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ALGEBRAIC SYSTEMS WITH LIPSCHITZ PERTURBATIONS

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Abstract. By using variational methods, the existence of infinitely many solutions for a nonlinear algebraic system with a parameter is established in presence of a perturbed Lipschitz term. Our goal was achieved requiring an appropriate behavior of the nonlinear term f, either at zero or at infinity, without symmetry conditions.

1. Introduction

In many cases a problem in a continuous framework can be handled by using a suitable method from discrete mathematics, and conversely. For instance, let us consider the following relations

$$[u(i+1,j)-2u(i,j)+u(i-1,j)]+[u(i,j+1)-2u(i,j)+u(i,j-1)]$$
$$+\lambda f((i,j),u(i,j))=0, \quad \forall\,(i,j)\in\mathbb{Z}[1,m]\times\mathbb{Z}[1,n],$$

under the Dirichlet boundary conditions

$$u(i,0)=u(i,n+1)=0, \quad \forall\, i\in\mathbb{Z}[1,m],$$
$$u(0,j)=u(m+1,j)=0, \quad \forall\, j\in\mathbb{Z}[1,n],$$

where $f:\mathbb{Z}[1,m]\times\mathbb{Z}[1,n]\times\mathbb{R}\to\mathbb{R}$ denotes a continuous function and λ is a positive parameter. As pointed out by Galewski and Orpel in [5], the above problem serves as the discrete counterpart of the following continuous one:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\lambda f((x,y),u(x,y))=0, \\ u(x,0)=u(x,n+1)=0, \ \forall x\in(0,m+1) \\ u(0,y)=u(m+1,y)=0, \ \forall y\in(0,n+1). \end{cases}$$

However, the results obtained here (see Theorem 1 below) cannot be directly achieved by proving the existence of solutions for the above equation. The modeling/simulation of certain nonlinear problems from economics, biological neural networks, optimal control and others, enforced in a natural manner a rapid development of the theory of discrete equations (see for instance [26] and references therein).

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In this paper, motivated by this increasing interest, we study the following algebraic system

$$Au = \lambda f(u) + h(u), \qquad (S^{f,h}_{A,\lambda})$$

in which $u = (u_1, ..., u_n)^t \in \mathbb{R}^n$ is a column vector, $A = (a_{ij})_{n\times n}$ is a positive-definite matrix, $f(u) := (f_1(u_1), ..., f_n(u_n))^t$, where the functions $f_k : \mathbb{R} \to \mathbb{R}$ are assumed to be continuous for every $k \in \mathbb{Z}[1, n] := \{1, 2, ..., n\}$, and λ is a positive parameter.

Moreover,

$$h(u) := (h_1(u_1), ..., h_n(u_n))^t,$$

where, for every $k \in \mathbb{Z}[1, n]$, the functions $h_k : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous with constants $L_k \geq 0$, that is:

$$|h_k(t_1) - h_k(t_2)| \leq L_k|t_1 - t_2|,$$

for every $t_1, t_2 \in \mathbb{R}$, and $h_k(0) = 0$.

A large number of discrete problems can be formulated as special cases of the non-perturbed ($h = 0$) algebraic system, namely $(S^f_{A,\lambda})$; see, for instance, the papers [23, 25, 26, 27, 28] and references therein. We also point out that the case

$$A := \begin{pmatrix} 2 & -1 & 0 & ... & 0 \\ -1 & 2 & -1 & ... & 0 \\ & & \ddots & & \\ 0 & ... & -1 & 2 & -1 \\ 0 & ... & 0 & -1 & 2 \end{pmatrix}_{n\times n},$$

has been considered in order to study the existence of nontrivial solutions of nonlinear second-order difference equations [12, 13, 15]. Moreover, as it is well-known, boundary value problems involving fourth-order difference equations such as

$$(D^f_\lambda) \qquad \begin{cases} \Delta^4 u_{k-2} = \lambda f_k(u_k), \quad \forall\, k \in \mathbb{Z}[1, n] \\ u_{-2} = u_{-1} = u_0 = 0, \\ u_{n+1} = u_{n+2} = u_{n+3} = 0, \end{cases}$$

can also be expressed as the problem $(S^f_{A,\lambda})$, where A is the real symmetric and positive definite matrix of the form

$$A := \begin{pmatrix} 6 & -4 & 1 & 0 & ... & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & ... & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & ... & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & ... & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & ... & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & ... & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & ... & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & -4 & 6 \end{pmatrix}.$$

Further, general references on difference equations and their applications can be found e.g. in [1, 10].

