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Orthogonal ⊥

EIGENVALUES OF THE LAPLACIAN IN A DOMAIN WITH A THIN TUBULAR HOLE

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Abstract. We deal with the eigenvalues of the Laplacian in a domain with a thin tubular hole. We impose the Robin or the Neumann B.C on the boundary of the hole and investigate the detailed asymptotic behavior of the eigenvalues when the hole becomes thinner and shrinks to a lower dimensional manifold.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ a bounded domain with a smooth boundary. We consider the eigenvalue problem of the Laplacian,

$$\Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega, \qquad \Phi = 0 \quad \text{on} \quad \partial\Omega. \tag{1.1}$$

It is known that (1.1) is a spectral problem of a self-adjoint operator with a compact resolvent and so the set of the eigenvalues is an unbounded discrete sequence of positive values $\{\lambda_k\}_{k=1}^{\infty}$ (cf. Courant-Hilbert [7], Edmunds-Evans [10], Davies [9]). For later use, we denote the corresponding orthonormal eigenfunctions by $\{\Phi_k\}_{k=1}^{\infty} \subset L^2(\Omega)$ i.e.

$$(\Phi_k, \Phi_\ell)_{L^2(\Omega)} = \delta(k, \ell) \quad (k, \ell \in \mathbb{N}). \tag{1.2}$$

Here $\delta(k, \ell)$ is the Kronecker delta symbol. It is known that each Φ_k is smooth in $\overline{\Omega}$ from the regularity theory for the elliptic equation with smooth coefficients (cf. Gilbarg-Trudinger [13], Evans [11]).

The basic subject in the present paper is to consider the perturbation of each eigenvalue under domain variation of a certain singular type. More precisely, we deal with the domain with a thin tubular hole and look into the asymptotic behavior of the eigenvalue when the tubular hole becomes thinner and shrinks to a low dimensional submanifold. For the formulation of the problem, we define the singularly peturbed domain $\Omega(\epsilon)$ as follows.

Let M be a $m-$dimensional smooth submanifold of $\mathbb{R}^n$. Assume that M is compact, orientable and satisfies $M \subset \Omega$. We assume that $m \leqq n-2$. Define the tubular neighbhorhood of M by

$$B(M, \epsilon) = \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, M) < \epsilon\}$$

($\epsilon > 0$: small) and define the domain $\Omega(\epsilon)$ by

$$\Omega(\epsilon) = \Omega \setminus \overline{B(M, \epsilon)}.$$

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Put $\Gamma = \partial\Omega$, $\Gamma(M,r) = \partial B(M,r)$ $(r > 0)$ for simplicity of notation. When $\epsilon > 0$ is small, the measure of $\Omega(\epsilon)$ is almost same as that of Ω while $\Omega(\epsilon)$ is topologically different from Ω since new boundary $\Gamma(M,\epsilon)$ emerges. So the detailed analysis of the relation of the eigenvalue problems of the Laplacian in $\Omega(\epsilon)$ and Ω seems to be complicated.

For the case that the Dirichlet B.C. is imposed on $\Gamma(M,\epsilon)$, a nice perturbation formula of the eigenvalue $\lambda_k^D(\epsilon)$ has been established through the works due to Besson [2] ('85), Chavel-Feldman [5] ('88), Courtois [8] ('95), i.e.

$$\lambda_k^D(\epsilon) - \lambda_k = \begin{cases} \left\{(n-m-2)|S^{n-m-1}| \int_M \Phi_k(\xi)^2 ds(\xi)\right\} \epsilon^{n-m-2} + \text{H.O.T.} & \text{for } n-m \geqq 3, \\ \left\{2\pi \int_M \Phi_k(\xi)^2 ds(\xi)\right\} / \log(1/\epsilon) + \text{H.O.T.} & \text{for } n-m=2. \end{cases}$$

where S^{n-m-1} is the unit sphere in $\mathbb{R}^{n-m}$ and "H.O.T." implies "a higher order term".

