymbol{\xi} \text{ in } L^2(\omega). \end{aligned} \tag{8.1}$$

We now show that the limits appearing in (8.1) satisfy in addition the relations

$$\begin{aligned} &|\boldsymbol{\eta}| = 1 \text{ and } \partial_\alpha\boldsymbol{\psi}\cdot\boldsymbol{\eta} = 0 \;\text{ a.e. in } \omega, \\ &\boldsymbol{\zeta} = \partial_1\boldsymbol{\psi}\wedge\partial_2\boldsymbol{\psi} \text{ and } \boldsymbol{\xi} = \partial_1\boldsymbol{\eta}\wedge\partial_2\boldsymbol{\eta} \;\text{ a.e. in } \omega, \\ &\boldsymbol{\psi} = \boldsymbol{\theta} \text{ and } \boldsymbol{\eta} = \boldsymbol{a}_3 \; d\gamma\text{-a.e. on } \gamma_0. \end{aligned} \tag{8.2}$$

The first two equalities of (8.2) follow from the relations

$$\begin{aligned} &|\boldsymbol{a}_3(\boldsymbol{\varphi}_n)| = 1 \text{ and } \boldsymbol{a}_3(\boldsymbol{\varphi}_n) \to \boldsymbol{\eta} \text{ a.e. in } \omega, \\ &\partial_\alpha\boldsymbol{\varphi}_n\cdot\boldsymbol{a}_3(\boldsymbol{\varphi}_n) = 0 \text{ and } \partial_\alpha\boldsymbol{\varphi}_n\cdot\boldsymbol{a}_3(\boldsymbol{\varphi}_n) \rightharpoonup \partial_\alpha\boldsymbol{\psi}\cdot\boldsymbol{\eta} \text{ in } L^2(\omega). \end{aligned}$$

The third equality of (8.2) follow from the convergence (see (8.1))

$$\partial_1\boldsymbol{\varphi}_n\wedge\partial_2\boldsymbol{\varphi}_n \rightharpoonup \boldsymbol{\zeta} \text{ in } L^2(\omega),$$

combined with the relations

$$\partial_1\boldsymbol{\varphi}_n \wedge \partial_2\boldsymbol{\varphi}_n = \frac{1}{2}\{\partial_1(\boldsymbol{\varphi}_n \wedge \partial_2\boldsymbol{\varphi}_n) + \partial_2(\partial_1\boldsymbol{\varphi}_n \wedge \boldsymbol{\varphi}_n)\}$$
$$\rightharpoonup \frac{1}{2}\{\partial_1(\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi}) + \partial_2(\partial_1\boldsymbol{\psi} \wedge \boldsymbol{\psi})\} = \partial_1\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi} \text{ in } D'(\omega;\mathbb{R}^3).$$

The fourth equality of (8.2) is proved in the same manner as the third above.

The fifth and sixth equalities of (8.2) follow from the relations

$$\boldsymbol{\varphi}_n = \boldsymbol{\theta} \text{ and } \boldsymbol{a}_3(\boldsymbol{\varphi}_n) = \boldsymbol{a}_3 \;\; d\gamma\text{-a.e. on } \gamma_0,$$

which hold for every $n \in \mathbb{N}$, by using that

$$\boldsymbol{\varphi}_n \rightharpoonup \boldsymbol{\psi} \text{ in } H^1(\omega;\mathbb{R}^3) \Rightarrow \boldsymbol{\varphi}_n|_{\gamma_0} \to \boldsymbol{\psi}|_{\gamma_0} \text{ in } L^2(\gamma_0;\mathbb{R}^3),$$
$$\boldsymbol{a}_3(\boldsymbol{\varphi}_n) \rightharpoonup \boldsymbol{\eta} \text{ in } H^1(\omega;\mathbb{R}^3) \Rightarrow \boldsymbol{a}_3(\boldsymbol{\varphi}_n)|_{\gamma_0} \to \boldsymbol{\eta}|_{\gamma_0} \text{ in } L^2(\gamma_0;\mathbb{R}^3),$$

where the notation $\boldsymbol{\psi}|_{\gamma_0}$ denotes the trace on γ_0 of a vector field $\boldsymbol{\psi} \in H^1(\omega;\mathbb{R}^3)$.

