JEPE Vol 1, 2015, p. 175-188

Orthogonal ⊥

HOMOGENIZATION OF HEAT TRANSFER PROCESS IN COMPOSITE MATERIALS

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Abstract. In this paper, we adapt the periodic unfolding method to study the asymptotic behavior, as ε tends to zero, of a class of stationary heat problems on composite materials consisting of two connected constituents which are ε-periodically distributed. The nonlinear transfer condition on the interface is assumed to depend on a real parameter γ. We first survey compactness results and the relationship between the traces of two unfolding operators corresponding to the two components. Then, we study the homogenization and corrector results for the problem for the different values. The homogenization result for the case $\gamma = 1$ completes the previous works in the literature.

1. Introduction

In this work, we investigate a heat diffusion problem in composite materials with a nonlinear transmission condition on the interfacial barrier depending on a real parameter γ. More precisely, we study the asymptotic behavior, as ε tends to zero, of the following problem:

$$(P)\quad \begin{cases} -\mathrm{div}\left(A^\varepsilon \nabla u_1^\varepsilon\right) + h_1^\varepsilon\left(x, u_1^\varepsilon\right) = f \quad \text{in } \Omega_1^\varepsilon, \\ -\mathrm{div}\left(A^\varepsilon \nabla u_2^\varepsilon\right) + h_2^\varepsilon\left(x, u_2^\varepsilon\right) = f \quad \text{in } \Omega_2^\varepsilon, \\ -A^\varepsilon \nabla u_1^\varepsilon . n_1^\varepsilon = A^\varepsilon \nabla u_2^\varepsilon . n_2^\varepsilon = \varepsilon^{\gamma+1} h^\varepsilon\left(x, u_1^\varepsilon - u_2^\varepsilon\right) \quad \text{on } \Gamma^\varepsilon, \\ u_i^\varepsilon = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_i^\varepsilon, \quad i = 1, 2, \end{cases}$$

where $\gamma \leq 1$ and n_i^ε are the unit outward normal to the two connected components Ω_i^ε of an open bounded set Ω in $\mathbb{R}^n$ $(n \geq 3)$ for $i = 1, 2$. These components are separated by an ε-periodic interface Γ^ε. We assume that the heat source $f \in L^2(\Omega)$.

The boundary condition $(P)_3$ means that the heat flux through the interfacial barrier is continuous and defined via a nonlinear function h of the temperature difference between the two components of the composite. This assumption is motivated by experimental results (see for instance [4]). The nonlinear terms are given by $h_i^\varepsilon\left(x, u_i^\varepsilon\right) = h_i\left(x/\varepsilon, u_i^\varepsilon\right)$ for $i = 1, 2$; $h^\varepsilon\left(x, u_1^\varepsilon - u_2^\varepsilon\right) = \varepsilon^{-1} h(x/\varepsilon, u_1^\varepsilon - u_2^\varepsilon)$ if $\gamma = 1$ and $h^\varepsilon\left(x, u_1^\varepsilon - u_2^\varepsilon\right) = h\left(x/\varepsilon, (u_1^\varepsilon - u_2^\varepsilon)/\varepsilon\right)$ if $\gamma \neq 1$, where the functions $h,\ h_1$ and h_2 satisfy some natural growth assumptions.

2010 *Mathematics Subject Classification.* Primary: 06B10; Secondary: 06D05.

Key words and phrases. Periodic homogenization, composite material, nonlinear interfacial jump, periodic unfolding method.

Received 24/10/2014, accepted 29/05/2015.

Research supported by a grant from Ministry of Education and Training of Vietnam.

In order to study this problem, we adapt the periodic unfolding method to a domain Ω consisting of two connected components Ω_1^ε and Ω_2^ε. Let us remind that the periodic unfolding method was first introduced for fixed domains by Cioranescu, Damlamian and Griso [5, 6] and then extended to perforated domains [9, 7]. Later, Donato, Le Nguyen and Tardieu [15] adapted this method to two-component domains (including a connected component and an unconnected component). In the latter paper, two unfolding operators were introduced: $\mathcal{T}_1^\varepsilon$ (originally denoted by $\mathcal{T}_\varepsilon^*$ in [9, 7]) acting on Ω_1^ε and $\mathcal{T}_2^\varepsilon$ acting on Ω_2^ε, including the relationship between their traces on the common boundary. One important feature of these operators is that they map functions defined on oscillating domains into functions defined on fixed domains. The results in [15] have been recently improved by Donato and Le Nguyen [16].

Here, we consider the case where both two components of the domain are connected. Such a domain appears in [18], where a linear model of diffusion in fissured porous media was studied and in [28], where a similar thermal diffusion problem was suggested. In contrast to these studies where only one of two components can reach the boundary of the domain Ω, we allow both components to meet the boundary. This is probably a more natural assumption.

In the present work, we study the homogenization and corrector results for problem (P) for $\gamma \leq 1$. The presence of the nonlinear terms h_1, h_2 and the nonlinear transmission condition on the interface coupling heat equations on two components make the main difficulties. The case $\gamma < 1$ was studied in [16] with the assumption that one of two components is unconnected. For $\gamma = 1$, the nonlinear jump condition is described somewhat differently from that in [16]. Homogenization result for this case was announced without any proof in [28]. Here, we show its detailed proof by unfolding.

Since Ω_2^ε has the same geometrical structure as Ω_1^ε, the operator $\mathcal{T}_2^\varepsilon$ inherits the compactness results associated with $\mathcal{T}_1^\varepsilon$ stated in [7, 16] (see Theorem 2). As a result, the homogenization results for problem (P) are not the same as those in [16] for some cases. In particular, the homogenized problem for $\gamma \leq -1$ keeps the same. The situation is different for the case $\gamma \in]-1, 1[$ where the unfolded limit of the problem contains an additional integral term since the unfolded limit of the gradient of the solution is different from that in [16] (see (3.1) in Theorem 2). For $\gamma = 1$, the corresponding homogenized problem is a system solved by the solution (u_1, u_2), whose existence and uniqueness are proved by the Minty-Browder theorem. Consequently, we will show in detail the results only for the cases $\gamma = 1$, $\gamma \in]-1, 1[$. For the other cases, we only point out different points. It should be also noted that in this work, the case $\gamma > 1$ is not considered due to the fact that the solution of the problem is not bounded. We refer to [14] for the idea of renormalization on the heat source f in the component Ω_2^ε to obtain a nontrivial limit behavior in this case.

For related linear homogenization problems of elliptic type, we refer the reader to [2, 3, 17, 18, 20, 23, 24, 25, 26, 27]. Parabolic problems can be found in [11, 12, 13, 19, 21, 1, 29].

The remainder of this paper is as follows. In Section 2, we introduce the problem and the assumptions. Section 3 is devoted to the periodic unfolding method for domains consisting of two connected components. Finally, we show the homogenization and corrector results for the problem for different values of the parameter γ in Section 4.