In this paper we deal with the case of the Robin B.C. or the Neumann B.C. on $\Gamma(M,\epsilon)$. So we consider the following eigenvalue problems,

$$\text{(1.3)} \qquad \begin{cases} \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \\ \Phi = 0 \quad \text{on} \quad \Gamma, \quad \dfrac{\partial\Phi}{\partial\nu} + \sigma\epsilon^\tau\Phi = 0 \quad \text{on} \quad \Gamma(M,\epsilon). \end{cases}$$

$$\text{(1.4)} \qquad \begin{cases} \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \\ \Phi = 0 \quad \text{on} \quad \Gamma, \quad \dfrac{\partial\Phi}{\partial\nu} = 0 \quad \text{on} \quad \Gamma(M,\epsilon). \end{cases}$$

Here ν is the unit outward normal vector on $\partial\Omega(\epsilon)$. Note that ν is pointing into $B(M,\epsilon)$ at a point on $\Gamma(M,\epsilon)$. In this problem, there are important two parameters $\tau \in (-\infty,\infty)$ and $\sigma > 0$, which divide the situation into several different cases.

We can prove that each k−th eigenvalue of $\Omega(\epsilon)$ (with the Robin B.C. or the Neumann B.C. on $\Gamma(M,\epsilon)$) approaches the original k−th eigenvalue of Ω (cf. Proposition 1) for $\epsilon \to 0$. We will look into a detailed behavior of this converegence as a perturbation formula and understand the dependencies of several parameters in the bondary condition and the geometric properties of the problem. The behaviors of eigenvalues depend on the several different cases of the boundary condition on $\Gamma(M,\epsilon)$.

Definition 1. We denote the eigenvalues of (1.3) by $\{\lambda_k^R(\epsilon)\}_{k=1}^\infty$ and the corresponding complete orthonormal system by $\{\Phi_{k,\epsilon}^R\}_{k=1}^\infty \subset L^2(\Omega(\epsilon))$, respectively.

$$(\Phi_{k,\epsilon}^R, \Phi_{\ell,\epsilon}^R)_{L^2(\Omega(\epsilon))} = \delta(k,\ell) \quad (k,\ell \geqq 1).$$

Definition 2. We denote the eigenvalues of (1.4) by $\{\lambda_k^N(\epsilon)\}_{k=1}^\infty$ and the corresponding complete orthonormal system $\{\Phi_{k,\epsilon}^N\}_{k=1}^\infty \subset L^2(\Omega(\epsilon))$, respectively.

$$(\Phi_{k,\epsilon}^N, \Phi_{\ell,\epsilon}^N)_{L^2(\Omega(\epsilon))} = \delta(k,\ell) \quad (k,\ell \geqq 1).$$

It should be noted that $\lambda_k^N(\epsilon) \leqq \lambda_k^R(\epsilon) \leqq \lambda_k^D(\epsilon)$ from the comparison principle of eigenvalues under different boundary conditions (cf. Courant-Hilbert [7](Chapter IV)). We can justify that the limit values of these eigenvalues for $\epsilon \to 0$, agree to λ_k.

Proposition 1. *For each $k \in \mathbb{N}$, it holds that*

$$\lim_{\epsilon\to 0} \lambda_k^R(\epsilon) = \lambda_k, \quad \lim_{\epsilon\to 0} \lambda_k^N(\epsilon) = \lambda_k. \tag{1.5}$$

Recall that λ_k is the $k-th$ eigenvalue of (1.1).

