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ASYMPTOTICS FOR DILUTE EMULSIONS WITH SURFACE TENSION

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Abstract. We consider an emulsion formed by two newtonian fluids, one being dispersed in the other under the form of droplets, in the presence of surface tension. We investigate the dilute case where the droplet size a_ϵ is much smaller than the distance ϵ between the droplets' centers. We prove that the limit behavior when $\epsilon \to 0$ is described by the unperturbed Stokes flow and estimate the order of convergence rate of the velocity to be $a_\epsilon^{n/2}$. We improve the convergence result and determine the first corrector in the velocity expansion. Taylor's and Einstein's viscosity formulas are recovered.

1. Introduction

A fundamental question in hydrodynamics is the characterization of the behavior of suspensions, emulsion and foams, in terms of the properties of the different phases, geometry and volume fraction.

The work on dilute suspensions of rigid spherical particles in a viscous fluid starts with the papers of A. Einstein [7], [8] who showed that dilute emulsions have a newtonian behavior and derived the well known effective viscosity formula:

$$\mu^{eff} = (1 + 2.5\phi + \mathcal{O}(\phi^2))\mu_0. \tag{1.1}$$

where the fluid viscosity is μ_0 and ϕ is the volume fraction of particles.

Emulsions of small fluid droplets were studied by G. I. Taylor [22]; he generalized Einstein's viscosity formula for suspensions of rigid particles to emulsions of spherical fluid droplets suspended in a viscous fluid

$$\mu^{eff} = \left(1 + \frac{2\mu_0 + 5\mu_d}{2(\mu_0 + \mu_d)}\phi + \mathcal{O}(\phi^2)\right)\mu_0, \tag{1.2}$$

where the fluid viscosity is μ_0, the droplet viscosity is μ_d and ϕ is the volume fraction of the droplets. From here Einstein's formula can be obtained for $\mu_d \to \infty$.

Using the homogenization method, T. Lévy and E. Sanchez-Palencia [14] obtained the velocity asymptotics for small concentrations and also derived the emulsion viscosity behavior in terms of the volume fraction ϕ

$$\mu^{eff} = \left(1 + \frac{5(\mu_d - \mu_0)}{2\mu_d + 3\mu_0}\phi + \mathcal{O}(\phi^2)\right)\mu_0. \tag{1.3}$$

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The same formula for spherical droplets (1.3) is obtained in the paper of H. Ammari et al. [1], where the authors extend their previous work on elastic composites to emulsions of droplets of arbitrary shape without surface tension. The latter paper is extended to the case of emulsions with surface tension in the work of E. Bonnetier et al. [4].

The two formulas (1.2) and (1.3) do not agree, and one can observe that in the case of similar viscosities, $\mu_0 = \mu_d$, in the framework of (1.3) one obtains that there is no perturbation of the flow in the presence of droplets, although keeping the droplets spherical should perturb the flow. This is due to the fact that in [14], [1], [4] the boundary conditions at the droplet interface are imposed in the Lagrangian frame, while the Stokes equations are written in the Eulerian frame.

The non-dilute case for emulsions with surface tension was studied in [16] in the framework of the homogenization method and the Stokes homogenized limit was obtained using two-scale convergence. Upper and lower bounds for the effective viscosity were derived in [16] and were shown to agree with Taylor's formula (1.2) in the dilute limit for spherical droplets.

We should also mention here that the case of non-convected droplets was studied in the framework of periodic homogenization in [15] and in [19]. In the former a Darcy-type flow is recovered in the limit, while in the latter a Stokes or a Brinkman type flow is obtained, depending on the speed of convergence to zero of the ratio between the particle size and the distance between the particles' centers.

In this paper we study the asymptotics of the velocity in the case of dilute emulsions of arbitrary shaped droplets of size a_ϵ, in the presence of surface tension. The system is modeled by a stationary Stokes flow in an open, bounded, Lipschitz domain Ω in $\mathbb{R}^n$ with $n \geq 3$. We assume that the distance between the droplets' centers to be ϵ, with $\epsilon \gg a_\epsilon$ and that the droplets do not intersect $\partial\Omega$.

In Section 2. we formulate the problem and explain the choice of boundary conditions on the surface of the droplets. The weak formulation and the setting for the periodic problem is given in Section 3. The corresponding local problem on a reference cell is given in Section 4. and the convergence of the solutions results from our previous work [19].

In Section 5. we derive the convergence of the emulsion velocity $\boldsymbol{v}^\epsilon$ to the velocity field $\boldsymbol{v}^0$ corresponding to the unperturbed flow; this is done using Γ- convergence of the corresponding functionals and in Section 6. the rate of convergence is shown to be

$$\|\boldsymbol{v}^\epsilon - \boldsymbol{v}\|_{\left(H_0^1(\Omega)\right)^n} \leq C\, a_\epsilon^{n/2}.$$

Furthermore, in section 7 we find the a_ϵ^n term in the expansion of the velocity, which involves bulk and surface polarization tensors. The formula for the effective viscosity of the dilute emulsion is derived in Section 8. and the first term in the volume fraction expansion recovers G. I. Taylor's result [22].

2. Viscous drops in a viscous fluid with surface tension

We will consider in which both the droplets and the ambient fluid are Newtonian and incompressible and the flow is at low Reynolds numbers; thus the flow will be governed by the Stokes equations and the velocity will be solenoidal.

To make things more precise, let us denote by Ω the domain occupied by the emulsion, by Ω_1 the domain occupied by the droplets, of viscosity μ_d, and by Ω_2 the domain occupied by the continuous liquid phase of viscosity μ_0 and $\Omega = \Omega_1 \cup \overline{\Omega_2}$. The droplets are denoted by T_ℓ and their surface by S_ℓ; the union of the bubble surfaces $S = \overline{\Omega_1} \cap \overline{\Omega_2}$. The problem is described by the balance of momentum and mass equations

$$-\text{div}\ \left(-pI + 2\,\mu(\boldsymbol{x})\,e(\boldsymbol{v})\right) = \boldsymbol{f} \quad \text{in } \Omega_1 \cup \Omega_2 \tag{2.1}$$

$$\text{div}\ \boldsymbol{v} = 0 \quad \text{in } \Omega \tag{2.2}$$

where $\boldsymbol{v}$ and p represent the fluid velocity and pressure, $\boldsymbol{f}$ are the body forces, $\mu(\boldsymbol{x}) = \mu_d$ if $\boldsymbol{x} \in \Omega_1$ and $\mu(\boldsymbol{x}) = \mu_0$ if $\boldsymbol{x} \in \Omega_2$ and the strain rate tensor is $e(\boldsymbol{v}) = \frac{1}{2}(\nabla v + (\nabla v)^T)$. The stress tensor will be denoted by $\sigma = -p\,I + 2\,\mu(\boldsymbol{x})\,e(\boldsymbol{v})$.

As was mentioned in the introduction, it is important to understand the boundary conditions on the interface between the two fluids in the emulsion problem.

i. First a kinematic boundary condition needs to be imposed on the droplet boundary to ensure that the boundary remains a material interphase boundary.

Let us assume that the shape of the droplets is given by the surface $F(t, \boldsymbol{x}) = 0$. Then the normal velocity of the droplet boundary is given by

$$\boldsymbol{v} \cdot \boldsymbol{n} = -\frac{\frac{\partial F}{\partial t}}{|\nabla_x F|}.$$

The kinematic boundary condition on the droplet surface imposes the normal velocity of each fluid to be equal to the normal velocity of the surface:

$$[\![\boldsymbol{v} \cdot \boldsymbol{n}]\!] = 0, \ \text{and } \boldsymbol{v} \cdot \boldsymbol{n} = -\frac{\frac{\partial F}{\partial t}}{|\nabla_x F|}. \tag{2.3}$$

Let us now assume that, although it moves, the droplet shape does not change in time, thus $F(t, \boldsymbol{x}) = G(\boldsymbol{x}')$, where $\boldsymbol{x}'$ are the coordinates of a point on the droplet surface in a moving frame, with orthonormal base $\{\boldsymbol{e}'_i\}$, centered at the center of mass $\boldsymbol{x}_c$ of the droplet. Thus