2. Preliminaries

2.1. **Notation.** Let Ω be an open bounded set with a Lipschitz continuous boundary in $\mathbb{R}^n$ $(n \geq 3)$ and $Y := \prod_{i=1}^n [0, l_i[$ be a reference cell with $l_i > 0$, $i = 1, ..., n$. We assume that

Y_1 and Y_2 are two disjoint connected open subsets of Y with the common boundary Γ such that Y_1 and Y_2 reach the boundary ∂Y (see Figure 1). Set

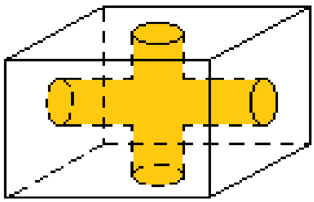

FIGURE 1. The reference cell Y

$$\partial Y_2 = \Gamma \cup \Gamma_2, \quad \partial Y_1 = \Gamma \cup \Gamma_1,$$

where Γ_i for $i = 1, 2$ are the intersections of the boundary ∂Y_2 with ∂Y and assume that Γ_i is identically reproduced on the opposite faces of Y. Then, the two connected components of Ω are defined as follows:

$$\Omega_1^\varepsilon = \Omega \setminus \bigcup_{k \in K_\varepsilon} \varepsilon \overline{Y_2^k}, \qquad \Omega_2^\varepsilon = \Omega \backslash \overline{\Omega_1^\varepsilon}, \qquad \Gamma^\varepsilon = \partial \Omega_1^\varepsilon \cap \Omega$$

where $K_\varepsilon = \left\{ k \in \mathbb{Z}^n \middle| \ \varepsilon Y_i^k \cap \Omega \neq \varnothing, \ i = 1, 2 \right\}$ and $Y_i^k = Y_i + (k_1 l_1, ..., k_n l_n)$, $i = 1, 2$.

In order to define the unfolding operators, we follow the notations introduced in [7, 15]:

- $\widehat{K}_\varepsilon = \left\{ k \in \mathbb{Z}^n \middle| \ \varepsilon Y^k \subset \Omega \right\}$, $\qquad \widehat{\Omega}_\varepsilon = int \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon \left(k_l + \overline{Y} \right)$, $\qquad \Lambda_\varepsilon = \Omega \backslash \widehat{\Omega}_\varepsilon$,
- $\widehat{\Omega}_i^\varepsilon = \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon Y_i^k$, $\qquad \Lambda_i^\varepsilon = \Omega_i^\varepsilon \backslash \widehat{\Omega}_i^\varepsilon$, $\quad i = 1, 2$.

In the sequel, we denote ε by a positive real sequence which tends to zero and c by a constant independent of ε. The following usual notations are also employed:

- $\theta_i = \dfrac{|Y_i|}{|Y|}$, $i = 1, 2$,
- $\widetilde{u}$: the zero extension to the whole Ω of a function u defined on Ω_1^ε or Ω_2^ε,
- χ_ω: the characteristic function of each open set ω of $\mathbb{R}^n$,
- $\mathcal{M}_\omega(f) := \dfrac{1}{|\omega|} \int_\omega f \, dx$, for any open set ω of $\mathbb{R}^n$ and for any $f \in L^1(\omega)$,
- $[z]_Y := (k_1 l_1, ..., k_n l_n)$ with $k_1, ..., k_n \in \mathbb{Z}$ such that $\{z\}_Y := z - [z]_Y \in Y$ for a.e. $z \in \mathbb{R}^n$,
- $M(\alpha, \beta, \mathcal{O})$: the set of the $n \times n$ matrix-valued functions A in $(L^\infty(\mathcal{O}))^{n^2}$ such that, for any $\lambda \in \mathbb{R}^n$,

$$\begin{cases} (A(x)\lambda, \lambda) \geq \alpha |\lambda|^2 & \text{a.e. in } \mathcal{O}, \\ |A(x)\lambda| \leq \beta |\lambda| & \text{a.e. in } \mathcal{O}. \end{cases}$$

2.2. **Problem.** Let the function $f \in L^2(\Omega)$ and the matrix field A belong to $M(\alpha, \beta, Y)$ with $\alpha,\ \beta \in \mathbb{R},\ 0 < \alpha \le \beta$. Assume that A is Y-periodic, then we define

$$A^\varepsilon(x) = A(x/\varepsilon) \qquad \text{in } \Omega.$$

Our goal is to describe the asymptotic behavior, as $\varepsilon \to 0$, of the following problem:

$$\begin{cases} -\mathrm{div}\left(A^\varepsilon \nabla u_1^\varepsilon\right) + h_1^\varepsilon\left(x, u_1^\varepsilon\right) = f & \text{in } \Omega_1^\varepsilon, \\ -\mathrm{div}\left(A^\varepsilon \nabla u_2^\varepsilon\right) + h_2^\varepsilon\left(x, u_2^\varepsilon\right) = f & \text{in } \Omega_2^\varepsilon, \\ -A^\varepsilon \nabla u_1^\varepsilon . n_1^\varepsilon = A^\varepsilon \nabla u_2^\varepsilon . n_2^\varepsilon = \varepsilon^{\gamma+1}\ h^\varepsilon\left(x, u_1^\varepsilon - u_2^\varepsilon\right) \quad \text{on } \Gamma^\varepsilon, \\ u_i^\varepsilon = 0 \ \text{ on } \partial\Omega \cap \partial\Omega_i^\varepsilon,\ \ i = 1, 2. \end{cases} \tag{2.1}$$

where $\gamma \le 1$ and n_i^ε are the unit outward normal to the two components Ω_i^ε for $i = 1, 2$. The nonlinear transmission condition on the interface coupling heat equations on two components is described via the function h^ε in $(2.1)_3$ defined by

$$\begin{cases} h^\varepsilon\left(x, u_1^\varepsilon - u_2^\varepsilon\right) = \varepsilon^{-1} h(x/\varepsilon, u_1^\varepsilon - u_2^\varepsilon) & \text{if } \gamma = 1, \\ h^\varepsilon\left(x, u_1^\varepsilon - u_2^\varepsilon\right) = h\big(x/\varepsilon, (u_1^\varepsilon - u_2^\varepsilon)/\varepsilon\big) & \text{if } \gamma \ne 1, \end{cases}$$

and the remaining nonlinear terms are given by $h_i^\varepsilon\left(x, u_i^\varepsilon\right) = h_i\left(x/\varepsilon, u_i^\varepsilon\right)$ for $i = 1, 2$, where the functions h, h_1 and h_2 satisfy the following assumptions:

$$\begin{cases} h \quad \text{satisfies assumptions}\,(\mathcal{H}_1)\,\text{-}\,(\mathcal{H}_3)\,, \\ h_1,\ h_2 \text{ satisfies assumptions}\,(\mathcal{H}_1)\ \text{ and }\ (\mathcal{H}_4)\,, \end{cases} \tag{2.2}$$

with $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ given below (see also in [16]).

Assumption $\mathcal{H}_1$: The function $g(y, s) : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies assumption $(\mathcal{H}_1)$ iff

(i) g is a Carathéodory function,
(ii) $g(\cdot, s)$ is $Y-$periodic for all $s \in \mathbb{R}$,
(iii) $g(y, \cdot)$ is an increasing function in $C^1(\mathbb{R})$ s.t. $g(y, 0) = 0$ for a.e. $y \in \mathbb{R}^n$.