Proposition 2. *For each $k \in \mathbb{N}$, there exist $\epsilon_0 > 0$ and $c(k) > 0$ such that*

$$|\Phi_{k,\epsilon}^R(x)| \leqq c(k), \quad |\Phi_{k,\epsilon}^N(x)| \leqq c(k) \quad (x \in \Omega(\epsilon),\, 0 < \epsilon \leqq \epsilon_0). \tag{1.6}$$

For any sequence of positive values $\{\epsilon_p\}_{p=1}^\infty$ with $\lim_{p\to\infty}\epsilon_p = 0$, there exists a subsequence $\{\zeta_p\}_{p=1}^\infty$ and orthonormal systems of eigenfunctions $\{\Phi'_k\}_{k=1}^\infty$ and $\{\Phi''_k\}_{k=1}^\infty$ of (1.1) *corresponding to $\{\lambda_k\}_{k=1}^\infty$, respectively such that*

$$(\Phi'_k, \Phi'_\ell)_{L^2(\Omega)} = \delta(k,\ell), \quad (\Phi''_k, \Phi''_\ell)_{L^2(\Omega)} = \delta(k,\ell) \quad (k,\ell \in \mathbb{N}), \tag{1.7}$$

$$\lim_{p\to\infty} \|\Phi_{k,\zeta_p}^R - \Phi'_k\|_{L^2(\Omega(\zeta_p))} = 0, \quad \lim_{p\to\infty} \|\Phi_{k,\zeta_p}^N - \Phi''_k\|_{L^2(\Omega(\zeta_p))} = 0. \tag{1.8}$$

Remark 1. From the regularity theory of elliptic equations with (1.6), Φ_{k,ζ_p}^R and Φ_{k,ζ_p}^N converge to Φ'_k and Φ''_k, respectively, in $C^s(\overline{\Omega} \setminus B(M,r))$ as $p \to \infty$ for any $r > 0$ and $s \in \mathbb{N}$.

Notation. ∇ is the gradient in $\mathbb{R}^n$. ∇_M is the tangential gradient and ∇_N is the normal gradient at a point of the manifold M.

Notation. We denote the mean curvature vector field on M by H. H is a kind of a normal vector field on M. So the vector $H(\xi)$ is orthogonal to the tangent space $T_\xi M$ at any point $\xi \in M$. For a function ϕ defined in a tubular neighborhood of M, H acts on ϕ as a differential in H direction. Actually

$$H[\phi](\xi) = \lim_{t\to 0}(\phi(\xi + tH(\xi)) - \phi(\xi))/t \quad \text{at each} \quad \xi \in M.$$

We present the main results of this paper.

Theorem 1. *Assume that $n - m = q \geqq 3$ and λ_k is simple in* (1.1).
(0) We have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^N(\epsilon) - \lambda_k}{\epsilon^q} = \frac{|S^{q-1}|}{q}\int_M \left\{-\frac{q}{q-1}|\nabla_N\Phi_k|^2 - |\nabla_M\Phi_k|^2 + \lambda_k\Phi_k^2 - \Phi_k H[\Phi_k]\right\} ds(\xi).$$

(1) Assume $\tau > 1$, then we have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^q} = \frac{|S^{q-1}|}{q}\int_M \left\{-\frac{q}{q-1}|\nabla_N\Phi_k|^2 - |\nabla_M\Phi_k|^2 + \lambda_k\Phi_k^2 - \Phi_k H[\Phi_k]\right\} ds(\xi).$$

(2) Assume $\tau = 1$, then we have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^q} = \frac{|S^{q-1}|}{q}\int_M \left\{-\frac{q}{q-1}|\nabla_N\Phi_k|^2 - |\nabla_M\Phi_k|^2 + (\lambda_k + q\sigma)\Phi_k^2 - \Phi_k H[\Phi_k]\right\} ds(\xi).$$

(3) Assume $-1 < \tau < 1$, then we have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{q+\tau-1}} = \sigma|S^{q-1}|\int_M \Phi_k(\xi)^2 ds(\xi).$$

(4) Assume $\tau = -1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{q-2}} = \frac{\sigma(q-2)}{q-2+\sigma} |S^{q-1}| \int_M \Phi_k(\xi)^2 ds(\xi).$$

(5) Assume $\tau < -1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{q-2}} = (q-2)|S^{q-1}| \int_M \Phi_k(\xi)^2 ds(\xi).$$

Here $|S^{q-1}| = 2\pi^{q/2}/\mathbf{\Gamma}(q/2)$, which is is the measure of S^{q-1} and $\mathbf{\Gamma}(s) = \int_0^\infty t^{s-1}e^{-t}\,dt$ is the standard Gamma function.