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{x}_c \ \text{ and } \ \frac{d\boldsymbol{e}'_i}{dt} = A_{ij}\boldsymbol{e}'_j,$$

with $A = (A_{ij})$ an antisymmetric matrix. Then

$$0 = \frac{\partial G}{\partial t} = \frac{\partial G}{\partial x'_i}\frac{dx'_i}{dt} = n'_i((\boldsymbol{v} - \boldsymbol{v}^c) \cdot \boldsymbol{e}'_i + A_{kl}(\boldsymbol{x} - \boldsymbol{x}_c)_k(\boldsymbol{e}'_i)_l)$$

and thus the kinematic boundary condition becomes

$$[\![\boldsymbol{v} \cdot \boldsymbol{n}]\!] = 0, \ \text{and } \boldsymbol{v} \cdot \boldsymbol{n} = (\boldsymbol{v}^c + A(\boldsymbol{x} - \boldsymbol{x}_c)) \cdot \boldsymbol{n}. \tag{2.4}$$

Let us observe as in [19] that the kinematic boundary condition (2.4) implies

$$\boldsymbol{v}^c = \frac{1}{|T_\ell|}\int_{T_\ell} \boldsymbol{v}\, d\boldsymbol{x}. \tag{2.5}$$

and the normal velocity in (2.4) can thus be written as

$$\boldsymbol{v} \cdot \boldsymbol{n} = \left(\frac{1}{|T_\ell|}\int_{T_\ell} \boldsymbol{v}\, d\boldsymbol{x} + A(\boldsymbol{x} - \boldsymbol{x}_c)\right) \cdot \boldsymbol{n}.$$

In the particular case of spherical droplets $\boldsymbol{x} - \boldsymbol{x}_c$ is parallel to $\boldsymbol{n}$ and thus the kinematic condition (2.4) reduces to

$$[\![\boldsymbol{v} \cdot \boldsymbol{n}]\!] = 0, \ \text{and } \boldsymbol{v} \cdot \boldsymbol{n} = \left(\frac{1}{|T_\ell|}\int_{T_\ell} \boldsymbol{v}\, d\boldsymbol{x}\right) \cdot \boldsymbol{n}. \tag{2.6}$$

ii. A second type of boundary condition connects the jump of the stress across the boundary to the surface tension. Indeed on the droplet surface there is a stress jump $[\![\sigma \boldsymbol{n}]\!] \neq 0$, and (2.1) is only valid in Ω_1 and Ω_2. The stress jump can be obtained from the principle that the forces on an element of interfacial area of arbitrary shape and size must be in equilibrium, because the interface is assumed to have zero thickness and thus zero mass. One can thus obtain [12]

$$[\![\sigma \boldsymbol{n}]\!] = \lambda\,(\nabla_s \cdot \boldsymbol{n})\,\boldsymbol{n} - \nabla_s \lambda$$

where λ is the surface tension and $\nabla_s = \nabla - \boldsymbol{n}(\boldsymbol{n} \cdot \nabla)$ is the surface gradient operator. If the surface tension is uniform, the stress has only a normal jump across the interface, that is proportional to the surface tension and the mean curvature κ:

$$[\![\sigma \boldsymbol{n}]\!] = \lambda\,\kappa\,\boldsymbol{n}. \tag{2.7}$$

iii. A third type of boundary condition that needs to be imposed is the balance of forces and torques for each droplet:

$$\int_{S_\ell} [\![\sigma \boldsymbol{n}]\!] ds = 0, \text{ and } \int_{S_\ell} (\boldsymbol{x} - \boldsymbol{x}_c) \times [\![\sigma \boldsymbol{n}]\!] ds = 0. \tag{2.8}$$

Let us show that in the case of uniform surface tension (2.8) is satisfied. Let us assume that T_ℓ has a $W^{2,\infty}$ boundary and that $g \in W^{2,1}(T_\ell)$ and $\boldsymbol{u} \in (C^1(\mathbb{R}^n))^n$ uniformly bounded and use the formula for integration by parts on $S_\ell = \partial T_\ell$ (see [10]):

$$\int_S (\boldsymbol{v} \cdot \nabla g + g\,\mathrm{div_s}\boldsymbol{v})\,ds = \int_S \left(\frac{\partial g}{\partial n} + \kappa g\right) \boldsymbol{v} \cdot \boldsymbol{n} ds \tag{2.9}$$

where $\mathrm{div_s}\boldsymbol{v} = \mathrm{div}\ \boldsymbol{v} - \nabla \boldsymbol{v}\,\boldsymbol{n} \cdot \boldsymbol{n}$.

From (2.9) for $g = \lambda$ and $\boldsymbol{v} = \boldsymbol{e}_i$ we get that

$$\int_{S_\ell} [\![\sigma \boldsymbol{n}]\!] ds = \int_{S_\ell} \lambda\,\kappa\,\boldsymbol{n} ds = 0.$$

Similarly by denoting $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}_c$ from (2.9) for $g = \lambda$ and $\boldsymbol{v} = \boldsymbol{e}_i \times \boldsymbol{y}$ we have

$$\int_{S_\ell} (\boldsymbol{x} - \boldsymbol{x}_c) \times [\![\sigma \boldsymbol{n}]\!] \cdot \boldsymbol{e}_i ds = \int_{S_\ell} \lambda \kappa (\boldsymbol{y} \times \boldsymbol{n}) \cdot \boldsymbol{e}_i ds =$$
$$= \int_{S_\ell} \lambda \kappa (\boldsymbol{e}_i \times \boldsymbol{y}) \cdot \boldsymbol{n} ds = \int_{S_\ell} \lambda \mathrm{div_s}(\boldsymbol{e}_\mathrm{i} \times \boldsymbol{y}) \mathrm{ds} = 0.$$

For simplicity on the exterior boundary we will consider a no-slip condition:

$$\boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega.$$

3. Periodic homogenization for droplets of arbitrary shape

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with Lipschitz boundary $\Gamma = \partial\Omega$, and let $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ be the unit cube in $\mathbb{R}^n$. For every $\epsilon > 0$, let N^ϵ be the set of all points $\ell \in \mathbb{Z}^n$ such that $\epsilon(\ell + Y)$ is strictly included in Ω and denote by $|N^\epsilon|$ their total number. Let T be the closure of an open connected set with Lipschitz boundary, compactly included in Y. For every $\epsilon > 0$ and $\ell \in N^\epsilon$ we consider the set $T_\ell^\epsilon \subset\subset \epsilon(\ell + Y)$, where $T_\ell^\epsilon = \epsilon\ell + a_\epsilon T$, and $a_\epsilon = \mathcal{O}(\epsilon^{n/(n-2)})$. The set T^ϵ_ℓ represents one of the droplets suspended in the fluid, and $S^\epsilon_\ell = \partial T^\epsilon_\ell$ denotes its surface. We now define the following subsets of Ω:

$$\Omega_{1\epsilon} = \bigcup_{\ell \in N^\epsilon} T^\epsilon_\ell \ , \quad \Omega_{2\epsilon} = \Omega \backslash \overline{\Omega_{1\epsilon}},$$

where $\Omega_{1\epsilon}$ is the domain occupied by the droplets of viscosity μ_d, and $\Omega_{2\epsilon}$ is the domain occupied by the surrounding fluid, of viscosity μ_0. Let $\boldsymbol{n}$ be the unit normal on the boundary of $\Omega_{2\epsilon}$ that points outside the domain.

The problem describing the flow in the presence of droplets is the following:

$$-\text{div}\, \boldsymbol{\sigma}^\epsilon = \boldsymbol{f} + \lambda_\epsilon \kappa_\epsilon \boldsymbol{n} \chi_{S^\epsilon} \text{ in } \Omega, \tag{3.1a}$$

$$\boldsymbol{\sigma}^\epsilon = -p^\epsilon I + 2\,\mu^\epsilon\, e(\boldsymbol{v}^\epsilon), \tag{3.1b}$$

$$\text{div}\, \boldsymbol{v}^\epsilon = 0 \text{ in } \Omega, \tag{3.1c}$$

with boundary conditions on the surface of each droplet T^ϵ_ℓ, $\ell \in N^\epsilon$:

$$[\![\boldsymbol{v}^\epsilon]\!] = \boldsymbol{0} \quad \text{ on } S^\epsilon_\ell \ , \tag{3.2a}$$

$$\boldsymbol{v}^\epsilon = \boldsymbol{V}^{\ell,\epsilon} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}^\ell_c) \quad \text{ on } S^\epsilon_\ell \ , \tag{3.2b}$$

and, for simplicity, no-slip condition on the exterior boundary

$$\boldsymbol{v}^\epsilon = \boldsymbol{0} \quad \text{ on } \Gamma \ , \tag{3.3}$$

where χ_{S^ϵ} represents the characteristic function of the union of the droplet surfaces S^ϵ_ℓ, $[\![\cdot]\!]$ denotes the jump across S^ϵ_ℓ, $\boldsymbol{\omega}$ is an unknown, constant vector in $\mathbb{R}^n$, $\boldsymbol{x}^\ell_c$ the position vector of the center of mass of the droplet T^ϵ_ℓ, and the viscosity μ^ϵ defined as

$$\mu^\epsilon(\boldsymbol{x}) = \begin{cases} \mu_d & \text{if } x \in \Omega_{1\epsilon}, \\ \mu_0 & \text{if } x \in \Omega_{2\epsilon}. \end{cases}$$