Here, by using variational methods, under the key assumption that

$$L := \max_{k\in\mathbb{Z}[1,n]} L_k < \lambda_1,$$

where λ_1 is the first eigenvalue of the matrix A, we determine open intervals of positive parameters such that problem $(S^{f,h}_{A,\lambda})$ admits either an unbounded sequence of solutions, provided that the nonlinearity f has a suitable behaviour at infinity (Theorem 3), or a sequence of pairwise distinct solutions that converges to zero, if a similar behaviour occurs at zero (see Theorem 4).

Our main tool is a recent critical point result obtained by Ricceri and recalled here in a convenient form (see Theorem 2).

A special case of our results reads as follows (see Remark 4).

Theorem 1. *Let $z : \mathbb{R} \to \mathbb{R}$ be a nonnegative and continuous function. Assume that*

$$\liminf_{t\to+\infty} \frac{\int_0^t z(\xi)d\xi}{t^2} = 0, \qquad \limsup_{t\to+\infty} \frac{\int_0^t z(\xi)d\xi}{t^2} = +\infty.$$

Then, for each $\lambda > 0$, and for every Lipschitz continuous function $h : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant $L_h < \lambda_A$ (where λ_A is the first eigenvalue of the matrix A defined in Section 4), the following discrete problem

$$\begin{gathered}[u(i+1,j) - 2u(i,j) + u(i-1,j)] + [u(i,j+1) - 2u(i,j) + u(i,j-1)] \\ +\lambda z(u(i,j)) + h(u(i,j)) = 0, \qquad \forall\, (i,j) \in \mathbb{Z}[1,m] \times \mathbb{Z}[1,n]\end{gathered}$$

with boundary conditions

$$\begin{aligned} u(i,0) = u(i,n+1) = 0, &\quad \forall\, i \in \mathbb{Z}[1,m], \\ u(0,j) = u(m+1,j) = 0, &\quad \forall\, j \in \mathbb{Z}[1,n], \end{aligned}$$

admits an unbounded sequence of solutions.

Finally, for completeness, we just mention here that there is a vast literature on nonlinear difference equations based on fixed point and upper and lower solution methods (see [2, 8]). For related topics see the works [3, 6, 7, 22]. For a complete and exhaustive overview on variational methods we refer the reader to the monographs [11, 20].

2. Abstract Setting

Let $(X, \|\cdot\|)$ be a finite-dimensional Banach space and let $J_\lambda : X \to \mathbb{R}$ be a function satisfying the following structure hypothesis:

(Λ) for all $u \in X$, $J_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ *where $\Phi, \Psi : X \to \mathbb{R}$ are two functions of class C^1 on X with Φ coercive, i.e.* $\lim_{\|u\|\to\infty} \Phi(u) = +\infty$, *and λ is a real positive parameter.*

Moreover, provided that $r > \inf_X \Phi$, put

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Clearly, $\gamma \geq 0$ and $\delta \geq 0$. When $\gamma = 0$ (or $\delta = 0$), in the sequel, we agree to read $1/\gamma$ (or $1/\delta$) as $+\infty$.

Theorem 2. *Assuming that the condition* (Λ) *holds, one has*

(a) *If* $\gamma < +\infty$ *then, for each* $\lambda \in]0, 1/\gamma[$, *the following alternative holds*:
either
(a_1) J_λ *possesses a global minimum,*
or
(a_2) *there is a sequence* $\{u_m\}$ *of critical points* (*local minima*) *of* J_λ *such that* $\lim_{m\to\infty} \Phi(u_m) = +\infty$.

(b) *If* $\delta < +\infty$ *then, for each* $\lambda \in]0, 1/\delta[$, *the following alternative holds*:
either
(b_1) *there is a global minimum of* Φ *which is a local minimum of* J_λ,
or
(b_2) *there is a sequence* $\{u_m\}$ *of pairwise distinct critical points* (*local minima*) *of* J_λ, *with* $\lim_{m\to\infty} \Phi(u_m) = \inf_X \Phi$, *which converges to a global minimum of* Φ.

Remark 1. Theorem 2 is the finite-dimensional version of the quoted multiplicity result of Ricceri from [21].