(iii) Let the functions $\mathbb{W}_M : \omega \times \mathbb{D}_+ \to \mathbb{R}$ and $\mathbb{W}_F : \omega \times \mathbb{D}_+ \to \mathbb{R}$ be respectively defined by

$$\mathbb{W}_M(y,\boldsymbol{q},\boldsymbol{u}_\alpha,\boldsymbol{e}) := \mu_*\Big[a^{\alpha\beta}(y)\boldsymbol{u}_\alpha \cdot \boldsymbol{u}_\beta - 2\Big] + \lambda_*\Big[\frac{(\boldsymbol{e}\cdot\boldsymbol{q})^2}{a(y)} - 1\Big]$$
$$- (\lambda_* + \mu_*)\log\Big(\frac{(\boldsymbol{e}\cdot\boldsymbol{q})^2}{a(y)}\Big)$$

and

$$\mathbb{W}_F(y,\boldsymbol{q},\boldsymbol{v}_\alpha,\boldsymbol{f}) := K\Big[\hat{\mu}_*\Big(\hat{c}^{\alpha\beta}(y)\boldsymbol{v}_\alpha \cdot \boldsymbol{v}_\beta - 2\Big) + \hat{\lambda}_*\Big(\frac{(\boldsymbol{f}\cdot\boldsymbol{q})^2}{c(y)} - 1\Big)$$
$$- (\hat{\lambda}_* + \hat{\mu}_*)\log\Big(\frac{(\boldsymbol{f}\cdot\boldsymbol{q})^2}{c(y)}\Big)\Big]$$

for each $y \in \omega$, each $(\boldsymbol{q},\boldsymbol{u}_\alpha,\boldsymbol{e}) \in \mathbb{D}_+$, and each $(\boldsymbol{q},\boldsymbol{v}_\alpha,\boldsymbol{f}) \in \mathbb{D}_+$.

Let $\mathbb{D} := \mathbb{R}^3 \times (\mathbb{R}^3)^2 \times \mathbb{R}^3$ and let the functions $\tilde{\mathbb{W}}_M : \omega \times \mathbb{D} \to \mathbb{R} \cup \{+\infty\}$ and $\tilde{\mathbb{W}}_F : \omega \times \mathbb{D} \to \mathbb{R} \cup \{+\infty\}$ be defined by letting, for each $y \in \omega$,

$$\tilde{\mathbb{W}}_M(y,\boldsymbol{q},\boldsymbol{u}_\alpha,\boldsymbol{e}) := \mathbb{W}_M(y,\boldsymbol{q},\boldsymbol{u}_\alpha,\boldsymbol{e}) \quad \text{if } (\boldsymbol{q},\boldsymbol{u}_\alpha,\boldsymbol{e}) \in \mathbb{D}_+,$$
$$+\infty \quad \text{if } (\boldsymbol{q},\boldsymbol{u}_\alpha,\boldsymbol{e}) \in \mathbb{D} - \mathbb{D}_+,$$

and

$$\tilde{\mathbb{W}}_F(y,\boldsymbol{q},\boldsymbol{v}_\alpha,\boldsymbol{f}) := \mathbb{W}_F(y,\boldsymbol{q},\boldsymbol{v}_\alpha,\boldsymbol{f}) \quad \text{if } (\boldsymbol{q},\boldsymbol{v}_\alpha,\boldsymbol{f}) \in \mathbb{D}_+,$$
$$+\infty \quad \text{if } (\boldsymbol{q},\boldsymbol{v}_\alpha,\boldsymbol{f}) \in \mathbb{D} - \mathbb{D}_+.$$