The boundary conditions (3.2) are those deduced in the previous section. Moreover, we have that the droplet velocities are related to the flow by the following:

$$\boldsymbol{V}^{\ell,\epsilon} = \frac{1}{|T^\epsilon_\ell|} \int_{T^\epsilon_\ell} \boldsymbol{v}^\epsilon \, d\boldsymbol{x}$$

3.1. **Weak formulation.** The emulsion flow problem in (3.1) – (3.3) has the equivalent variational formulation: For any $\boldsymbol{f} \in \left(L^2(\Omega)\right)^n$, find $\boldsymbol{v}^\epsilon \in V^\epsilon$ such that,

$$\int_\Omega 2\,\mu^\epsilon\, e(\boldsymbol{v}^\epsilon) : e(\boldsymbol{w})\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}\, d\boldsymbol{x}, \text{ for any } \boldsymbol{w} \in V^\epsilon, \tag{3.4}$$

where V^ϵ is the closed subspace of $\left(H^1_0(\Omega)\right)^n$ given by:

$$V^\epsilon = \left\{ \boldsymbol{w} \in \left(H^1_0(\Omega)\right)^n \mid \text{div}\, \boldsymbol{w} = 0 \text{ in } \Omega,\ \boldsymbol{w} = \boldsymbol{W}^{\ell,\epsilon} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}^\ell_c) \text{ on } S^\epsilon_\ell,\ \boldsymbol{\omega} \in \mathbb{R}^3 \right\}.$$

The proof of the equivalence is similar to the one in [19] and relies on the fact that, for uniform surface tension, the jump of the stress is of the form (2.7) and thus (2.8) is automatically satisfied.

The existence and uniqueness of a weak solution of the emulsion flow problem follows from the Lax-Milgram lemma. Furthermore, $\boldsymbol{v}^\epsilon$ is the unique solution of the problem:

$$\left\{\begin{array}{l}\text{Find } \boldsymbol{v}^\epsilon \in \left(H_0^1(\Omega)\right)^n \text{ such that}\\ J^\epsilon(\boldsymbol{v}^\epsilon) = \min\limits_{\boldsymbol{u}\in\left(H_0^1(\Omega)\right)^n} J^\epsilon(\boldsymbol{u})\end{array}\right. \tag{3.5}$$

where

$$J^\epsilon(\boldsymbol{u}) = \int_\Omega \mu^\epsilon\, e(\boldsymbol{u})\colon e(\boldsymbol{u})\, d\boldsymbol{x} + I_{V^\epsilon}(\boldsymbol{u}) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\, d\boldsymbol{x} \tag{3.6}$$

and I_D represents the indicator function of the set D, defined by

$$I_D(s) = \left\{\begin{array}{ll} 0 & \text{if } s\in D\\ +\infty & \text{if } s\notin D\end{array}\right.$$

4. The local problem

Let us consider for some $\ell \in N^\epsilon$ the reference cell, $Y_\ell^\epsilon\colon = \epsilon(\ell + Y)$ and the ball B_ℓ^ϵ with center in the center of cell $\ell \in N^\epsilon$ and radius $\epsilon/2$. The local problem is defined as finding $(\boldsymbol{w}^{k\epsilon}, q^{k\epsilon})$ solutions of

$$\left\{\begin{array}{ll} -\text{div}\, \boldsymbol{\sigma}^{k\epsilon} = \lambda_\epsilon \kappa_\epsilon \boldsymbol{n}\chi_{S_\ell^\epsilon} & \text{in } B_\ell^\epsilon\backslash S_\ell^\epsilon,\\ \boldsymbol{\sigma}^{k\epsilon} = -q^{k\epsilon} I + 2\,\mu\, e(\boldsymbol{w}^{k\epsilon}), & \\ \text{div}\, \boldsymbol{w}^{k\epsilon} = 0 & \text{in } B_\ell^\epsilon,\\ [\![\boldsymbol{w}^{k\epsilon}]\!] = \boldsymbol{0} & \text{on } S_\ell^\epsilon,\\ \boldsymbol{w}^{k\epsilon} = \boldsymbol{W}^{\ell,\epsilon} + \boldsymbol{\omega}\times(\boldsymbol{y} - \boldsymbol{y}_c^\ell) & \text{on } S_\ell^\epsilon,\\ \boldsymbol{w}^{k\epsilon} = \boldsymbol{e}_k & \text{on } \partial B_\ell^\epsilon,\end{array}\right. \tag{4.1}$$

where $\boldsymbol{e}_k$ is the k^{th} - unit vector of the Cartesian base and $\mu = \mu_d$ in T_ℓ^ϵ and $\mu = \mu_0$ in $B_\ell^\epsilon - \overline{T_\ell^\epsilon}$. Existence and uniqueness of a solution to (4.1) is established as in [19] and furthermore the following result is also proved:

Lemma 1. *If $a_\epsilon = \mathcal{O}(\epsilon^{n/(n-2)})$ then the solution to the local problem* (4.1)*, $\boldsymbol{w}^{k\epsilon}$ converges strongly to $\boldsymbol{e}_k$ in $(H^1(\Omega))^n$.*

5. A convergence result

Using (3.6), let us define the energy functional $E^\epsilon\colon \left(H_0^1(\Omega)\right)^n \mapsto \mathbb{R}\cup\{+\infty\}$ by

$$E^\epsilon(\boldsymbol{u}) = \int_\Omega \mu^\epsilon\, e(\boldsymbol{u})\colon e(\boldsymbol{u})\, d\boldsymbol{x} + I_{V^\epsilon}(\boldsymbol{u}). \tag{5.1}$$

Theorem 1. *The sequence $\left(E^\epsilon\right)_\epsilon$ defined by* (5.1) *Γ - converges in the strong topology of $\left(H_0^1(\Omega)\right)^n$ to the functional E defined by*

$$E(\boldsymbol{u}) = \int_\Omega \mu_0\, e(\boldsymbol{u})\colon e(\boldsymbol{u})\, d\boldsymbol{x} + I_V(\boldsymbol{u}), \tag{5.2}$$

where V is the closed subspace of $\left(H_0^1(\Omega)\right)^n$ defined by

$$V = \left\{ \boldsymbol{w} \in \left(H_0^1(\Omega)\right)^n \mid \text{div } \boldsymbol{w} = 0 \right\},$$

Proof. We first remark that for every $\boldsymbol{u} \in \left(H_0^1(\Omega)\right)^n$ which is not divergence-free in Ω we have

$$\Gamma - \liminf_{\epsilon \to 0} E^\epsilon(\boldsymbol{u}) = \Gamma - \limsup_{\epsilon \to 0} E^\epsilon(\boldsymbol{u}) = +\infty.$$

and thus we only have to deal with divergence-free functions.

In order to prove the theorem we have to prove the following two assertions

(a) For all $\boldsymbol{v}^0 \in V$ there exists a $\boldsymbol{v}^\epsilon \in V^\epsilon$, $\boldsymbol{v}^\epsilon \to \boldsymbol{v}^0$ in $\left(H_0^1(\Omega)\right)^n$ such that $\lim_{\epsilon \to 0} E^\epsilon(\boldsymbol{v}^\epsilon) = E(\boldsymbol{v}^0)$,

(b) For all $\boldsymbol{u}^0 \in V$, for all $\boldsymbol{u}^\epsilon \in V^\epsilon$, $\boldsymbol{u}^\epsilon \to \boldsymbol{u}^0$ in $\left(H_0^1(\Omega)\right)^n$ such that $\liminf_{\epsilon \to 0} E^\epsilon(\boldsymbol{u}^\epsilon) \geq E(\boldsymbol{u}^0)$.