Theorem 2. *Assume that $n - m = q = 2$ and λ_k is simple in (1.1).*
(0) We have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^N(\epsilon) - \lambda_k}{\epsilon^2} = \pi \int_M \left(-2|\nabla_N \Phi_k|^2 - |\nabla_M \Phi_k|^2 + \lambda_k \Phi_k^2 - \Phi_k H[\Phi_k]\right) ds(\xi).$$

(1) Assume $\tau > 1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^2} = \pi \int_M \left(-2|\nabla_N \Phi_k|^2 - |\nabla_M \Phi_k|^2 + \lambda_k \Phi_k^2 - \Phi_k H[\Phi_k]\right) ds(\xi).$$

(2) Assume $\tau = 1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^2} = \pi \int_M \left(-2|\nabla_N \Phi_k|^2 - |\nabla_M \Phi_k|^2 + (\lambda_k + 2\sigma) \Phi_k^2 - \Phi_k H[\Phi_k]\right) ds(\xi).$$

(3) Assume $-1 < \tau < 1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{1+\tau}} = 2\pi\sigma \int_M \Phi_k(\xi)^2 ds(\xi).$$

(4) Assume $\tau \leqq -1$, then we have

$$\lim_{\epsilon \to 0} (\lambda_k^R(\epsilon) - \lambda_k) \log(1/\epsilon) = 2\pi \int_M \Phi_k(\xi)^2 ds(\xi).$$

Remark 2. It should be noted that in the case $\tau < -1$ in Theorem 1 and Theorem 2, the formula takes the same form as $\lambda_k^D(\epsilon)$ (the case of the Dirichlet B.C. on $\Gamma(M,\epsilon)$). In this case the Robin B.C. is close to the Dirichlet B.C. On the other hand, the formula for $\lambda_k^R(\epsilon)$ for $\tau > 1$ (in (1)) takes the same form as $\lambda_k^N(\epsilon)$ (in (0)).

Remark 3. For the case that M is a point, the hole is a ball or a small set around the point, there are many results. For the case of Dirichlet B.C., we refer to Swanson [36], [37], Rauch-Taylor [33], Ozawa [25],[26],[28], Flucher [12], Maz'ya-Nazarov-Plamenevskij [23].

For the case of Robin B.C and a spherical hole, more closely related results are Ozawa [27],[29],[30], Roppongi [34], Ozawa-Roppongi [30]. Actually Theorem 1, Theorem 2 for $M =$ a point ($m = 0$) and $n = 2, 3$ agree to their results. More recently, quite elaborate (higher order) $\epsilon-$expansion of the eigenvalue for the Neumann B.C. case are studies by Maz'ya-Nazarov-Plamenevskij [23], Lanza de Cristoforis [21], Ammari-Kang-Lim-Zribi [1]. It seems difficult to generalize the main results to higher order expansion only by the mathods in this paper and so it remains as an future subject. For a similar problem about the Lamé operator concerning in a domain with a small spherical hole, see Maz'ya-Nazarov-Plamenevskij [23].

Remark 4. For the case $n = 3$, $\dim M = 1$, Theorem 2-(3) agrees to Theorem 1 in Ozawa [31]. His method relies on approximate Green function. So our main results can be regarded a generalization of works due to Ozawa, Roppongi and others, to general M by an improved and direct method.

We briefly mention some other related works. There have been many works on the eigenvalues of the Laplacian in relation with the domain perturbation problems. Among them, some are motivated by phisical phenomena and others are from the problems of geometric analysis, etc (cf. Courant-Hilbert [7], Grebenkov-Nguyen [15] with its references therein). For the regular domain pertubation, J. Hadamard deduced the famous variational formula for the eigenvalue which is a pioneering work in the research field of PDEs on perturbed domains (see also Hadamard [16] for the pertubation formula of Green function). Later his formulas were rigorously justified from PDE theory point of view and extended in several directions.