As ambient space X, consider the n-dimensional Banach space $\mathbb{R}^n$ endowed by the norm

$$\|u\| := \Big(\sum_{k=1}^{n} u_k^2\Big)^{1/2}.$$

Set $\mathfrak{X}_n$ be the class of all symmetric and positive-definite matrices of order n. Further, we denote by $\lambda_1, ..., \lambda_n$ the eigenvalues of A, ordered as follows $0 < \lambda_1 \leq ... \leq \lambda_n$.

It is well-known that if $A \in \mathfrak{X}_n$, then for every $u \in X$, one has

$$\lambda_1\|u\|^2 \leq u^t A u \leq \lambda_n \|u\|^2, \tag{2.1}$$

and

$$\|u\|_\infty \leq \frac{1}{\sqrt{\lambda_1}}(u^t A u)^{1/2}, \tag{2.2}$$

where $\|u\|_\infty := \max\limits_{k\in\mathbb{Z}[1,n]} |u_k|$.

Set

$$\Phi(u) := \frac{u^t A u}{2} - \sum_{k=1}^{n} H_k(u_k), \tag{2.3}$$

and

$$\Psi(u) := \sum_{k=1}^{n} F_k(u_k), \qquad J_\lambda(u) := \Phi(u) - \lambda\Psi(u), \tag{2.4}$$

for every $u \in X$, where $H_k(t) := \displaystyle\int_0^t h_k(\xi)d\xi$ and $F_k(t) := \displaystyle\int_0^t f_k(\xi)d\xi$, for every $(k,t) \in \mathbb{Z}[1,n] \times \mathbb{R}$.

Standard arguments show that $J_\lambda \in C^1(X,\mathbb{R})$, as well as that critical points of J_λ are exactly the solutions of problem $(S^{f,h}_{A,\lambda})$; see, for instance, the paper [24].

Lemma 1. *Set*

$$L := \max_{k\in\mathbb{Z}[1,n]} L_k < \lambda_1. \tag{2.5}$$

Then the functional Φ is coercive.

Proof. Bearing in mind (2.1), since h_k is a Lipschitz continuous function (for every $k \in \mathbb{Z}[1,n]$) with constant $L_k \geq 0$ and $h_k(0) = 0$, we have

$$\begin{aligned}\Phi(u) &\geq \frac{\lambda_1}{2}\|u\|^2 - \sum_{k=1}^{n}|H_k(u_k)| \geq \frac{1}{2}\|u\|^2 - \sum_{k=1}^{n}\left(\int_0^{u_k}|h_k(t)|dt\right)\\ &\geq \frac{\lambda_1}{2}\|u\|^2 - L\sum_{k=1}^{n}\int_0^{u_k}|t|dt = \frac{1}{2}\|u\|^2 - \frac{L}{2}\sum_{k=1}^{n}u_k^2\\ &= \left(\frac{\lambda_1 - L}{2}\right)\|u\|^2.\end{aligned}$$

Hence, by (2.5), the above relation implies that the functional Φ is coercive. ☐

3. Main results

Set

$$A_\infty := \liminf_{t\to+\infty}\frac{\sum_{k=1}^{n}\max_{|\xi|\leq t}F_k(\xi)}{t^2}, \qquad \text{and} \qquad B^\infty := \limsup_{t\to+\infty}\frac{\sum_{k=1}^{n}F_k(t)}{t^2}.$$

From now on we shall assume that the functions $h_k : \mathbb{R} \to \mathbb{R}$, for every $k \in \mathbb{Z}[1,n]$, are Lipschitz continuous with constants $L_k > 0$ such that condition (2.5) holds.

Theorem 3. *Let $A \in \mathfrak{X}_n$ and assume that the following inequality holds*

$$A_\infty < \frac{\lambda_1 - L}{\mathrm{Tr}(A) + 2\sum_{i<j}a_{ij} + nL}B^\infty. \tag{h_∞^L}$$

Then, for each

$$\lambda \in \left]\frac{\mathrm{Tr}(A) + 2\sum_{i<j}a_{ij} + nL}{2B^\infty}, \frac{\lambda_1 - L}{2A_\infty}\right[,$$

problem $(S_{A,\lambda}^{f,h})$ admits an unbounded sequence of solutions.

Proof. Fix λ as in the assertion of the theorem and put Φ, Ψ, J_λ as in (2.3) and (2.4). Since the critical points of J_λ are the solutions of problem $(S_{A,\lambda}^{f,h})$, our aim is to apply Theorem 2 part (a) to function J_λ. Clearly (Λ) holds.