Assumption $\mathcal{H}_2$: The function $g(y, s) : Y \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies assumption $(\mathcal{H}_2)$ iff there exists a constant $c > 0$ and an exponent q, with $1 \le q < \min\left\{2, \frac{n}{n-2}\right\}$ such that

$$\left|\frac{\partial g}{\partial s}(y, s)\right| \le c\left(1 + |s|^{q-1}\right) \quad \text{for a.e. } y \in Y \text{ and for all } s \in \mathbb{R}.$$

Assumption $\mathcal{H}_3$: The function $g(y, s) : Y \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies assumption $(\mathcal{H}_3)$ iff there exists a constant $c > 0$ such that

$$s g(y, s) \ge c\,|s|^2 \qquad \text{for a.e. } y \in Y \text{ and for all } s \in \mathbb{R}.$$

Assumption $\mathcal{H}_4$: The function $g(y, s) : Y \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies assumption $(\mathcal{H}_4)$ iff there exists a constant $c > 0$ and an exponent p, with $1 \le p < +\infty$ if $n = 2$ and $1 \le p \le \frac{n+2}{n-2}$ if $n > 2$, such that

$$\left|\frac{\partial g}{\partial s}(y, s)\right| \le c\left(1 + |s|^{p-1}\right) \quad \text{for a.e. } y \in Y \text{ and for all } s \in \mathbb{R}.$$

2.3. **Existence-uniqueness result and a priori estimates.** Let us first recall some necessary functional spaces introduced in [16].

Definition 1. For every $\gamma \in \mathbb{R}$,

$$H_\gamma^\varepsilon \doteq \{u = (u_1, u_2) |\ u_1 \in V_1^\varepsilon,\ u_2 \in V_2^\varepsilon\}$$

equipped with the norm

$$\|u\|^2_{H_\gamma^\varepsilon} = \|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} + \|\nabla u_2\|^2_{L^2(\Omega_2^\varepsilon)} + \varepsilon^\gamma \|u_1 - u_2\|^2_{L^2(\Gamma^\varepsilon)},$$

where $V_i^\varepsilon \doteq \left\{ v \in H^1(\Omega_i^\varepsilon) \middle|\ v = 0 \text{ on } \partial\Omega \cap \partial\Omega_i^\varepsilon \right\}$, for $i = 1, 2$, endowed with the norms

$$\|v\|_{V_i^\varepsilon} = \|\nabla v\|_{L^2(\Omega_i^\varepsilon)}.$$

Remark 1. (i) As seen in [7], it should be noted that the Lipschitz condition of $\partial\Omega$ implies that for every open subset Ω_{0i} of $\mathbb{R}^n$ such that $\Omega \subset \Omega_{0i}$ and $\partial\Omega \cap \partial\Omega_i^\varepsilon = \partial\Omega \cap \Omega_{0i}$,

$$V_i^\varepsilon = \left\{ v \in H^1(\Omega_i^\varepsilon) \middle| \exists\ v' \in H^1(\Omega_{0i}^\varepsilon),\ v' = 0 \text{ in } \Omega_{0i}^\varepsilon \backslash \overline{\Omega_i^\varepsilon} \text{ and } v = v'|_{\Omega_i^\varepsilon} \right\},$$

where $\Omega_{0i}^\varepsilon = \Omega_{0i} \backslash \bigcup_{k \in \mathbb{Z}^n} \varepsilon \overline{Y_i^k}$, for $i = 1, 2$.

(ii) The norm $\|\cdot\|_{V_i^\varepsilon}$ is equivalent to $\|\cdot\|_{H^1(\Omega_i^\varepsilon)}$ by constants independent of ε since the Poincaré inequality holds in the space V_i^ε with a constant c independent of ε, i.e.

$$\|v\|_{L^2(\Omega_i^\varepsilon)} \le c \|\nabla v\|_{L^2(\Omega_i^\varepsilon)} \quad \forall v \in V_i^\varepsilon, \text{ for } i = 1, 2,$$

(see [7, Theorem 2.9] for more details).

We now provide an existence and uniqueness result and a priori estimates for the solution of the problem in the spirit of Theorem 2.6 and Proposition 2.7 in [16].

Theorem 1 ([16])**.** *Let the function h satisfy assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and the functions h_1, h_2 satisfy assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_4)$. Suppose further that h or h_2 fulfills assumption $(\mathcal{H}_3)$. Then, for every fixed ε, the variational formulation of problem* (2.1) *given by*

$$\text{(2.3)} \quad \begin{cases} \text{Find } u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \in H_\gamma^\varepsilon \text{ such that} \\ \displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla v_1\ dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla v_2\ dx + \int_{\Omega_1^\varepsilon} v_1 h_1^\varepsilon(x, u_1^\varepsilon)\ dx \\ \qquad \displaystyle + \int_{\Omega_2^\varepsilon} v_2 h_2^\varepsilon(x, u_2^\varepsilon)\ dx + \varepsilon^{\gamma+1} \int_{\Gamma^\varepsilon} (v_1 - v_2) h^\varepsilon(x, u_1^\varepsilon - u_2^\varepsilon)\ d\sigma \\ \displaystyle = \int_{\Omega_1^\varepsilon} f v_1\ dx + \int_{\Omega_2^\varepsilon} f v_2\ dx\ \forall (v_1, v_2) \in H_\gamma^\varepsilon \end{cases}$$

has a unique solution $u^\varepsilon \in H_\gamma^\varepsilon$.

Remark 2. On contrary to the problem studied in [16], where Ω_2^ε is not connected, the uniqueness result for the solution of problem (2.3) does not require the assumption that h or h_2 is strictly increasing thanks to Remark 1.

Proposition 1 ([25]). *Let $\gamma \leq 1$ and assumptions* (2.2) *hold. If $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is a weak solution of problem* (2.1)*, then there exists a positive constant c, independent of ε, such that*

$$\begin{cases} \|(\nabla u_1^\varepsilon, \nabla u_2^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)\times L^2(\Omega_2^\varepsilon)} \leq c, \\ \|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \leq c\, \varepsilon^{-\gamma/2}. \end{cases} \tag{2.4}$$

3. Unfolding method for domains consisting of two connected components

In this section, we first recall the definition of the unfolding operators $\mathcal{T}_1^\varepsilon$, $\mathcal{T}_2^\varepsilon$ and $\mathcal{T}_b^\varepsilon$ introduced in [7, 15, 16]. As mentioned before, since the component Ω_2^ε has the same geometrical structure as Ω_1^ε, the operator $\mathcal{T}_2^\varepsilon$ possesses the same compactness results as those of $\mathcal{T}_1^\varepsilon$ presented in [7, 15, 16] (see Theorem 2). This leads to some simplifications in the results concerning the traces of the two unfolding operators comparing to the one in [15, 16] (see Theorem 3). The result concerning the composed operators $\mathcal{T}_1^\varepsilon \circ h_1^\varepsilon$ still keep the same, but the one related to $\mathcal{T}_2^\varepsilon \circ h_2^\varepsilon$ is stronger than that in [16], due to the connectedness of Ω_2^ε.