Part (a). Let $\boldsymbol{v}^0 \in (\mathcal{D}(\Omega))^n$ such that div $\boldsymbol{v}^0 = 0$. Define the sequence $\boldsymbol{v}^\epsilon$ in the following way (see [5], [23]):

$$\boldsymbol{v}^\epsilon(\boldsymbol{x}) = \begin{cases} \boldsymbol{v}^0(\boldsymbol{x}) & \text{in } Y_\ell^\epsilon - B_\ell^\epsilon, \\ \boldsymbol{v}^0(\boldsymbol{x}) + (\boldsymbol{w}^{k\epsilon}(\boldsymbol{x}) - \boldsymbol{e}_k) v_k^0(\boldsymbol{x}_c^\ell) - \text{curl } (\widetilde{\boldsymbol{v}}_{\epsilon\ell} \phi_{\epsilon\ell}) & \text{in } B_\ell^\epsilon - T_\ell^\epsilon, \\ v_k^0(\boldsymbol{x}_c^\ell) \boldsymbol{w}^{k\epsilon}(\boldsymbol{x}) & \text{in } T_\ell^\epsilon, \end{cases} \tag{5.3}$$

where $\widetilde{\boldsymbol{v}}_{\epsilon\ell}$ is the vector valued function associated with $\boldsymbol{v}^0(\boldsymbol{x}) - \boldsymbol{v}^0(\boldsymbol{x}_c^\ell)$ with some relevant properties listed below:

(5.4)

1. div $\widetilde{\boldsymbol{v}}_{\epsilon\ell} = 0$,
2. curl $(\widetilde{\boldsymbol{v}}_{\epsilon\ell}) = \boldsymbol{v}^0(\boldsymbol{x}) - \boldsymbol{v}^0(\boldsymbol{x}_c^\ell)$,
3. $\|\widetilde{\boldsymbol{v}}_{\epsilon\ell}\|_{(L^2(\Omega))^n} \leq C a_\epsilon \left\|\boldsymbol{v}^0(\boldsymbol{x}) - \boldsymbol{v}^0(\boldsymbol{x}_c^\ell)\right\|_{(L^2(\Omega))^n}$,
4. $\|\nabla \widetilde{\boldsymbol{v}}_{\epsilon\ell}\|_{(L^2(B_\ell^{a_\epsilon}))^{n\times n}} \leq C \left\|\boldsymbol{v}^0(\boldsymbol{x}) - \boldsymbol{v}^0(\boldsymbol{x}_c^\ell)\right\|_{(L^2(\Omega))^n}$,
5. $\sum_{\ell \in N^\epsilon}$ curl $(\widetilde{\boldsymbol{v}}_{\epsilon\ell} \phi_{\epsilon\ell}) \to \boldsymbol{0}$ strongly in $\left(H_0^1(\Omega)\right)^n$ as $\epsilon \to 0$.

and where

$$\phi_{\epsilon\ell}(\boldsymbol{x}) = \phi_\ell\left(\frac{\boldsymbol{x} - \boldsymbol{x}_c^\ell}{\epsilon}\right), \quad \phi_{\epsilon\ell}(\boldsymbol{x}) \in \mathcal{D}(B_\ell^\epsilon), \quad \phi_{\epsilon\ell}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in T_\ell^\epsilon, \\ 0 & \text{if } \boldsymbol{x} \in B_\ell^\epsilon - B_\ell^{a_\epsilon}, \end{cases}$$

with $supp(\phi_{\epsilon\ell}) \subset B_\ell^{a_\epsilon}$, where $B_\ell^{a_\epsilon}$ is the ball with center the center of cell $\ell \in N^\epsilon$ and radius $2a_\epsilon$.

One can now verify, that the sequence $\boldsymbol{v}^\epsilon$ is divergence free, belongs in $\left(H_0^1(\Omega)\right)^n$, $\boldsymbol{v}^\epsilon = \boldsymbol{W}^{\ell,\epsilon} + \boldsymbol{\omega} \times (\boldsymbol{y} - \boldsymbol{y}_c^\ell)$ on S_ℓ^ϵ and $\boldsymbol{v}^\epsilon \to \boldsymbol{v}^0$ in $\left(H_0^1(\Omega)\right)^n$. Hence, computing $E^\epsilon(\boldsymbol{v}^\epsilon)$ we obtain:

$$
\begin{aligned}
E^\epsilon(\boldsymbol{v}^\epsilon) &= \sum_{\ell \in N^\epsilon} \int_{Y_\ell^\epsilon - T_\ell^\epsilon} \mu\, e(\boldsymbol{v}^0) : e(\boldsymbol{v}^0)\, d\boldsymbol{x} + \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon} \mu\, v_k^0(\boldsymbol{x}_c^\ell) v_m^0(\boldsymbol{x}_c^\ell) e(\boldsymbol{w}^{k\epsilon}) : e(\boldsymbol{w}^{m\epsilon})\, d\boldsymbol{x} \\
&+ \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon - T_\ell^\epsilon} 2\,\mu\, v_k^0(\boldsymbol{x}_c^\ell) e(\boldsymbol{v}^0) : e(\boldsymbol{w}^{k\epsilon})\, d\boldsymbol{x} - \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon - T_\ell^\epsilon} 2\,\mu\, e(\boldsymbol{v}^0) : e(\text{curl }\ (\tilde{\boldsymbol{v}}_{\epsilon\ell}\phi_{\epsilon\ell}))\, d\boldsymbol{x} \\
&- \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon - T_\ell^\epsilon} 2\,\mu\, v_k^0(\boldsymbol{x}_c^\ell) e(\boldsymbol{w}^{k\epsilon}) : e(\text{curl }\ (\tilde{\boldsymbol{v}}_{\epsilon\ell}\phi_{\epsilon\ell}))\, d\boldsymbol{x} \\
&+ \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon - T_\ell^\epsilon} \mu\, e(\text{curl }\ (\tilde{\boldsymbol{v}}_{\epsilon\ell}\phi_{\epsilon\ell})) : e(\text{curl }\ (\tilde{\boldsymbol{v}}_{\epsilon\ell}\phi_{\epsilon\ell}))\, d\boldsymbol{x} \\
&= \int_{\Omega \setminus \Omega_{1\epsilon}} \mu\, e(\boldsymbol{v}^0) : e(\boldsymbol{v}^0)\, d\boldsymbol{x} + o(1)
\end{aligned}
$$

Passing to the limit we obtain $\lim_{\epsilon\to 0} E^\epsilon(\boldsymbol{v}^\epsilon) = E(\boldsymbol{v}^0)$. For $\boldsymbol{v}^0 \in \left(H_0^1(\Omega)\right)^n$ we use a diagonalization process to complete the argument.

Part (b). Let $\boldsymbol{u}^\epsilon \in V^\epsilon$ such that $\boldsymbol{u}^\epsilon \to \boldsymbol{u}^0$ in $\left(H_0^1(\Omega)\right)^n$, where $\boldsymbol{u}^0 \in V$. Moreover, assume that $\boldsymbol{v}^0 \in (\mathcal{D}(\Omega))^n$ such that div $\boldsymbol{v}^0 = 0$ and define the sequence $\boldsymbol{v}^\epsilon$ as before. Using a sub-differential type inequality we get:

$$
E^\epsilon(\boldsymbol{u}^\epsilon) \geq E^\epsilon(\boldsymbol{v}^\epsilon) + \int_\Omega 2\,\mu^\epsilon\, e(\boldsymbol{v}^\epsilon) : e(\boldsymbol{u}^\epsilon - \boldsymbol{v}^\epsilon)\, d\boldsymbol{x}. \tag{5.5}
$$

Due to the strong convergence of $\boldsymbol{u}^\epsilon \to \boldsymbol{u}^0$, and $\boldsymbol{v}^\epsilon \to \boldsymbol{v}^0$ in $\left(H_0^1(\Omega)\right)^n$ we can pass to the limit first as $\epsilon \to 0$ then using the continuity of the functional E^ϵ and a diagonalization argument to make $\boldsymbol{v}^0 \to \boldsymbol{u}^0$ in the strong topology of $\left(H_0^1(\Omega)\right)^n$ we get, $\liminf_{\epsilon\to 0} E^\epsilon(\boldsymbol{u}^\epsilon) \geq E(\boldsymbol{u}^0)$. □