Let the function $\mathbb{W} : \omega \times \mathbb{R}^3 \times \mathbb{R}^9 \to \mathbb{R} \cup \{+\infty\}$ be defined for each $(y,\boldsymbol{q},\boldsymbol{F}) \in \omega \times \mathbb{R}^3 \times \mathbb{R}^9$ by

$$\mathbb{W}(y,\boldsymbol{q},\boldsymbol{F}) = \tilde{\mathbb{W}}_M(y,\boldsymbol{q},\boldsymbol{u}_\alpha,\boldsymbol{e})\sqrt{a(y)}, \text{ where } \boldsymbol{F} := (\boldsymbol{u}_\alpha,\boldsymbol{e}),$$

and let the sequences $(\boldsymbol{\eta}_n)$ and $(\boldsymbol{F}_n)$ be defined by

$$\boldsymbol{\eta}_n := \boldsymbol{a}_3(\boldsymbol{\varphi}_n) \text{ and } \boldsymbol{F}_n := (\partial_\alpha\boldsymbol{\varphi}_n, \partial_1\boldsymbol{\varphi}_n \wedge \partial_2\boldsymbol{\varphi}_n).$$

Since

$$\boldsymbol{\eta}_n \to \boldsymbol{\eta} \text{ a.e. in } \omega,$$
$$\boldsymbol{F}_n \rightharpoonup \boldsymbol{F} := (\partial_\alpha\boldsymbol{\psi}, \partial_1\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi}) \text{ in } L^2(\omega;\mathbb{R}^9),$$

it is easy to see that the function $\mathbb{W}$ and the sequences $(\boldsymbol{\eta}_n)$ and $(\boldsymbol{F}_n)$ satisfy the assumptions of Theorem 1. Therefore,

$$\begin{aligned}
&\int_\omega \tilde{\mathbb{W}}_M(\cdot, \boldsymbol{\eta}, \partial_\alpha \boldsymbol{\psi}, \partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi})\sqrt{a}\, dy \\
&\qquad \le \liminf_{n\to\infty} \int_\omega \tilde{\mathbb{W}}_M(\cdot, \boldsymbol{a}_3(\boldsymbol{\varphi}_n), \partial_\alpha \boldsymbol{\varphi}_n, \partial_1 \boldsymbol{\varphi}_n \wedge \partial_2 \boldsymbol{\varphi}_n)\sqrt{a}\, dy \\
&\qquad = \liminf_{n\to\infty} \int_\omega W_M^\sharp[\boldsymbol{\varphi}_n]\sqrt{a}\, dy.
\end{aligned} \tag{8.3}$$

Likewise, let the function $\mathbb{W}^* : \omega \times \mathbb{R}^3 \times \mathbb{R}^9 \to \mathbb{R} \cup \{+\infty\}$ be defined for each $(y, \boldsymbol{q}, \boldsymbol{F}) \in \omega \times \mathbb{R}^3 \times \mathbb{R}^9$ by

$$\mathbb{W}^*(y, \boldsymbol{q}, \boldsymbol{F}) = \tilde{\mathbb{W}}_F(y, \boldsymbol{q}, \boldsymbol{v}_\alpha, \boldsymbol{f})\sqrt{a(y)}, \text{ where } \boldsymbol{F} := (\boldsymbol{v}_\alpha, \boldsymbol{f}),$$

and let the sequences $(\boldsymbol{\eta}_n)$ and $(\boldsymbol{F}_n^*)$ be defined by

$$\boldsymbol{\eta}_n := \boldsymbol{a}_3(\boldsymbol{\varphi}_n) \text{ and } \quad \boldsymbol{F}_n^* := (\partial_\alpha \boldsymbol{a}_3(\boldsymbol{\varphi}_n), \partial_1 \boldsymbol{a}_3(\boldsymbol{\varphi}_n) \wedge \partial_2 \boldsymbol{a}_3(\boldsymbol{\varphi}_n)).$$