3.1. **Definition.**

Definition 2. For any function ϕ Lebesgue-measurable on Ω_i^ε, the periodic unfolding operators $\mathcal{T}_i^\varepsilon$, $i = 1, 2$ are defined by the formula

$$\mathcal{T}_i^\varepsilon(\phi)(x,y) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & a.e.\ (x,y) \in \widehat{\Omega}_\varepsilon \times Y_i, \\ 0 & a.e.\ (x,y) \in \Lambda_\varepsilon \times Y_i. \end{cases}$$

For any function ϕ Lebesgue-measurable on Γ^ε, the periodic boundary unfolding operator $\mathcal{T}_b^\varepsilon$ is defined by the formula

$$\mathcal{T}_b^\varepsilon(\phi)(x,y) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & a.e.\ (x,y) \in \widehat{\Omega}_\varepsilon \times \Gamma, \\ 0 & a.e.\ (x,y) \in \Lambda_\varepsilon \times \Gamma. \end{cases}$$

For the sake of simplicity, we write $\mathcal{T}_i^\varepsilon(\varphi)$ instead of $\mathcal{T}_i^\varepsilon\left(\varphi|_{\Omega_i^\varepsilon}\right)$, $i = 1, 2$, for any function φ defined on Ω.

The following proposition states the important properties of the unfolding operators, whose proof can be consulted in [7, 15].

Proposition 2 ([7, 15]). *For $p \in [1, +\infty]$, the operators $\mathcal{T}_i^\varepsilon$, $i = 1, 2$, are linear and continuous from $L^p(\Omega_i^\varepsilon)$ to $L^p(\Omega \times Y)$ and*

(i) $\mathcal{T}_i^\varepsilon(\varphi\psi) = \mathcal{T}_i^\varepsilon(\varphi)\mathcal{T}_i^\varepsilon(\psi)$, *for every functions φ, ψ Lebesgue-measurable on Ω_i^ε,*

(ii) *for every $\varphi \in L^1(\Omega_i^\varepsilon)$,*

$$\frac{1}{|Y|}\int_{\Omega\times Y_i} \mathcal{T}_i^\varepsilon(\varphi)(x,y)\ dx\ dy = \int_{\widehat{\Omega}_i^\varepsilon} \varphi(x)\ dx = \int_{\Omega_i^\varepsilon} \varphi(x)\ dx - \int_{\Lambda_i^\varepsilon} \varphi(x)\ dx,$$

(iii) *for every $\varphi \in L^p(\Omega_i^\varepsilon)$,*

$$\|\mathcal{T}_i^\varepsilon(\varphi)\|_{L^p(\Omega\times Y_i)} \leq |Y|^{1/p}\|\varphi\|_{L^p(\Omega_i^\varepsilon)},$$

Moreover, for $p \in [1, +\infty[$, one has

(iv) *for every* $\varphi \in L^p(\Omega)$,

$$\mathcal{T}_i^\varepsilon(\varphi) \longrightarrow \varphi \quad \text{strongly in } L^p(\Omega \times Y_i),$$

(v) *if* $\{\varphi_\varepsilon\}$ *is a sequence in* $L^p(\Omega)$ *such that* $\varphi_\varepsilon \longrightarrow \varphi$ *strongly in* $L^p(\Omega)$, *then,*

$$\mathcal{T}_i^\varepsilon(\varphi^\varepsilon) \longrightarrow \varphi \quad \text{strongly in } L^p(\Omega \times Y_i),$$

(vi) *if* $\varphi \in L^p(Y_i)$ *is a* Y*-periodic function and* $\varphi^\varepsilon(x) = \varphi(x/\varepsilon)$, *then*

$$\mathcal{T}_i^\varepsilon(\varphi^\varepsilon) \longrightarrow \varphi \quad \text{strongly in } L^p(\Omega \times Y_i),$$

(vii) *if* $\varphi_\varepsilon \in L^p(\Omega_i^\varepsilon)$ *satisfy* $\mathcal{T}_i^\varepsilon(\varphi_\varepsilon) \rightharpoonup \widehat{\varphi}$ *weakly in* $L^p(\Omega \times Y_i)$, *then*

$$\widetilde{\varphi}_\varepsilon \rightharpoonup \theta_i \mathcal{M}_{Y_i}(\widehat{\varphi}) \quad \text{weakly in } L^p(\Omega),$$

(for $p = +\infty$, *the above result holds in the weak* topology),*

(viii) *if* $\varphi \in W^{1,p}(\Omega_i^\varepsilon)$, *then* $\nabla_y[\mathcal{T}_i^\varepsilon(\varphi)] = \varepsilon \mathcal{T}_i^\varepsilon(\nabla\varphi)$ *and* $\mathcal{T}_i^\varepsilon(\varphi) \in L^p\big(\Omega, W^{1,p}(Y_i)\big)$.

Let us now recall the adjoints of $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$, which are needed to study the corrector results for the solution of the problem. Their properties can be found in [7, 15].

Definition 3 (the averaging operators)**.** For $p \in [1, +\infty]$, the averaging operators $\mathcal{U}_i^\varepsilon : L^p(\Omega \times Y_i) \longmapsto L^p(\Omega_i^\varepsilon)$, $i = 1, 2$, are defined as follows:

$$\mathcal{U}_i^\varepsilon(\Phi)(x) = \begin{cases} \dfrac{1}{|Y|} \int_Y \Phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\dfrac{x}{\varepsilon}\right\}_Y\right) dz & \text{a.e. } x \in \widehat{\Omega}_i^\varepsilon, \\ 0 & \text{a.e. } x \in \Lambda_i^\varepsilon. \end{cases}$$

Remark 3. $\mathcal{U}_i^\varepsilon$ are almost left-inverses of $\mathcal{T}_i^\varepsilon$, $i = 1, 2$, which means that, for any $\varphi \in L^p(\Omega_i^\varepsilon)$

$$\mathcal{U}_i^\varepsilon(\mathcal{T}_i^\varepsilon(\varphi))(x) = \begin{cases} \varphi(x) & \text{a.e. } x \in \widehat{\Omega}_i^\varepsilon, \\ 0 & \text{a.e. } x \in \Lambda_i^\varepsilon. \end{cases}$$

3.2. **Compactness results.** Due to the connectedness of Ω_2^ε, the operator $\mathcal{T}_2^\varepsilon$ possesses the same compactness results as those of $\mathcal{T}_1^\varepsilon$ presented in [16] as follows:

Theorem 2. *For any* $\gamma \in \mathbb{R}$, *if* $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ *is a bounded sequence in* H_γ^ε, *then, there exists a subsequence (still denoted* ε*),* $u_i \in H_0^1(\Omega)$ *and* $\widehat{u}_i \in L^2\big(\Omega, H^1_{per}(Y_i)\big)$, $i = 1, 2$ *such that*

$$\begin{cases} \mathcal{T}_i^\varepsilon(u_i^\varepsilon) \longrightarrow u_i & \text{strongly in} \quad L^2\big(\Omega, H^1(Y_i)\big), \\ \mathcal{T}_i^\varepsilon(\nabla u_i^\varepsilon) \rightharpoonup \nabla u_i + \nabla_y \widehat{u}_i & \text{weakly in} \quad L^2(\Omega \times Y_i), \\ Z_i^\varepsilon = \dfrac{1}{\varepsilon}\left[\mathcal{T}_i^\varepsilon(u_i^\varepsilon) - \mathcal{M}_\Gamma(\mathcal{T}_i^\varepsilon(u_i^\varepsilon))\right] \rightharpoonup y_\Gamma \nabla u_i + \widehat{u}_i & \text{weakly in} \quad L^2\big(\Omega, H^1(Y_i)\big), \end{cases} \tag{3.1}$$

where $\mathcal{M}_\Gamma(\widehat{u}_i) = 0$ *a.e. in* Ω.