An immediate consequence of **Theorem** 1 is

Corollary 1. *The sequence* $\{\boldsymbol{v}^\epsilon\}$ *of solutions to* (3.5) *is convergent in the strong topology of* $(H_0^1(\Omega))^n$ *to* $\boldsymbol{v}$ *solution of*

$$
\begin{cases} \textit{Find } \boldsymbol{v} \in \left(H_0^1(\Omega)\right)^n \textit{ such that,} \\ J(\boldsymbol{v}) = \min\limits_{\boldsymbol{u} \in \left(H_0^1(\Omega)\right)^n} J(\boldsymbol{u}), \end{cases} \tag{5.6}
$$

where

$$
J(\boldsymbol{u}) = \int_\Omega \mu_0\, e(\boldsymbol{u}) : e(\boldsymbol{u})\, d\boldsymbol{x} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}\, d\boldsymbol{x} + I_V(\boldsymbol{u}), \tag{5.7}
$$

and

$$
V = \left\{ \boldsymbol{w} \in \left(H_0^1(\Omega)\right)^n \mid \text{div } \boldsymbol{w} = 0 \textit{ in } \Omega \right\}.
$$

6. Asymptotics of the velocity field for a single droplet

We now consider the emulsion flow problem for a single droplet T^ϵ centered at $\boldsymbol{x}_c$ in Ω and of size a_ϵ

$$-\text{div}\ (2\,\mu^\epsilon\, e(\boldsymbol{v}^\epsilon) - p^\epsilon I) = \boldsymbol{f} + \lambda_\epsilon \kappa_\epsilon \boldsymbol{n}_x \chi_{S^\epsilon} \quad \text{in } \Omega, \tag{6.1a}$$

$$\text{div}\ \boldsymbol{v}^\epsilon = 0 \quad \text{in } \Omega, \tag{6.1b}$$

$$[\![\boldsymbol{v}^\epsilon]\!] = \boldsymbol{0} \quad \text{on } S^\epsilon, \tag{6.1c}$$

$$\boldsymbol{v}^\epsilon = \boldsymbol{V}^\epsilon + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_c) \quad \text{on } S^\epsilon, \tag{6.1d}$$

$$\boldsymbol{v}^\epsilon = \boldsymbol{0} \quad \text{on } \Gamma. \tag{6.1e}$$

where as before $S^\epsilon = \partial T^\epsilon$ and $\mu^\epsilon(\boldsymbol{x}) = \mu_d$ if $x \in T^\epsilon$, and $\mu^\epsilon(\boldsymbol{x}) = \mu_0$ if $x \in \Omega \backslash T^\epsilon$.

The weak formulation of this problem is as in (3.4), where the number of droplets is reduced to one. In the limit as $\epsilon \to 0$ the velocity vector field $\boldsymbol{v}^\epsilon$ will converge to $\boldsymbol{v}$ solution to the unperturbed Stokes flow problem

$$\begin{aligned} -\text{div}\ (2\,\mu_0\, e(\boldsymbol{v}) - p\, I) &= \boldsymbol{f} \quad \text{in } \Omega, \\ \text{div}\ \boldsymbol{v} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{v} &= \boldsymbol{0} \quad \text{on } \Gamma. \end{aligned} \tag{6.2}$$

Theorem 2. *There exists a constant $C > 0$ independent of ϵ such that*

$$\|\boldsymbol{v}^\epsilon - \boldsymbol{v}\|_{\left(H_0^1(\Omega)\right)^n} \leq C\, a_\epsilon^{n/2}.$$

Proof. See [4], theorem 3.2. □

Next we want to improve on **Theorem** 2 and find the next term in the expansion $\boldsymbol{v}^\epsilon$. The main result will be shown in the **Main Theorem**. To prove this theorem we will adapt the method used in Ammari et al. [1] and Bonnetier et all. [4] to our problem that includes the kinematic boundary condition and use some results from Maris and Vernescu [18].

First let us denote by $(\boldsymbol{G}, F)$ the Green's tensors associated with the homogenized flow in (6.2)

$$\begin{aligned} -\text{div}\ (2\,\mu\, e_x(\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z})) - F_i(\boldsymbol{x},\boldsymbol{z})I) &= \boldsymbol{e}_i \delta(\boldsymbol{x}-\boldsymbol{z}) \quad \text{in } \Omega, \\ \text{div}\ _x\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}) &= 0 \quad \text{in } \Omega, \\ \boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}) &= \boldsymbol{0} \quad \text{on } \Gamma. \end{aligned} \tag{6.3}$$

Moreover, we introduce the following problems centered at the origin,

$$\begin{aligned} -\text{div}\ \left(2\,\mu\, e(\boldsymbol{\phi}^{kl}) - s^{kl} I\right) &= \lambda\, \kappa\, \boldsymbol{n}_y \chi_S \quad \text{in } \mathbb{R}^n, \\ \text{div}\ \boldsymbol{\phi}^{kl} &= 0 \quad \text{in } \mathbb{R}^n, \\ [\![\boldsymbol{\phi}^{kl}]\!] &= \boldsymbol{0} \quad \text{on } S, \\ \boldsymbol{\phi}^{kl} &= \boldsymbol{\phi}^{c,kl} + \boldsymbol{\omega}^{kl} \times \boldsymbol{y} \quad \text{on } S, \\ \boldsymbol{\phi}^{kl} &\to \boldsymbol{B}^{kl} \quad \text{at } \infty, \end{aligned} \tag{6.4}$$

$$
\begin{aligned}
-\text{div }_y(2\,\mu\, e_y(\boldsymbol{V}^\epsilon) - q^\epsilon\, I) &= \lambda\,\kappa\,\boldsymbol{n}_y\,\chi_S && \text{in } \Omega_\epsilon,\\
\text{div }_y\boldsymbol{V}^\epsilon &= 0 && \text{in } \Omega_\epsilon,\\
[\![\boldsymbol{V}^\epsilon]\!] &= \boldsymbol{0} && \text{on } S,\\
\boldsymbol{V}^\epsilon &= \boldsymbol{V}^{c,\epsilon} + \boldsymbol{\omega}\times\boldsymbol{y} && \text{on } S,\\
\boldsymbol{V}^\epsilon &= e(\boldsymbol{v})(\boldsymbol{x}_c)\boldsymbol{y} && \text{on } \partial\Omega_\epsilon.
\end{aligned}
\tag{6.5}
$$

$$
\begin{aligned}
-\text{div }_y(2\,\mu\, e_y(\boldsymbol{V}) - q\, I) &= \lambda\,\kappa\,\boldsymbol{n}_y\chi_S && \text{in } \mathbb{R}^n,\\
\text{div }_y\boldsymbol{V} &= 0 && \text{in } \mathbb{R}^n,\\
[\![\boldsymbol{V}]\!] &= \boldsymbol{0} && \text{on } S,\\
\boldsymbol{V} &= \boldsymbol{V}^{c} + \boldsymbol{\omega}\times\boldsymbol{y} && \text{on } S,\\
\boldsymbol{V} &\to e(\boldsymbol{v})(\boldsymbol{x}_c)\,\boldsymbol{y} && \text{at } \infty.
\end{aligned}
\tag{6.6}
$$

where $\boldsymbol{B}^{kl} = \big(\boldsymbol{e}_l n_k + \boldsymbol{e}_k n_l - \frac{2}{n}\boldsymbol{n}_y\delta_{kl}\big)$, $\boldsymbol{\omega}^{kl}$, $\boldsymbol{\omega}$ are unknown constant vectors in $\mathbb{R}^n$, $\boldsymbol{\phi}^{c,kl}$, $\boldsymbol{V}^{c,\epsilon}$ and $\boldsymbol{V}^c$ are defined in similar spirit as in formula (2.5), and Ω_ϵ is the scaled domain defined the following way

$$
\Omega_\epsilon := \left\{\boldsymbol{y} = \frac{\boldsymbol{x}-\boldsymbol{x}_c}{a_\epsilon} \mid \boldsymbol{x}\in\Omega\right\}.
$$

For simplicity we have assumed Ω_ϵ to be a ball in $\mathbb{R}^n$.

The balance of forces and torques for each problem is automatically satisfied by virtue of formula (2.9), whilst existence and uniqueness of the above problems follows from the theory of pseudo-monotone operators in Hilbert spaces [17].