Since

$$\begin{aligned}
&\boldsymbol{\eta}_n \to \boldsymbol{\eta} \text{ a.e. in } \omega, \\
&\boldsymbol{F}_n^* \rightharpoonup \boldsymbol{F}^* := (\partial_\alpha \boldsymbol{\eta}, \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta}) \text{ in } L^2(\omega; \mathbb{R}^9),
\end{aligned}$$

it is easy to see that the function $\mathbb{W}^*$ and the sequences $(\boldsymbol{\eta}_n)$ and $(\boldsymbol{F}_n^*)$ also satisfy the assumptions of Theorem 1. Therefore,

$$\begin{aligned}
&\int_\omega \tilde{\mathbb{W}}_F(\cdot, \boldsymbol{\eta}, \partial_\alpha \boldsymbol{\eta}, \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta})\sqrt{a}\, dy \\
&\qquad \le \liminf_{n\to\infty} \int_\omega \tilde{\mathbb{W}}_F(\cdot, \boldsymbol{a}_3(\boldsymbol{\varphi}_n), \partial_\alpha \boldsymbol{a}_3(\boldsymbol{\varphi}_n), \partial_1 \boldsymbol{a}_3(\boldsymbol{\varphi}_n) \wedge \partial_2 \boldsymbol{a}_3(\boldsymbol{\varphi}_n))\sqrt{a}\, dy \\
&\qquad = \liminf_{n\to\infty} \int_\omega W_F^\sharp[\boldsymbol{\varphi}_n]\sqrt{a}\, dy.
\end{aligned} \tag{8.4}$$

(iv) Since

$$\begin{aligned}
&\liminf_{n\to\infty} \int_\omega \varepsilon W_M^\sharp[\boldsymbol{\varphi}_n]\sqrt{a}\, dy + \liminf_{n\to\infty} \int_\omega \varepsilon^3 W_F^\sharp[\boldsymbol{\varphi}_n]\sqrt{a}\, dy \\
&\le \liminf_{n\to\infty} \int_\omega W_K^\sharp[\boldsymbol{\varphi}_n]\sqrt{a}\, dy = \liminf_{n\to\infty} J[\boldsymbol{\varphi}_n] + L[\boldsymbol{\psi}] < \infty,
\end{aligned}$$

we infer from inequalities (8.3) and (8.4) that

$$\int_\omega \tilde{\mathbb{W}}_M(\cdot, \boldsymbol{\eta}, \partial_\alpha \boldsymbol{\psi}, \partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi})\sqrt{a}\, dy < \infty$$

and

$$\int_\omega \tilde{\mathbb{W}}_F(\cdot, \boldsymbol{\eta}, \partial_\alpha \boldsymbol{\eta}, \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta})\sqrt{a}\, dy < \infty.$$

Hence, for almost all $y \in \omega$,

$$\begin{aligned}
&\tilde{\mathbb{W}}_M(y, \boldsymbol{\eta}(y), \partial_\alpha \boldsymbol{\psi}(y), \partial_1 \boldsymbol{\psi}(y) \wedge \partial_2 \boldsymbol{\psi}(y)) < +\infty, \\
&\tilde{\mathbb{W}}_F(y, \boldsymbol{\eta}(y), \partial_\alpha \boldsymbol{\eta}(y), \partial_1 \boldsymbol{\eta}(y) \wedge \partial_2 \boldsymbol{\eta}(y)) < +\infty,
\end{aligned}$$

which combined with the above definition of the functions $\tilde{\mathbb{W}}_M$ and $\tilde{\mathbb{W}}_F$ imply that, for almost all $y \in \omega$,

$$(\boldsymbol{\eta}(y), \partial_\alpha \boldsymbol{\psi}(y), \partial_1 \boldsymbol{\psi}(y) \wedge \partial_2 \boldsymbol{\psi}(y)) \in \mathbb{D}_+,$$
$$(\boldsymbol{\eta}(y), \partial_\alpha \boldsymbol{\eta}(y), \partial_1 \boldsymbol{\eta}(y) \wedge \partial_2 \boldsymbol{\eta}(y)) \in \mathbb{D}_+.$$