There have been also many works for the eigenvalue problems of singularly perturbed domains. Roughly speaking, there are two typical cases of singular perturbation of domains. The first case is the domain with holes or cavities like ones in this paper. The second case is a domain with a thin handle like a Dumbbell shaped domain (or thin domains) (cf. Jimbo-Kosugi [17] with its references and also Kozlov-Maz'ya-Movchan [20]). For such singular domain perturbation, if we try to look into the detailed behaviors of eigenvalues and eigenfunctions, there appear difficulties due to that the perturbed domain $\Omega(\epsilon)$ can not be parametrized by a diffeomorphism from Ω smoothly up to the limit $\epsilon \to 0$ and we find that the situations seriously depend on each individual situation and the eigenfunctions may behave singularly near a new boundary which arise due to singular perturbation of domains. The interesting point in this subject is that a lot of mathematical techniques and methods are involved to overcome the problems and see the solutions of PDE in a singularly perturbed domain. In this paper we construct a good approxomate eigenfunction (cf. $\widetilde{\Phi}_{k,\epsilon}$ in the section 4 and the section 5) by using some special functions which have sharp layers near M. See also Maz'ya-Nazarov-Plamenevskij [23] for several techniques to have good modification of solution of PDE under several kinds of singular perturbation of domains.

2. Preliminaries

In this section we prepare several notations and facts for the proofs of the main results.

[**Coordinate system in $B(M, r_0), \Gamma(M, r_0)$ and the metric tensor**]

For calculation and estimation of auxiliary functions, approximate eigenfunctions, we prepare a coordinate system in a neighborhood of M. Since M is a compact $m-$dimensional manifold, it has a union of a finite number of patches $\{M_\alpha\}_\alpha$ and each M_α has a local coordinate $(\xi_1, \xi_2, \cdots, \xi_m)$ and the metric tensor of M is expressed as $\{\overline{g}_{ij}(\xi)\}_{ij}$ in terms of this coordinate. From the regularity of M, for a point x in a tubular neighborhood $B(M, r_0)$, there exists a unique $\xi \in M$ such that $\overrightarrow{\xi x}$ is normal to M provided that $r_0 > 0$ is taken adequately small. For any $\xi \in M$, the vector space $\mathbb{R}^n$ has the orthogonal decompoposition into the tangent space $T_\xi M$ and the normal space $N_\xi M$.i.e.

$$\mathbb{R}^n = T_\xi M \oplus N_\xi M.$$

Note that $\dim(T_\xi M) = m, \quad \dim(N_\xi M) = q.$

Let $(e_1(\xi), e_2(\xi), \cdots, e_q(\xi))$ be an orthonormal frame in $N_\xi M$. We can choose this frame which depends smoothly on ξ in each local patch M_α. Using this frame, we can express x uniquely in a neighborhood of M as

$$x = \xi + \sum_{\ell=1}^{q} \eta_\ell \, e_\ell(\xi). \tag{2.1}$$

Denote the second term by $\eta \cdot e(\xi)$ and note that this second term and also $|\eta \cdot e(\xi)| = |\eta|$ do not depend on the choice of the coordinate system $(\xi_1, \cdots, \xi_m)$ and the frame $\{e_\ell(\xi)\}_{\ell=1}^q$. Using the coordinate $(\xi_1, \xi_2, \cdots, \xi_m)$ for $\xi \in M_\alpha$, we can introduce the coordinate

$$(\xi_1, \xi_2, \cdots, \xi_m, \eta_1, \eta_2, \cdots, \eta_q)$$

in the tubular neighborhood $B(M, r_0)$ provided that $r_0 > 0$ is taken adequately small (cf. Gray [14]). Actually this is justified by the implicit function theorem. We can also assume that $\overline{B(M, r_0)} \subset \Omega$. Under these conditions, $\Gamma(M, r_0) = \partial B(M, r_0)$ becomes a smooth manifold of $n-1$ dimension, which is the set expressed by the condition $|\eta| = r_0$ in the above local coordinate. We can take the (symmetric) metric tensor as $(g_{i,j}(\xi, \eta))_{1 \leqq i,j \leqq n}$ in terms of this coodinate at this point $x = \xi + \eta \cdot e(\xi)$. For the indicies $m+1 \leqq i \leqq n$ or $m+1 \leqq j \leqq n$, we are regarading as $\xi_{m+\ell} = \eta_\ell$ $(1 \leqq \ell \leqq q)$. Putting