Furthermore, if $\gamma < 1$, *we have*

$$u_1 = u_2. \tag{3.2}$$

Proof. The convergences (3.1) are contained in Theorem 2.13 in [7]. The proof of equality (3.2) is similar to that in [15, Theorem 2.18]. □

Let us emphasize that the convergence of $\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon)$ given in (3.1) is not the same as that in [15, 16], where the limit of $\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon)$ in $L^2(\Omega \times Y_2)$ is $\nabla_y \overline{u}_2$ with $\overline{u}_2 \in L^2\big(\Omega, H^1(Y_2)\big)$. And then, by similar arguments as in [15], the relation between the traces of the unfolded limit of u_1^ε and u_2^ε is given below.

Theorem 3. *Let $\gamma \in \mathbb{R}$ and $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a bounded sequence in H_γ^ε. Then, there exists a subsequence (still denoted ε), $u_i \in H_0^1(\Omega)$ and $\widehat{u}_i \in L^2\left(\Omega, H_{per}^1(Y_i)\right)$ such that $\mathcal{M}_\Gamma(\widehat{u}_i) = 0$ a.e. in Ω for $i = 1, 2$ and the convergences (3.1) hold.*

Moreover, if $\gamma < 1$, then $u_1 = u_2$ and

(i) *if $\gamma < -1$, we have*

$$\widehat{u}_1 = \widehat{u}_2 \quad \text{on } \Omega \times \Gamma, \tag{3.3}$$

(ii) *if $\gamma = -1$, there exists $\xi_\Gamma \in L^2(\Omega)$ such that*

$$\frac{\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)}{\varepsilon} \rightharpoonup \widehat{u}_1 - \widehat{u}_2 + \xi_\Gamma \text{ weakly in } L^2(\Omega \times \Gamma). \tag{3.4}$$

Remark 4. The results here is not the same as the one in the case of [15, 16], where there is an additional term $y_\Gamma \nabla u_1$ in the limit (3.4).

We conclude this section by showing the limit behavior of the unfolded Nemytskii operators related to the nonlinear terms h_1 and h_2, which are crucial to prove homogenization results. The assumption that both components can meet the boundary makes the proof more technical, the same as that in [16, Proposition 4.7(i)].

Proposition 3. *For $\gamma \in \mathbb{R}$, let the function h_1 and h_2 satisfy assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_4)$. If $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is a bounded sequence in H_γ^ε, then there exists a subsequence (still denoted ε) and $u_1,\ u_2 \in H_0^1(\Omega)$ such that*

(i) $\mathcal{T}_1^\varepsilon(h_1^\varepsilon(x, u_1^\varepsilon)) \longrightarrow h_1(y, u_1)$ *strongly in* $L^{t/p}(\Omega \times Y_1)$,

(ii) $\mathcal{T}_2^\varepsilon(h_2^\varepsilon(x, u_2^\varepsilon)) \longrightarrow h_2(y, u_2)$ *strongly in* $L^{t/p}(\Omega \times Y_2)$,

where

$$\begin{cases} t = max\{2, p\} & \text{if } \ n = 2, \\ t \in \left[max\left\{2, \frac{n+2}{n-2}\right\}, 2^*\right[& \text{if } \ n > 2. \end{cases}$$

Moreover, if $\gamma < 1$, then $u_1 = u_2$.

Remark 5. Note that the convergence result concerning h_2 is different from that in [16], where the convergence (ii) takes place in $L^{2/p_2}(\Omega \times Y_2)$ only for $\gamma < 1$ with $1 \leq p_2 \leq \min\{2, \frac{n+2}{n-2}\}$.

4. Homogenization and corrector results for $\gamma \in]-1, 1]$

We present here only the homogenization results for $\gamma = 1$ and $\gamma \in]-1, 1[$ separately, which are different from those obtained in [16].

For the case $\gamma \leq -1$, the results keep the same as those in [16], although the convergence results in Theorem 3 are different from those in the case where Ω_2^ε is unconnected. As in [16], the presence of the nonlinear function h of the solution jump is a challenging point in the homogenization process and Theorem 3 is essential to overcome this difficulty. Note that for $\gamma = -1$, in the unfolded limit problem the function $\widehat{u}_2$ belongs to $L^2(\Omega; H_{per}^1(Y_2))$ instead of $L^2(\Omega; H^1(Y_2))$. Let us also remind that this case is a difficult one due to identifying the limit of the unfolded interface term since we have only the weak convergence of $\mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)/\varepsilon$ in $L^2(\Omega \times \Gamma)$. Then, as studied in [16] we choose a suitable sequence of test functions to overcome this and then the homogenized matrix is described in a more complicated way via a nonlinear function related to the correctors.

4.1. **Homogenization result for the case $\gamma = 1$.** For this case, thanks to the strong convergence result of $\mathcal{T}_2^\varepsilon(u_2^\varepsilon)$, we can pass to the limit in the unfolded term concerning the nonlinear function h_2, while in [16], we cannot do that and therefore assume that $h_2 = 0$. As in [18, 28], the homogenized problem is a system in the solution (u_1, u_2). In order to prove the existence and the uniqueness of this solution, we apply here the Minty-Browder theorem together with the lemma about the ellipticity of some homogenized matrices as follows:

Lemma 1. *Let A_γ^0 and B_γ^0 be the matrix field given by:*

$$A_\gamma^0 e_j = \theta_1 \mathcal{M}_{Y_1}\Big(Ae_j - A\nabla\chi_{1j}\Big), \quad B_\gamma^0 e_j = \theta_2 \mathcal{M}_{Y_2}\Big(Ae_j - A\nabla\chi_{2j}\Big), \tag{4.1}$$

where the correctors χ_{1j} and χ_{2j}, $j = 1, ..., n$, are the unique solutions of the cell problems, for $i = 1, 2$,

$$\begin{cases} -div\Big(A(y)\nabla\big(\chi_{ij} - y_j\big)\Big) = 0 & \text{in } Y_i, \\ A(y)\nabla\big(\chi_{ij} - y_j\big).n_i = 0 & \text{on } \Gamma, \\ \chi_{ij} \ \ Y - periodic, \ \mathcal{M}_\Gamma\big(\chi_{ij}\big) = 0. \end{cases} \tag{4.2}$$