Therefore, our conclusion follows provided that $\gamma < +\infty$ as well as that J_λ turns out to be unbounded from below. To this end, let $\{c_m\}$ be a real sequence such that $\lim_{m\to\infty} c_m = +\infty$ and

$$\lim_{m\to\infty}\frac{\sum_{k=1}^{n}\max_{|\xi|\leq c_m}F_k(\xi)}{c_m^2} = A_\infty,$$

Write

$$r_m := \frac{\lambda_1 - L}{2}c_m^2,$$

for every $m \in \mathbb{N}$.

Since, owing to (2.2), it follows that

$$\{v \in X : v^t A v < 2r_m\} \subset \{v \in X : |v_k| \le c_m \ \forall\, k \in \mathbb{Z}[1,n]\},$$

and we obtain

$$\varphi(r_m) \le \frac{\sup\limits_{v^t A v < 2r_m} \sum\limits_{k=1}^n F_k(v_k)}{r_m} \le \frac{\sum\limits_{k=1}^n \max\limits_{|t|\le c_m} F_k(t)}{r_m} = \frac{2}{\lambda_1 - L} \frac{\sum\limits_{k=1}^n \max\limits_{|t|\le c_m} F_k(t)}{c_m^2}.$$

Hence, it follows that

$$\gamma \le \lim_{m\to\infty} \varphi(r_m) \le \frac{2}{\lambda_1 - L} A_\infty < \frac{1}{\lambda} < +\infty.$$

Now, we verify that J_λ is unbounded from below. First, assume that $B^\infty = +\infty$. Accordingly, fix such M that

$$M > \frac{\mathrm{Tr}(A) + 2\sum_{i<j} a_{ij} + nL}{2\lambda}$$

and let $\{b_m\}$ be a sequence of positive numbers, with $\lim\limits_{m\to\infty} b_m = +\infty$, such that

$$\sum_{k=1}^n F_k(b_m) > M b_m^2, \qquad (\forall\, m \in \mathbb{N}).$$

Thus, taking in X the sequence $\{s_m\}$ which, for each $m \in \mathbb{N}$, is given by $(s_m)_k := b_m$ for every $k \in \mathbb{Z}[1,n]$, owing to (2.1) and noting that

$$\begin{aligned}
\Phi(u) &\le \frac{u^t A u}{2} + \sum_{k=1}^n \left(\int_0^{u_k} |h_k(t)| dt \right) \\
&\le \frac{u^t A u}{2} + \frac{L}{2} \sum_{k=1}^n u_k^2 \\
&= \frac{u^t A u}{2} + \frac{L}{2} \|u\|^2.
\end{aligned}$$

one immediately has

$$\begin{aligned}
J_\lambda(s_m) &= \frac{s_m^t A s_m}{2} - \lambda \sum_{k=1}^n F_k(b_m) \\
&\le \frac{\mathrm{Tr}(A) + 2\sum_{i<j} a_{ij} + nL}{2} b_m^2 - \lambda \sum_{k=1}^n F_k(b_m) \\
&< \left(\frac{\mathrm{Tr}(A) + 2\sum_{i<j} a_{ij} + nL}{2} - \lambda M \right) b_m^2.
\end{aligned}$$

that is, $\lim\limits_{m\to\infty} J_\lambda(s_m) = -\infty$.

Next, assume that $B^\infty < +\infty$. Since

$$\lambda > \frac{\mathrm{Tr}(A) + 2\sum_{i<j} a_{ij} + nL}{2B^\infty},$$

we can fix $\varepsilon > 0$ such that

$$\varepsilon < B^{\infty} - \frac{\operatorname{Tr}(A) + 2\sum_{i<j} a_{ij} + nL}{2\lambda}.$$

Therefore, also calling $\{b_m\}$ a sequence of positive numbers such that $\lim_{m\to\infty} b_m = +\infty$ and

$$(B^{\infty} - \varepsilon)b_m^2 < \sum_{k=1}^{n} F_k(b_m) < (B^{\infty} + \varepsilon)b_m^2, \qquad (\forall\, m \in \mathbb{N})$$

arguing as before and by choosing $\{s_m\}$ in X as above, one has

$$J_\lambda(s_m) < \left(\frac{\operatorname{Tr}(A) + 2\sum_{i<j} a_{ij} + nL}{2} - \lambda(B^{\infty} - \varepsilon)\right) b_m^2.$$

So, $\lim_{m\to\infty} J_\lambda(s_m) = -\infty$.