In what follows, it will be convenient to re-define problems (6.4), (6.5), (6.6). In accordance we define $\hat{\boldsymbol{\phi}}^{kl} := \boldsymbol{\phi}^{kl} - \boldsymbol{B}^{kl}$, $\boldsymbol{W}^\epsilon := \boldsymbol{V}^\epsilon - e(\boldsymbol{v})(\boldsymbol{x}_c)\boldsymbol{y}$, and $\boldsymbol{W} := \boldsymbol{V} - e(\boldsymbol{v})(\boldsymbol{x}_c)\boldsymbol{y}$. Hence, we obtain the following corresponding problems for (6.4), (6.5), and (6.6),

$$
\begin{aligned}
-\text{div }\Big(2\,\mu\, e(\hat{\boldsymbol{\phi}}^{kl}) - s^{kl} I\Big) &= \lambda\,\kappa\,\boldsymbol{n}_y\,\chi_S\\
&\quad - (\mu_d - \mu_0)\,(\boldsymbol{e}_l n_k + \boldsymbol{e}_k n_l - \frac{2}{n}\boldsymbol{n}_y\delta_{kl})\chi_S && \text{in } \mathbb{R}^n,\\
\text{div }\hat{\boldsymbol{\phi}}^{kl} &= 0 && \text{in } \mathbb{R}^n,\\
[\![\hat{\boldsymbol{\phi}}^{kl}]\!] &= 0 && \text{on } S,\\
\hat{\boldsymbol{\phi}}^{kl} &= -\boldsymbol{B}^{kl} + \boldsymbol{\phi}^{c,kl} + \boldsymbol{\omega}^{kl}\times\boldsymbol{y} && \text{on } S,\\
\hat{\boldsymbol{\phi}}^{kl} &\to \boldsymbol{0} && \text{at } \infty,
\end{aligned}
\tag{6.7}
$$

$$
\begin{aligned}
-\text{div }_y(2\,\mu\, e_y(\boldsymbol{W}^\epsilon) - q^\epsilon\, I) &= (\lambda\,\kappa\, I - 2\,(\mu_0-\mu_d)\, e(\boldsymbol{u})(\boldsymbol{x}_c))\boldsymbol{n}_y\chi_S && \text{in } \Omega_\epsilon,\\
\text{div }_y\boldsymbol{W}^\epsilon &= 0 && \text{in } \Omega_\epsilon,\\
[\![\boldsymbol{W}^\epsilon]\!] &= \boldsymbol{0} && \text{on } S,\\
\boldsymbol{W}^\epsilon &= -e(\boldsymbol{u})(\boldsymbol{x}_c)\,\boldsymbol{y} + \boldsymbol{V}^{c,\epsilon} + \boldsymbol{\omega}\times\boldsymbol{y} && \text{on } S,\\
\boldsymbol{W}^\epsilon &= \boldsymbol{0} && \text{on } \partial\Omega_\epsilon.
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
-\mathrm{div}_{\,y}(2\,\mu\, e_y(\boldsymbol{W}) - q\, I) &= (\lambda\,\kappa\, I - 2\,(\mu_0 - \mu_d)\, e(\boldsymbol{u})(\boldsymbol{x}_c))\boldsymbol{n}_y \chi_S && \text{in } \mathbb{R}^n,\\
\mathrm{div}_{\,y}\boldsymbol{W} &= 0 && \text{in } \mathbb{R}^n,\\
[\![\boldsymbol{W}]\!] &= \boldsymbol{0} && \text{on } S,\\
\boldsymbol{W} = -e(\boldsymbol{u})(\boldsymbol{x}_c)\,\boldsymbol{y} + \boldsymbol{V}^c &+ \boldsymbol{\omega}\times\boldsymbol{y} && \text{on } S,\\
\boldsymbol{W} &\to \boldsymbol{0} && \text{at } \infty.
\end{aligned}
\tag{6.9}
$$

Lemma 2. *There exists a constant C (independent of ϵ) such that*

$$
\|e_y\left(\boldsymbol{v}^\epsilon(a_\epsilon \boldsymbol{y} + \boldsymbol{x}_c) - \boldsymbol{v}(a_\epsilon \boldsymbol{y} + \boldsymbol{x}_c) - a_\epsilon \boldsymbol{W}^\epsilon(\boldsymbol{y})\right)\|_{(L^2(\Omega_\epsilon))^{n\times n}} \le C a_\epsilon^2
$$

Proof. See [4], lemma 3.3. □

Lemma 3. *There exists a constant C (independent of ϵ) such that*

$$
\|e_y\left(\boldsymbol{v}^\epsilon(a_\epsilon \boldsymbol{y} + \boldsymbol{x}_c) - \boldsymbol{v}(a_\epsilon \boldsymbol{y} + \boldsymbol{x}_c) - a_\epsilon \boldsymbol{W}(\boldsymbol{y})\right)\|_{(L^2(\Omega_\epsilon))^{n\times n}} \le C\, a_\epsilon^{3/2}
$$

Proof. In view of **Lemma** 2 it is enough to show that

$$
\|e_y(\boldsymbol{W} - \boldsymbol{W}^\epsilon)\|_{(L^2(\Omega_\epsilon))^{n\times n}} \le C\, a_\epsilon^{1/2}
$$

This follows from a result in [18]. □

In this subsection we derive the first corrector of the velocity expansion, expressed in terms of the Green's tensors and bulk and surface polarization tensors.

Main Theorem. *Under the above hypotheses, for any $\boldsymbol{z} \in \Omega$ at a distance $d > 0$ away from T^ϵ we have*

$$
\begin{aligned}
v_i^\epsilon(\boldsymbol{z}) &= v_i(\boldsymbol{z}) + 2\,(\mu_0 - \mu_d)\, a_\epsilon^n\, e_x(\boldsymbol{G}_i)(\boldsymbol{x}_c, \boldsymbol{z})\colon \int_T \left\{ \sum_{k,l=1}^n e_{xkl}(\boldsymbol{v})(\boldsymbol{x}_c)\, e_y(\boldsymbol{\phi}^{kl}(\boldsymbol{y})) \right\} d\boldsymbol{y}\\
&- a_\epsilon^n\, \lambda\, e_x(\boldsymbol{G}_i)(\boldsymbol{x}_c, \boldsymbol{z})\colon \int_S \boldsymbol{n}_y \otimes \boldsymbol{n}_y\, ds_y + \mathcal{O}\left(a_\epsilon^{n+\frac{1}{2}}\right)
\end{aligned}
$$

7. The proof of the main theorem

Using (6.3) and (6.2) we have

$$
v_i(\boldsymbol{z}) = \int_\Omega 2\,\mu_0\, e_x(\boldsymbol{G}_i)\colon e_x(\boldsymbol{v})\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot \boldsymbol{G}_i\, d\boldsymbol{x}
$$

Similarly, using (6.3) and (6.1a) – (6.1e) we have

$$
\begin{aligned}
v_i^\epsilon(\boldsymbol{z}) &= \int_\Omega 2\,\mu_0\, e_x(\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}))\colon e_x(\boldsymbol{v}^\epsilon)\,d\boldsymbol{x} = \int_\Omega 2\,\mu^\epsilon\, e_x(\boldsymbol{v}^\epsilon)\colon e_x(\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}))\,d\boldsymbol{x} \\
&+ \int_{T^\epsilon} 2\,(\mu_0-\mu_d)\, e_x(\boldsymbol{v}^\epsilon)\colon e_x(\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}))\,d\boldsymbol{x} \\
&= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{G}_i\,d\boldsymbol{x} + \int_{S^\epsilon} \lambda_\epsilon\,\kappa_\epsilon\,\boldsymbol{n}_x\cdot\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z})\,ds_x + \int_{T^\epsilon} 2\,(\mu_0-\mu_d)\, e_x(\boldsymbol{v}^\epsilon)\colon e_x(\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}))\,d\boldsymbol{x}
\end{aligned}
$$

Thus we get,

$$
(v_i^\epsilon - v_i)(\boldsymbol{z}) = \int_{S^\epsilon} \lambda_\epsilon\,\kappa_\epsilon\,\boldsymbol{n}_x\cdot\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z})\,d\boldsymbol{x} + \int_{T^\epsilon} 2\,(\mu_0-\mu_d)\, e_x(\boldsymbol{v}^\epsilon)\colon e_x(\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}))\,d\boldsymbol{x} \tag{7.1}
$$