Then there exist two positive numbers α_1, α_2 such that for any $\lambda \in \mathbb{R}^n$,

$$\begin{aligned} (A_\gamma^0\lambda, \lambda) &\geq \alpha_1 |\lambda|^2, \\ (B_\gamma^0\lambda, \lambda) &\geq \alpha_2 |\lambda|^2. \end{aligned}$$

Proof. The proof is similar to that of Proposition 6.12 in [8] with the remark that Γ_1, Γ_2 are identically reproduced on opposite faces of Y. □

Theorem 4. *For $\gamma = 1$, let assumptions (2.2) hold. If $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is the solution of problem (2.1), then there exist $u_i \in H_0^1(\Omega)$, $\widehat{u}_i \in L^2\left(\Omega, H^1_{per}(Y_i)\right)$ with $\mathcal{M}_\Gamma(\widehat{u}_i) = 0$ a.e. in Ω, for $i = 1, 2$, such that the convergences (3.1) hold and $(u_1, u_2, \widehat{u}_1, \widehat{u}_2)$ uniquely satisfies:*

$$\begin{cases} \dfrac{1}{|Y|}\displaystyle\sum_{i=1,2}\int_{\Omega\times Y_i} A(y)\left(\nabla u_i + \nabla_y \widehat{u}_i\right)\left(\nabla\varphi_i + \nabla_y\Phi_i\right)\,dx\,dy \\ +\dfrac{1}{|Y|}\displaystyle\sum_{i=1,2}\int_{\Omega\times Y_i} h_i(y, u_i)\,\varphi_i\,dx\,dy + \dfrac{1}{|Y|}\int_{\Omega\times\Gamma} h(y, u_1 - u_2)(\varphi_1 - \varphi_2)\,dx\,d\sigma_y \\ = \displaystyle\int_\Omega \theta_1 f\varphi_1\,dx + \int_\Omega \theta_2 f\varphi_2\,dx \quad \forall\varphi_i \in H_0^1(\Omega),\ \forall\Phi_i \in L^2\left(\Omega, H^1_{per}(Y_i)\right),\ i = 1, 2. \end{cases} \tag{4.3}$$

The pair (u_1, u_2) is the unique solution of the following homogenized system:

$$\begin{cases} -div\left(A_\gamma^0\nabla u_1\right) + \theta_1\mathcal{M}_{Y_1}\left(h_1(\cdot, u_1)\right) + \frac{|\Gamma|}{|Y|}\mathcal{M}_\Gamma(h(\cdot, u_1 - u_2)) = \theta_1 f & \text{in } \Omega, \\ -div\left(B_\gamma^0\nabla u_2\right) + \theta_2\mathcal{M}_{Y_2}\left(h_2(\cdot, u_2)\right) - \frac{|\Gamma|}{|Y|}\mathcal{M}_\Gamma(h(\cdot, u_1 - u_2)) = \theta_2 f & \text{in } \Omega, \\ u_1 = u_2 = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

with the matrices A_γ^0 and B_γ^0 defined by (4.1).

Proof. Let us take $v_i(x) = \varphi_i(x) + \varepsilon\omega_i(x)\psi_i^\varepsilon(x)$, $i = 1, 2$, as test functions in (2.3), where φ_i, $\omega_i \in \mathcal{D}(\Omega)$, $\psi_i \in H^1_{per}(Y_i)$, and $\psi_i^\varepsilon(x) = \psi_i(x/\varepsilon)$.

From the definition of the unfolding operators, one has, for $i = 1, 2$

$$\left\{ \begin{array}{lcl} \mathcal{T}_i^\varepsilon (v_i) & \longrightarrow & \varphi_i \quad \text{strongly in } L^2(\Omega \times Y_i), \\ \mathcal{T}_i^\varepsilon (\nabla v_i) & = & \mathcal{T}_i^\varepsilon (\nabla \varphi_i) + \varepsilon \, \psi_i \mathcal{T}_i^\varepsilon (\nabla \omega_i) + \nabla_y \psi_i \mathcal{T}_i^\varepsilon (\omega_i) \\ & \longrightarrow & \nabla \varphi_i + \nabla_y \Phi_i \quad \text{strongly in } L^2(\Omega \times Y_i), \end{array} \right. \tag{4.5}$$

where $\Phi_i(x, y) = \omega_i(x) \psi_i(y)$.

Then, by unfolding

$$\begin{aligned} \sum_{i=1,2} \int_{\Omega_i^\varepsilon} A^\varepsilon \nabla u_i^\varepsilon \nabla v_i \, dx &= \sum_{i=1,2} \frac{1}{|Y|} \int_{\Omega \times Y_i} A(y) \mathcal{T}_i^\varepsilon (\nabla u_i^\varepsilon) \mathcal{T}_i^\varepsilon (\nabla v_i) \, dx \, dy \\ &\longrightarrow \sum_{i=1,2} \frac{1}{|Y|} \int_{\Omega \times Y_i} A(y) (\nabla u_i + \nabla_y \widehat{u}_i)(\nabla \varphi_i + \nabla_y \Phi_i) \, dx dy, \end{aligned} \tag{4.6}$$

and

$$\sum_{i=1,2} \int_{\Omega_i^\varepsilon} f v_i \, dx = \sum_{i=1,2} \frac{1}{|Y|} \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon (f) \mathcal{T}_i^\varepsilon (v_i) \, dx \, d\sigma_y \longrightarrow \sum_{i=1,2} \int_\Omega \theta_i f \varphi_i \, dx. \tag{4.7}$$

On the other hand, using the convergence results given in Proposition 3 provides

$$\begin{aligned} \sum_{i=1,2} \int_{\Omega_i^\varepsilon} h_i^\varepsilon (x, u_i^\varepsilon) v_i \, dx &= \sum_{i=1,2} \frac{1}{|Y|} \int_{\Omega \times Y_i} h_i (y, \mathcal{T}_i^\varepsilon(u_i^\varepsilon)) \mathcal{T}_i^\varepsilon (v_i) \, dx \, dy \\ &\longrightarrow \sum_{i=1,2} \frac{1}{|Y|} \int_{\Omega \times Y_i} h_i(y, u_i) \, \varphi_i \, dx \, dy. \end{aligned} \tag{4.8}$$

Taking into account the trace properties and the convergences $(3.1)_1$ in Theorem 2, we have

$$\mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) \longrightarrow u_1 - u_2 \quad \text{strongly in } L^2(\Omega \times \Gamma),$$

which implies, by the classical result in [22],

$$h(y, \mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)) \longrightarrow h(y, u_1 - u_2) \quad \text{strongly in } L^{2/q}(\Omega \times \Gamma).$$

Hence,

$$\begin{aligned} \varepsilon \int_{\Gamma^\varepsilon} h\left(\frac{x}{\varepsilon}, u_1^\varepsilon - u_2^\varepsilon\right)(v_1 - v_2) \, d\sigma_x &= \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y, \mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)) \mathcal{T}_b^\varepsilon (v_1 - v_2) \, dx \, d\sigma_y \\ &\longrightarrow \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y, u_1 - u_2)(\varphi_1 - \varphi_2) \, dx \, d\sigma_y. \end{aligned} \tag{4.9}$$

By virtue of the convergences (4.6)-(4.9), we pass to the limit as $\varepsilon \to 0$ in the variational formulation (2.3) for (v_1, v_2) chosen as above. Then using the density arguments, we obtain the limit problem (4.3).