Hence, in both cases J_λ is unbounded from below. The proof is thus complete. □

Remark 2. If f_k are nonnegative continuous functions, condition $(\mathrm{h}_{\infty}^{L})$ reads as follows

$$\liminf_{t\to+\infty} \frac{\sum_{k=1}^{n} F_k(t)}{t^2} < \frac{\lambda_1 - L}{\operatorname{Tr}(A) + 2\sum_{i<j} a_{ij} + nL} \limsup_{t\to+\infty} \frac{\sum_{k=1}^{n} F_k(t)}{t^2}.$$

Arguing as in the proof of Theorem 3 and applying part (b) of Theorem 2, we obtain the following result.

Theorem 4. *Let $A \in \mathfrak{X}_n$ and assume that the following inequality holds*

$$(\mathrm{h}_0^L) \qquad A_0 < \frac{\lambda_1 - L}{\operatorname{Tr}(A) + 2\sum_{i<j} a_{ij} + nL} B^0.$$

Then, for each

$$\lambda \in \left]\frac{\operatorname{Tr}(A) + 2\sum_{i<j} a_{ij} + nL}{2B^0}, \frac{\lambda_1 - L}{2A_0}\right[,$$

problem $(S_{A,\lambda}^{f})$ admits a sequence of nontrivial solutions $\{u_m\}$ such that $\lim_{m\to\infty} \|u_m\| = \lim_{m\to\infty} \|u_m\|_\infty = 0$.

4. Application

In this section we consider a discrete system, namely $(E_\lambda^{f,h})$, given as follows

$$[u(i+1,j) - 2u(i,j) + u(i-1,j)] + [u(i,j+1) - 2u(i,j) + u(i,j-1)]$$
$$+\lambda f((i,j), u(i,j)) + h(u(i,j)) = 0, \quad \forall\, (i,j) \in \mathbb{Z}[1,m] \times \mathbb{Z}[1,n],$$

with boundary conditions

$$u(i,0) = u(i,n+1) = 0, \quad \forall\, i \in \mathbb{Z}[1,m],$$
$$u(0,j) = u(m+1,j) = 0, \quad \forall\, j \in \mathbb{Z}[1,n],$$

where $f : \mathbb{Z}[1,m] \times \mathbb{Z}[1,n] \times \mathbb{R} \to \mathbb{R}$ denotes a continuous function, λ is a positive real parameter and $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with constant L_h.

As ambient space X, we consider the mn-dimensional Banach space $\mathbb{R}^{mn}$ endowed by the norm

$$\|u\| := \Big(\sum_{k=1}^{mn} u_k^2\Big)^{1/2}.$$

Further, if $\ell \in \mathbb{N}$, the symbol $\mathfrak{M}_{\ell\times\ell}(\mathbb{R})$ stands for the linear space of all the matrices of order ℓ with real entries.

Let $v : \mathbb{Z}[1,m]\times\mathbb{Z}[1,n] \to \mathbb{Z}[1,mn]$ be the bijection defined by $v(i,j) := i + m(j-1)$, for every $(i,j) \in \mathbb{Z}[1,m]\times\mathbb{Z}[1,n]$.

Let us denote $w_k := u(v^{-1}(k))$ and $g_k(w_k) := f(v^{-1}(k), w_k)$, for every $k \in \mathbb{Z}[1,mn]$. With the above notations, problem $(E_\lambda^{f,h})$ can be written as a nonlinear algebraic system of the form

$$Aw = \lambda g(w) + \widetilde{h}(w), \qquad (S_{A,\lambda}^{g,\widetilde{h}})$$

where A is given by

$$A := \begin{pmatrix} D & -I_m & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -I_m & D & -I_m & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -I_m & D & -I_m & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & D & \dots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \dots & D & -I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & -I_m & D & -I_m & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -I_m & D & -I_m \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -I_m & D \end{pmatrix} \in \mathfrak{M}_{mn\times mn}(\mathbb{R}),$$

in which D is defined by

$$D := \begin{pmatrix} 4 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & \dots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \dots & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 4 \end{pmatrix} \in \mathfrak{M}_{m\times m}(\mathbb{R}),$$

$I_m \in \mathfrak{M}_{m\times m}(\mathbb{R})$ is the identity matrix and $g(w) := (g_1(w_1), ..., g_{mn}(w_{mn}))^t$, $\widetilde{h}(w) := (h(w_1), ..., h(w_{mn}))^t$, for every $w \in X$.