Consequently, for almost all $y \in \omega$,

$$(\partial_1 \boldsymbol{\psi}(y) \wedge \partial_2 \boldsymbol{\psi}(y)) \cdot \boldsymbol{\eta}(y) > 0 \text{ and } (\partial_1 \boldsymbol{\eta}(y) \wedge \partial_2 \boldsymbol{\eta}(y)) \cdot \boldsymbol{\eta}(y) > 0.$$

Combined with the relations (see (8.2))

$$|\boldsymbol{\eta}(y)| = 1 \text{ and } \partial_\alpha \boldsymbol{\psi}(y) \cdot \boldsymbol{\eta}(y) = 0 \text{ a.e. in } \omega,$$

the previous inequalities show that

$$\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi} \neq \mathbf{0} \text{ and } (\partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta}) \cdot \boldsymbol{\eta} > 0 \text{ a.e. in } \omega,$$

and then that

$$\boldsymbol{\eta} = \boldsymbol{a}_3(\boldsymbol{\psi}) := \frac{\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}}{|\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}|} \quad \text{a.e. in } \omega.$$

Besides,

$$a(\boldsymbol{\psi}) = |\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}|^2 = |\boldsymbol{\zeta}|^2 \in L^1(\omega)$$

and

$$c(\boldsymbol{\psi}) = |\partial_1 \boldsymbol{a}_3(\boldsymbol{\psi}) \wedge \partial_2 \boldsymbol{a}_3(\boldsymbol{\psi})|^2 = |\boldsymbol{\xi}|^2 \in L^1(\omega).$$

This shows that $\boldsymbol{\psi} \in \boldsymbol{U}(\omega)$. Hence the left-hand sides of the inequalities (8.3) and (8.4) satisfy

$$\int_\omega \tilde{\mathbb{W}}_M(\cdot, \boldsymbol{\eta}, \partial_\alpha \boldsymbol{\psi}, \partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}) \sqrt{a}\, dy = \int_\omega W_M^\sharp[\boldsymbol{\psi}] \sqrt{a}\, dy$$

and

$$\int_\omega \tilde{\mathbb{W}}_F(\cdot, \boldsymbol{\eta}, \partial_\alpha \boldsymbol{\eta}, \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta}) \sqrt{a}\, dy = \int_\omega W_F^\sharp[\boldsymbol{\psi}] \sqrt{a}\, dy.$$

Therefore inequalities (8.3) and (8.4) together imply that

$$\int_\omega \left\{ \varepsilon W_M^\sharp[\boldsymbol{\psi}] + \varepsilon^3 W_F^\sharp[\boldsymbol{\psi}] \right\} \sqrt{a}\, dy \leq \liminf_{n \to \infty} \int_\omega \left\{ \varepsilon W_M^\sharp[\boldsymbol{\varphi}_n] + \varepsilon^3 W_F^\sharp[\boldsymbol{\varphi}_n] \right\} \sqrt{a}\, dy,$$

which implies in turn that

$$J[\boldsymbol{\psi}] + L[\boldsymbol{\psi}] \leq \liminf_{n \to \infty} (J[\boldsymbol{\varphi}_n] + L[\boldsymbol{\varphi}_n]) = \inf_{\boldsymbol{\varphi} \in \boldsymbol{U}(\omega)} J[\boldsymbol{\varphi}] + L[\boldsymbol{\psi}].$$

Thus

$$J[\boldsymbol{\psi}] = \inf_{\boldsymbol{\varphi} \in \boldsymbol{U}(\omega)} J[\boldsymbol{\varphi}],$$

which completes the proof. □

Remark. It is likely that the specific nonlinear shell model appearing in Theorem 5 could be also justified by comparison with the three-dimensional model of nonlinear elasticity, by means of Γ-convergence theory as in [6] and [7]. This objective will be addressed in a future work.

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