We are now in position to identify the homogenized problem solved by u_1.

Firstly, we choose $\varphi_1 = \varphi_2 \equiv 0$ in (4.3) and get

$$\frac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} A(y)(\nabla u_i + \nabla_y \widehat{u}_i) \nabla_y \Phi_i \, dx \, dy = 0, \quad \forall \Phi_i \in L^2\left(\Omega; H^1_{per}(Y_i)\right),$$

which provides, for $i = 1, 2$,

$$\begin{cases} \operatorname{div}_y \left[A(y) \left(\nabla_y \widehat{u}_i(x,y) + \nabla u_i(x) \right) \right] = 0 & \text{a.e. in } \Omega \times Y_i, \\ A(y) \left[\nabla_y \widehat{u}_i(x,y) + \nabla u_i(x) \right] n_i = 0 & \text{a.e. on } \Omega \times \Gamma, \\ \widehat{u}_i(x,\cdot) \ Y\text{-periodic}, \ \mathcal{M}_\Gamma(\widehat{u}_i) = 0 & \text{a.e. in } \Omega. \end{cases}$$

Then, the forms of $\widehat{u}_1$ and $\widehat{u}_2$ are given as follows:

$$\widehat{u}_1(x,y) = -\sum_{j=1}^{n} \frac{\partial u_1}{\partial x_j}(x) \chi_{1j}(y), \quad \widehat{u}_2(x,y) = -\sum_{j=1}^{n} \frac{\partial u_2}{\partial x_j}(x) \chi_{2j}(y), \tag{4.10}$$

where χ_{1j} and χ_{2j}, $j = 1, ..., n$, are the unique solutions of the cell problems (4.2).

Next, taking $\Phi_1 = \Phi_2 \equiv 0$ in (4.3) we have

$$\frac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} A(y) \left(\nabla u_i + \nabla_y \widehat{u}_i \right) \nabla \varphi_i \, dx \, dy + \frac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} h_i(y, u_i) \, \varphi_i \, dx \, dy$$
$$+ \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y, u_1 - u_2)(\varphi_1 - \varphi_2) \, dx \, d\sigma_y = \sum_{i=1,2} \int_\Omega \theta_i f \varphi_i dx \quad \forall \varphi_1, \varphi_2 \in H_0^1(\Omega).$$

By substituting (4.10) into (4.11), we deduce

$$\begin{aligned} \int_\Omega A_\gamma^0 \nabla u_1 \nabla \varphi_1 \, dx + \int_\Omega B_\gamma^0 \nabla u_2 \nabla \varphi_2 \, dx + \frac{|\Gamma|}{|Y|} \int_\Omega (\varphi_1 - \varphi_2) \mathcal{M}_\Gamma h(\cdot, u_1 - u_2) \, dx \\ + \theta_1 \int_\Omega \varphi_1 \mathcal{M}_{Y_1} h_1(\cdot, u_1) \, dx + \theta_2 \int_\Omega \varphi_2 \mathcal{M}_{Y_2} h_2(\cdot, u_2) \, dx \\ = \int_\Omega \theta_1 f \varphi_1 \, dx + \int_\Omega \theta_2 f \varphi_2 \, dx, \quad \forall \varphi_1, \varphi_2 \in H_0^1(\Omega). \end{aligned} \tag{4.11}$$

Then, (4.4) follows from (4.11) by choosing $\varphi_1 \equiv 0$ and $\varphi_2 \equiv 0$ successively.

Now, let us pass to proving the existence and the uniqueness of the solution (u_1, u_2) of problem (4.11), which implies that the convergences (3.1) hold for the whole sequence ε. We apply the Minty-Browder theorem for the operator κ defined by

$$\kappa : u = (u_1, u_2) \in \mathcal{Q} = H_0^1(\Omega) \times H_0^1(\Omega) \longmapsto \mathcal{Q}'$$

where

$$\begin{aligned} \langle \kappa(u), \varphi \rangle_{\mathcal{Q}', \mathcal{Q}} = \int_\Omega A_\gamma^0 \nabla u_1 \nabla \varphi_1 \, dx + \int_\Omega B_\gamma^0 \nabla u_2 \nabla \varphi_2 \, dx + \frac{|\Gamma|}{|Y|} \int_\Omega (\varphi_1 - \varphi_2) \mathcal{M}_\Gamma h(\cdot, u_1 - u_2) \, dx \\ + \theta_1 \int_\Omega \varphi_1 \mathcal{M}_{Y_1} h_1(\cdot, u_1) \, dx + \theta_2 \int_\Omega \varphi_2 \mathcal{M}_{Y_2} h_2(\cdot, u_2) \, dx \\ - \int_\Omega \theta_1 f \varphi_1 \, dx - \int_\Omega \theta_2 f \varphi_2 \, dx \quad \forall \varphi = (\varphi_1, \varphi_2) \in \mathcal{Q}, \end{aligned}$$

with the space $\mathcal{Q}$ equipped by the following norm

$$\|\varphi\|_{\mathcal{Q}} = \left(\|\nabla \varphi_1\|_{L^2(\Omega)}^2 + \|\nabla \varphi_2\|_{L^2(\Omega)}^2 \right)^{1/2} \quad \forall \varphi = (\varphi_1, \varphi_2) \in \mathcal{Q}.$$

It is necessary to remark that if $h(y, \cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and monotone for a.e. $y \in \mathbb{R}^n$, so is the function $\delta : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\delta(s) = \mathcal{M}_\Gamma(h(\cdot, s))$. Then, the continuity and monotonicity of $\mathcal{M}_{Y_1} h_1(\cdot, s)$ and $\mathcal{M}_{Y_2} h_2(\cdot, s)$ in s follow from assumption $(\mathcal{H}_1)$. Moreover, Lemma 1 gives the ellipticity of the matrices A^0_γ and B^0_γ. Thus, it is straightforward to prove that the operator κ is bounded, continuous, monotone and coercive so that problem (4.11) has a solution $(u_1, u_2) \in H^1_0(\Omega) \times H^1_0(\Omega)$. The uniqueness of (u_1, u_2) comes from the fact that if u and v are two solutions of (4.11), we have $\langle \kappa(u) - \kappa(v), \varphi \rangle_{\mathcal{Q}', \mathcal{Q}} = 0$ for any $\varphi \in \mathcal{Q}$. Then, taking $\varphi \equiv u - v$ and using the ellipticity of A^0_γ and B^0_γ together with the Poincaré inequality in $H^1_0(\Omega)$, one derives that $u = v$.