In [9], Ji and Yang studied the structure of the spectrum of the above (non-perturbed) Dirichlet problem. By their result we have that $A \in \mathfrak{X}_{mn}$.

It is easy to observe that the solutions of $(E_\lambda^{f,h})$ are the critical points of the C^1-functional

$$J_\lambda(w) := \frac{w^t A w}{2} - \lambda \sum_{k=1}^{mn} \int_0^{w_k} g_k(t)dt - \sum_{k=1}^{mn} \int_0^{w_k} h(t)dt, \quad \forall\, w \in X.$$

Denote by λ_A the first eigenvalue of the matrix A. By using the above variational framework, Theorem 3 assumes the following form.

Theorem 5. *Assume that* $\lambda_A < L_h$, *in addition to*

$$(\mathrm{h}_\infty^h) \qquad \liminf_{t\to+\infty} \frac{\sum_{k=1}^{mn} \max_{|\xi|\le t} \int_0^\xi g_k(s)ds}{t^2} < \frac{\lambda_A - L_h}{(2+L_h)(m+n)} \limsup_{t\to+\infty} \frac{\sum_{k=1}^{mn} \int_0^t g_k(s)ds}{t^2}.$$

Then for each

$$\lambda \in \left]\frac{(2+L_h)(m+n)}{2B^\infty}, \frac{\lambda_A - L_h}{2A_\infty}\right[,$$

problem $(E_\lambda^{f,h})$ *admits an unbounded sequence of solutions.*

Remark 3. Substituting $\xi \to +\infty$ with $\xi \to 0^+$ in Theorem 5, the same statement as Theorem 4 is easily proved.

Remark 4. We just point out that Theorem 1 in Introduction directly follows by Theorem 5 assuming that $L_h < \lambda_A$.

In conclusion we present here a direct consequence of Theorem 5.

Example 1. Let $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with constant $L_h < \lambda_A$ and let

$$a_n := \frac{2n!(n+2)! - 1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)! + 1}{4(n+1)!},$$

for every $n \in \mathbb{N}$.
Let $\{g_n\}$ be a sequence of non-negative functions given by

$$g_n(\xi) := \sqrt{\frac{1}{16(n+1)!^2} - \left(\xi - \frac{n!(n+2)}{2}\right)^2}, \quad \forall n \in \mathbb{N}.$$

and define $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(\xi) := \begin{cases} [(n+1)!^2 - n!^2]\dfrac{g_n(\xi)}{\int_{a_n}^{b_n} g_n(t)dt} & \text{if } \xi \in \bigcup_{n=1}^{\infty}[a_n, b_n] \\ 0 & \text{otherwise.} \end{cases}$$

One has

$$\int_{n!}^{(n+1)!} f(t)dt = \int_{a_n}^{b_n} f(t)dt = (n+1)!^2 - n!^2$$

and

$$F(a_n) = n!^2 - 1, \quad F(b_n) = (n+1)!^2 - 1$$

for every $n \in \mathbb{N}$.
Hence

$$\lim_{n\to+\infty} \frac{F(b_n)}{b_n^2} = 4, \quad \lim_{n\to+\infty} \frac{F(a_n)}{a_n^2} = 0.$$

Therefore, we can prove that $\liminf_{\xi\to+\infty} \dfrac{F(\xi)}{\xi^2} = 0$ and $\limsup_{\xi\to+\infty} \dfrac{F(\xi)}{\xi^2} = 4$.

Then, for every

$$\lambda > \frac{(2+L_h)(m+n)}{8mn},$$

the following problem

$$[u(i+1,j)-2u(i,j)+u(i-1,j)]+[u(i,j+1)-2u(i,j)+u(i,j-1)]$$
$$+\lambda f(u(i,j))+h(u(i,j))=0, \quad \forall\,(i,j)\in\mathbb{Z}[1,m]\times\mathbb{Z}[1,n]$$

with boundary conditions

$$u(i,0)=u(i,n+1)=0, \quad \forall\,i\in\mathbb{Z}[1,m],$$
$$u(0,j)=u(m+1,j)=0, \quad \forall\,j\in\mathbb{Z}[1,n],$$

admits an unbounded sequence of solutions.

Remark 5. We refer to the paper of Galewski and Orpel [5] for some multiplicity results on discrete partial difference equations as well as to the monograph of Cheng [4] for their discrete geometrical interpretation. See also the papers [14, 16, 17, 18, 19] for recent contributions on discrete problems.

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