We compute the integral over the surface first and we get,

$$
\begin{aligned}
\int_{S^\epsilon} \lambda_\epsilon\,\kappa_\epsilon\,\boldsymbol{n}_x\cdot\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z})\,ds_x &= a_\epsilon^{n-1}\int_S \lambda\,\kappa\,\boldsymbol{n}_y\cdot\boldsymbol{G}_i(a_\epsilon\,\boldsymbol{y}+\boldsymbol{x}_c,\boldsymbol{z})\,ds_y \\
&= a_\epsilon^{n-1}\int_S \lambda\,\kappa\,\boldsymbol{n}_y\cdot\boldsymbol{G}_i(\boldsymbol{x}_c,\boldsymbol{z})\,ds_y + a_\epsilon^{n}\int_S \lambda\,\kappa\,\boldsymbol{n}_y\cdot\nabla_x\boldsymbol{G}_i(\boldsymbol{x}_c,\boldsymbol{z})\boldsymbol{y}\,ds_y + \mathcal{O}(a_\epsilon^{n+1}) \\
&= a_\epsilon^{n}\int_S \lambda\,\kappa\,\boldsymbol{n}_y\cdot\nabla_x\boldsymbol{G}_i(\boldsymbol{x}_c,\boldsymbol{z})\boldsymbol{y}\,ds_y + \mathcal{O}(a_\epsilon^{n+1}) \\
&= -a_\epsilon^{n}\int_S \lambda\,\nabla_x\boldsymbol{G}_i(\boldsymbol{x}_c,\boldsymbol{z})\boldsymbol{n}_y\cdot\boldsymbol{n}_y\,ds_y + \mathcal{O}(a_\epsilon^{n+1}) \\
&= -a_\epsilon^{n}\,\lambda\, e_x(\boldsymbol{G}_i(\boldsymbol{x}_c,\boldsymbol{z}))\colon \int_S \boldsymbol{n}_y\otimes\boldsymbol{n}_y\,ds_y + \mathcal{O}(a_\epsilon^{n+1})
\end{aligned} \tag{7.2}
$$

where we used an expansion on $\boldsymbol{G}_i$ and formula (2.9).

To compute the term over the droplet T^ϵ we define the following,

$$
\boldsymbol{R}_\epsilon(\boldsymbol{y}) = \boldsymbol{v}^\epsilon(a_\epsilon\boldsymbol{y}+\boldsymbol{x}_c) - \boldsymbol{v}(a_\epsilon\boldsymbol{y}+\boldsymbol{x}_c) - a_\epsilon\,\boldsymbol{W}(\boldsymbol{y}), \quad \boldsymbol{r}_\epsilon(\boldsymbol{x}) = \boldsymbol{R}_\epsilon\left(\frac{\boldsymbol{x}-\boldsymbol{x}_c}{a_\epsilon}\right) = \boldsymbol{R}_\epsilon(\boldsymbol{y})
$$

Thus,

$$
\begin{aligned}
\int_{T^\epsilon} 2\,(\mu_0-\mu_d)\, e_x(\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}))\colon e_x(\boldsymbol{v}^\epsilon)\,d\boldsymbol{x} &= \int_{T^\epsilon} 2\,(\mu_0-\mu_d)\, e_x(\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}))\colon e_x(\boldsymbol{r}^\epsilon(\boldsymbol{x}))\,d\boldsymbol{x} \\
+ \int_{T^\epsilon} 2\,(\mu_0-\mu_d)\, e_x(\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}))\colon e_x\left(\boldsymbol{v}(\boldsymbol{x}) + a_\epsilon\,\boldsymbol{W}\left(\frac{\boldsymbol{x}-\boldsymbol{x}_c}{a_\epsilon}\right)\right)\,d\boldsymbol{x}
\end{aligned}
$$

The first integral above, in view of **Lemma** 3, becomes

$$
\begin{aligned}
&\left|\int_{T^\epsilon} 2\,(\mu_0-\mu_d)\, e_x(\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}))\colon e_x(\boldsymbol{r}^\epsilon(\boldsymbol{x}))\,d\boldsymbol{x}\right| \\
&= a_\epsilon^{n-1}\left|\int_T 2\,(\mu_0-\mu_d)\, e_x(\boldsymbol{G}_i(a_\epsilon\boldsymbol{y}+\boldsymbol{x}_c,\boldsymbol{z}))\colon e_y(\boldsymbol{R}^\epsilon(\boldsymbol{y}))\,d\boldsymbol{y}\right| \\
&\le C\,a_\epsilon^{n+\frac{1}{2}}
\end{aligned} \tag{7.3}
$$

For the second integral we have,

$$\begin{aligned}(7.4)\qquad &\int_{T^\epsilon} 2\,(\mu_0-\mu_d)\, e_x(\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}))\colon e_x\left(\boldsymbol{v}(\boldsymbol{x}) + a_\epsilon\, \boldsymbol{W}\left(\frac{\boldsymbol{x}-\boldsymbol{x}_c}{a_\epsilon}\right)\right)\, d\boldsymbol{x}\\
&= 2\,(\mu_0-\mu_d)\, a_\epsilon^n \int_T e_x(\boldsymbol{G}_i(a_\epsilon \boldsymbol{y}+\boldsymbol{x}_c,\boldsymbol{z}))\colon\ (e_x(\boldsymbol{v}(a_\epsilon\boldsymbol{y}+\boldsymbol{x}_c)) + e_y\,(\boldsymbol{W}(\boldsymbol{y})))\ d\boldsymbol{y}\\
&= 2\,(\mu_0-\mu_d)\, a_\epsilon^n \int_T e_x(\boldsymbol{G}_i(\boldsymbol{x}_c,\boldsymbol{z}))\colon\ \{e_x(\boldsymbol{v})(\boldsymbol{x}_c) + e_y\,(\boldsymbol{W}(\boldsymbol{y}))\}\ d\boldsymbol{y} + \mathcal{O}(a_\epsilon^{n+1})\end{aligned}$$

To complete the proof, we write $\boldsymbol{W}(\boldsymbol{y})$ as a linear combination of $\hat{\boldsymbol{\phi}}^{kl}$, solution to (6.7), the following way,

$$\boldsymbol{W}(\boldsymbol{y}) = \sum_{k,l=1}^{n} e_{xkl}(\boldsymbol{v})(\boldsymbol{x}_c)\, \hat{\boldsymbol{\phi}}^{kl}(\boldsymbol{y})$$

Replacing $\boldsymbol{W}(\boldsymbol{y})$ in (7.4) we get,

$$\begin{aligned}(7.5)\qquad &\int_{T^\epsilon} 2\,(\mu_0-\mu_d)\, e_x(\boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{z}))\colon e_x\left(\boldsymbol{v}(\boldsymbol{x}) + a_\epsilon\, \boldsymbol{W}\left(\frac{\boldsymbol{x}-\boldsymbol{x}_c}{a_\epsilon}\right)\right)\, d\boldsymbol{x}\\
&= 2\,(\mu_0-\mu_d)\, a_\epsilon^n \int_T e_x(\boldsymbol{G}_i(\boldsymbol{x}_c,\boldsymbol{z}))\colon\ \left\{e_x(\boldsymbol{v})(\boldsymbol{x}_c) + \sum_{k,l=1}^{n} e_{xkl}(\boldsymbol{v}(\boldsymbol{x}_c))\, e_y(\hat{\boldsymbol{\phi}}^{kl}(\boldsymbol{y}))\right\}\ d\boldsymbol{y}\\
&+ \mathcal{O}(a_\epsilon^{n+1})\end{aligned}$$

Combining (7.1), (7.2), (7.3), (7.4), (7.5), and substituting $\hat{\boldsymbol{\phi}}^{kl}(\boldsymbol{y}) = \boldsymbol{\phi}^{kl}(\boldsymbol{y}) - \boldsymbol{B}^{kl}$ we get

$$\begin{aligned}(v_i^\epsilon - v_i)(\boldsymbol{z}) &= 2\,(\mu_0-\mu_d)\, a_\epsilon^n\, e_x(\boldsymbol{G}_i(\boldsymbol{x}_c,\boldsymbol{z}))\colon \int_T \left\{\sum_{k,l=1}^{n} e_{xkl}(\boldsymbol{v})(\boldsymbol{x}_c)\, e_y(\boldsymbol{\phi}^{kl}(\boldsymbol{y}))\right\}\ d\boldsymbol{y}\\
&- a_\epsilon^n\,\lambda\, e_x(\boldsymbol{G}_i(\boldsymbol{x}_c,\boldsymbol{z}))\colon \int_S \boldsymbol{n}_y\otimes\boldsymbol{n}_y\, ds_y + \mathcal{O}\left(a_\epsilon^{n+\frac{1}{2}}\right)\end{aligned}$$