Hence, (4.4) admits a unique solution (u_1, u_2) and (4.10) provides the uniqueness of $\widehat{u}_i$ for $i = 1, 2$, which implies the convergences of the whole sequence in (3.1). □

4.2. Homogenization result for the case $\gamma \in \,]-1, 1[$.

Theorem 5. *Let $\gamma \in \,]-1, 1[$ and assumptions (2.2) hold. If $u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2)$ is the solution of problem (2.1), then there exist $u_1 \in H^1_0(\Omega)$ and $\widehat{u}_i \in L^2\left(\Omega, H^1_{per}(Y_i)\right)$, $i = 1, 2$, such that*

$$(4.12)\quad \begin{cases} \mathcal{T}^\varepsilon_i(u^\varepsilon_i) \longrightarrow u_1 & \text{strongly in } L^2\left(\Omega, H^1(Y_i)\right), \\ \mathcal{T}^\varepsilon_i(\nabla u^\varepsilon_i) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_i & \text{weakly in } L^2(\Omega \times Y_i), \end{cases}$$

where the triplet $(u_1, \widehat{u}_1, \widehat{u}_2)$ is the unique solution of the problem

$$(4.13)\quad \begin{cases} \text{Find } u_1 \in H^1_0(\Omega), \widehat{u}_i \in L^2\left(\Omega, H^1_{per}(Y_i)\right) \\ \qquad \text{with } \mathcal{M}_\Gamma(\widehat{u}_i) = 0 \text{ a.e. in } \Omega \text{ such that,} \\ \dfrac{1}{|Y|}\displaystyle\sum_{i=1,2}\int_{\Omega \times Y_i} A(y)(\nabla u_1 + \nabla_y \widehat{u}_i)(\nabla\varphi + \nabla_y \Phi_i)\, dx\, dy \\ + \displaystyle\sum_{i=1,2}\frac{1}{|Y|}\int_{\Omega \times Y_i} h_i(y, u_1)\varphi\, dx\, dy = \int_\Omega f(x)\varphi(x)\, dx \\ \forall \varphi \in H^1_0(\Omega),\ \forall \Phi_i \in L^2\left(\Omega, H^1_{per}(Y_i)\right),\ i = 1, 2 \end{cases}$$

and u_1 is the unique solution of the following homogenized problem:

$$(4.14)\quad \begin{cases} -div\left((A^0_\gamma + B^0_\gamma)\nabla u_1\right) + \theta_1 \mathcal{M}_{Y_1}(h_1(\cdot, u_1)) + \theta_2 \mathcal{M}_{Y_2}(h_2(\cdot, u_1)) = f & \text{in } \Omega, \\ u_1 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

with the matrices A^0_γ and B^0_γ defined by (4.1).

Proof. The proof is similar to that in [16]. The only difference is that here, the limit of $\mathcal{T}^\varepsilon_2(\nabla u^\varepsilon_2)$ in $L^2(\Omega \times Y_2)$ given by Theorem 2 is not zero anymore, so that an additional integral term over $\Omega \times Y_2$ appears in the unfolded limit (4.13). Then, the coefficient matrix of the homogenized problem is the sum of the ones in the two independent Neuman problems (4.2), which is different from the one given in [16, Corollary 5.10]. □

Remark 6. For $\gamma \in \,]-1, 1]$, by unfolding, we have the convergences of the temperature field and the flux as follows:

$$(4.15)\quad \begin{cases} \widetilde{u^\varepsilon_i} \rightharpoonup \theta_i u_i & \text{weakly in } L^2(\Omega),\ i = 1, 2, \\ A^\varepsilon \widetilde{\nabla u^\varepsilon_1} \rightharpoonup A^0_\gamma \nabla u_1 & \text{weakly in } \left(L^2(\Omega)\right)^n, \\ A^\varepsilon \widetilde{\nabla u^\varepsilon_2} \rightharpoonup B^0_\gamma \nabla u_2 & \text{weakly in } \left(L^2(\Omega)\right)^n. \end{cases}$$

Moreover, if $\gamma \in \,]-1, 1[$, then $u_1 = u_2$.

4.3. Convergence of the energy and corrector results.

Theorem 6. *Under the assumptions of Theorem 4 for* $\gamma = 1$ *and the assumptions of Theorem 5 for* $\gamma \in]-1,1[$*, if* $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ *is the solution of problem* (2.1)*, then we have the convergence of the energy*

$$\begin{aligned}\lim_{\varepsilon\to 0}\Big(\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla u_2^\varepsilon \, dx\Big) \\ = \frac{1}{|Y|}\sum_{i=1,2}\int_{\Omega\times Y_i} A(y)(\nabla u_i + \nabla_y \widehat{u}_i)(\nabla u_i + \nabla_y \widehat{u}_i)\, dx\, dy,\end{aligned} \tag{4.16}$$

and

$$\begin{cases}\lim_{\varepsilon\to 0}\Big(\int_{\Lambda_1^\varepsilon} |\nabla u_1^\varepsilon|^2 dx + \int_{\Lambda_2^\varepsilon} |\nabla u_2^\varepsilon|^2 dx\Big) = 0, \\ \mathcal{T}_i^\varepsilon(\nabla u_i^\varepsilon) \longrightarrow \nabla u_i + \nabla_y \widehat{u}_i \quad \text{strongly in } L^2(\Omega\times Y_i), \quad \text{for } i = 1,2.\end{cases} \tag{4.17}$$

Moreover, the following corrector results hold

$$\begin{cases}\Big\|\nabla u_1^\varepsilon - \nabla u_1 + \sum_{i=1}^n \mathcal{U}_1^\varepsilon\Big(\frac{\partial u_1}{\partial x_i}\Big)\nabla_y \chi_{1i}(\{\tfrac{\cdot}{\varepsilon}\}_Y)\Big\|_{L^2(\Omega_1^\varepsilon)} \longrightarrow 0, \\ \Big\|\nabla u_2^\varepsilon - \nabla u_2 + \sum_{i=1}^n \mathcal{U}_2^\varepsilon\Big(\frac{\partial u_2}{\partial x_i}\Big)\nabla_y \chi_{2i}(\{\tfrac{\cdot}{\varepsilon}\}_Y)\Big\|_{L^2(\Omega_2^\varepsilon)} \longrightarrow 0.\end{cases} \tag{4.18}$$

Furthermore, if $\gamma \in]-1,1[$*, then* $u_1 = u_2$.

Proof. The above results are proved due to similar arguments as those used in [6, 7, 15]. For the case $\gamma = 1$, the strong convergence of the unfolded solution sequence $\mathcal{T}_2^\varepsilon(u_2^\varepsilon)$ makes the proof simplier than that in [15]. □

Remark 7. For the case $\gamma < -1$, the convergence of the energy and the corrector result for the solution is the same as those in the linear case [15]. The case $\gamma = -1$ remains an open problem due to the fact that the weak convergence of $\big(\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)\big)/\varepsilon$ does not allow to pass straightforward to the limit in the nonlinear term h.

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