8. Asymptotics of the viscosity for emulsions with spherical droplets

This section demonstrates that the kinematic condition, given by (6.1d), is essential to obtaining the proper solution to problem (6.1). The use of the kinematic condition leads to the recovery of the proper effective viscosity given by Taylor's formula (1.2). The results of this section rely on the work of [11] and [16] to compute the effective viscosity of the emulsion. We assume that the droplets have a spherical shape with fixed centers of mass. Moreover, we define the following tensors

$$A_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{n}\delta_{ij}\delta_{kl}, \quad B_k^{ij} = \frac{1}{2}(y_j\,\delta_{ik} + y_i\,\delta_{jk}) - \frac{1}{n}\delta_{ij}y_k$$

In periodic homogenization, the local problem for the suspension of spherical droplets of viscosity μ_d in a viscous fluid of viscosity μ_0 has the following form [16]:

(8.1)
$$
\begin{aligned}
-\text{div}_{\,y}\big(\tau^{ij}\big) &= \mathbf{0} && \text{in } Y - S,\\
\text{div}_{\,y}\boldsymbol{v}^{ij} &= 0 && \text{in } Y,\\
[\![\boldsymbol{v}^{ij}]\!] &= \mathbf{0} && \text{on } S,\\
\big(B^{ij}_m + v^{ij}_m\big)\, n_m &= \left(\frac{1}{B}\int_B v^{ij}_m\, d\boldsymbol{y}\right) n_m && \text{on } S,\\
[\![\tau^{ij}\,\boldsymbol{n}]\!] &= [\![\tau^{ij}\,\boldsymbol{n}]\!]\cdot \boldsymbol{n}\boldsymbol{n} && \text{on } S,\\
\boldsymbol{v}^{ij} & \text{ is Y-periodic,}\\
\int_{\partial B} [\![\tau^{ij}\,\boldsymbol{n}]\!]\, ds_y &= \mathbf{0}
\end{aligned}
$$

where Y is the unit period cell, S is the surface of the spherical drop, and

$$\tau^{ij}_{ml} = 2\,\mu(A_{ijml} + e_{ml}(v^{ij})) - p^{ij}\delta_{lm}$$

We remark that since the shape of the drop has already been determined, one cannot impose the normal stress balance.

We define the Hilbert space $V_{per} = \big\{\boldsymbol{w} \in (H^1_{per}(Y))^n \mid \text{div}\,\boldsymbol{w} = 0\big\}/\mathbb{R}$. Moreover, we define the closed, convex, and unbounded subset of V_{per} by

$$H^{ij} = \left\{\boldsymbol{w} \in V_{per} \mid \big(B^{ij}_m + w_m\big)\, n_m = \left(\frac{1}{|B|}\int_B w_m\, d\boldsymbol{y}\right) n_m \text{ on } S\right\}/\mathbb{R}.$$

The weak formulation of (8.1) yields the variational inequality: Find $\boldsymbol{v}^{ij} \in V_{per}$ such that

$$\int_Y 2\,\mu\, e_{kl}(\boldsymbol{B}^{ij} + \boldsymbol{v}^{ij}) e_{kl}(\boldsymbol{\phi} - \boldsymbol{v}^{ij})\, d\boldsymbol{y} = 0 \text{ for any } \boldsymbol{\phi} \in H^{ij} \tag{8.2}$$

for which local existence and uniqueness results can be found in [17].

According to [16], the formula for the homogenized coefficients is

$$2\mu^H_{ijkl} = \frac{1}{|Y|}\int_Y (\tau^{ij}_{kl} - \frac{1}{3}\delta_{kl}\,\tau^{ij}_{pp})\, d\boldsymbol{y} - \int_S [\![\frac{1}{2}\,\Big(\tau^{ij}_{kq}\, y_l\, n_q + \tau^{ij}_{lq}\, y_k\, n_q\Big) - \frac{1}{3}\,\delta_{kl}\tau^{ij}_{pq}\, y_p\, n_q]\!]\, ds_y \tag{8.3}$$

Using the divergence free feature of the vector field $\boldsymbol{B}^{kl} + \boldsymbol{v}^{kl}$, we can re-write (8.3) as follows:

$$
\begin{aligned}
2\,\mu^H_{ijkl}\,|Y| &= \int_Y (\tau^{ij}_{kl} - \frac{1}{3}\delta_{kl}\,\tau^{ij}_{pp})\, d\boldsymbol{y} - \int_S [\![\frac{1}{2}\,\Big(\tau^{ij}_{kq}\, y_l\, n_q + \tau^{ij}_{lq}\, y_k\, n_q\Big) - \frac{1}{3}\,\delta_{kl}\tau^{ij}_{pq}\, y_p\, n_q]\!]\, ds_y\\
&= \int_Y (\tau^{ij}_{kl} - \frac{1}{3}\delta_{kl}\,\tau^{ij}_{pp})\, d\boldsymbol{y} - \int_S [\![\tau^{ij}_{pq}\, n_q]\!]\, B^{kl}_p\, ds_y\\
&= \int_Y (\tau^{ij}_{kl} - \frac{1}{3}\delta_{kl}\,\tau^{ij}_{pp})\, d\boldsymbol{y} - \int_S [\![\tau^{ij}_{\alpha\beta}\, n_\beta]\!]\, n_\alpha\, B^{kl}_p\, n_p\, ds_y\\
&= \int_Y \tau^{ij}_{pq}\, e_{pq}(\boldsymbol{B}^{kl} + \boldsymbol{v}^{kl})\, d\boldsymbol{y} = \int_Y 2\,\mu\, e_{pq}(\boldsymbol{B}^{ij} + \boldsymbol{v}^{ij})\, e_{pq}(\boldsymbol{B}^{kl} + \boldsymbol{v}^{kl})\, d\boldsymbol{y}
\end{aligned}
$$

Since the shape of the drop is spherical we can solve problem (8.1) explicitly for two concentric spheres of radius a and b with $b > a$ as in [11] and obtain

(8.4)
$$\mu^H_{ijkl} = \mu_0\, A_{ijkl} \left(1 - \frac{\pi}{3}\, \frac{\lambda^3(5\,\lambda^7(1-\eta) + (5\eta+2))}{4\,\lambda^{10}(1-\eta) + 5\,\lambda^7(5\eta-2) - 42\,\eta\,\lambda^5 + 5\,\lambda^3(5\eta+2) - 4\,(\eta+1)}\right)$$

where $\lambda = a/b$ and $\eta = \mu_d/\mu_0$. Moreover, we remark that the concentration, ϕ, of the spherical droplets is given by:

$$\phi = \frac{(4/3)\,\pi\, a^3}{8\, b^3} = \frac{\pi}{6}\,\lambda^3$$

For low concentrations (i.e. λ small) we can expand (8.4) to yield

$$\mu^H_{ijkl} = \mu_0\, A_{ijkl} \left(1 + \frac{5\eta+2}{2\,(1+\eta)}\,\phi + \mathcal{O}\left(\phi^2\right)\right) \tag{8.5}$$

Hence, the *effective scalar viscosity* μ^*, is the viscosity of the Newtonian fluid that behaves as the mixture of arbitrarily oriented spherical droplets of viscosity μ_d, that are invariant under suspension rotation. The effective scalar viscosity is computed as the angular averaging of the tensor in (8.5)

$$\mu^* = \mu_0 \left(1 + \frac{5\eta+2}{2\,(1+\eta)}\,\phi + \mathcal{O}\left(\phi^2\right)\right) \tag{8.6}$$

which agrees with Taylor's result [22].

9. Conclusions

We consider an emulsion formed by two Newtonian fluids, one being dispersed in the other under the form of droplets, in the presence of surface tension. We investigate the dilute case where the droplet size a_ϵ is much smaller than the distance ϵ between the droplets' centers. We prove using Γ-convergence in **Theorem** 1 that the limit behavior when $\epsilon \to 0$ is described by the unperturbed Stokes flow and in **Theorem** 2 we estimate the order of convergence of the velocity to be $a_\epsilon^{n/2}$. In the **Main Theorem** we improve the convergence result and determine the first corrector in the velocity expansion. Our results differ from [4] due to the fact that the corrector $\boldsymbol{\phi}^{kl}$ is the solution to a problem that takes into account the kinematic condition, and as a result Taylor's and Einstein's viscosity formulas are recovered in (8.6).

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