$$g(\xi, \eta) = \det(g_{ij}(\xi, \eta))_{1 \leqq i,j \leqq n}, \quad \overline{g}(\xi) = \det(\overline{g}_{ij}(\xi))_{1 \leqq i,j \leqq m}$$

The volume element dx in $\mathbb{R}^n$ and the volume element $ds(\xi)$ in M are expressed as

$$dx = \sqrt{g(\xi, \eta)}\, d\xi_1 \cdots d\xi_m d\eta_1 \cdots d\eta_q, \quad ds(\xi) = \sqrt{\overline{g}(\xi)}\, d\xi_1 \cdots d\xi_m,$$

respectively. We see that

$$g_{ij}(\xi, \eta) = \delta(i, j) \quad \text{for} \quad m+1 \leqq i, j \leqq n,$$
$$g_{ij}(\xi, \eta) = \overline{g}_{i,j}(\xi) + O(|\eta|) \quad \text{for} \quad \xi \in M, |\eta| \leqq r_0 \quad 1 \leqq i, j \leqq m,$$
$$g(\xi, \eta) = \overline{g}(\xi) + O(|\eta|) \quad (\xi \in M,\ |\eta| \leqq r_0).$$

From these properties, we can express the measure dx in $B(M, r_0)$ and the surface measure dS on $\Gamma(M, r)$ in terms of the local coordinate as

$$dx = \rho_1(\xi, \eta) d\eta ds(\xi) \quad \rho_1(\xi, \eta) = (g(\xi, \eta)/\overline{g}(\xi))^{1/2}$$
$$dS = \rho_2(\xi, \eta) d\widehat{s}_\eta \, ds(\xi)$$

where $d\widehat{s}_\eta$ is the $q-1$ dimensional measure in the sphere $|\eta| = r$ in $\mathbb{R}^q$. $\rho_1(\xi, \eta)$ and $\rho_2(\xi, \eta)$ are positive smooth functions. It should be noted that

$$\rho_1(\xi, \eta) = 1 + O(|\eta|), \rho_2(\xi, \eta) = 1 + O(|\eta|)$$

in smooth sense for small $|\eta|$.

Let $(g^{ij}(\xi, \eta))_{ij}$ be the inverse matrix of $(g_{ij}(\xi, \eta))_{ij}$ and let $(\overline{g}^{ij}(\xi))_{ij}$ be the inverse matrix of $(\overline{g}_{ij}(\xi))_{ij}$. Then g^{ij} and $\overline{g}^{ij}$ have the similar relation for $1 \leqq i, j \leqq m$.

From the definition of this coordinate system $(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_q)$, we have the following properties of $g_{ij}(\xi, \eta)$ near M.

Lemma 1. *We have*

$$g_{i,m+\ell}(\xi,\eta)_{|M} = g_{m+\ell,i}(\xi,\eta)_{|M} = 0 \ (1 \leqq i \leqq m, 1 \leqq \ell \leqq q), \tag{2.2}$$

$$\frac{\partial g_{i,m+\ell}(\xi,\eta)}{\partial \eta_\ell} = 0 \quad in \quad B(M,r_0) \quad (1 \leqq i \leqq n, 1 \leqq \ell \leqq q), \tag{2.3}$$

$$\left(\frac{\partial g_{i,m+\ell}(\xi,\eta)}{\partial \xi_\ell}\right)_{|M} = 0 \quad (1 \leqq i \leqq n, m+1 \leqq j \leqq n, 1 \leqq \ell \leqq q). \tag{2.4}$$

Proof. We use (2.1). For a point $x = \xi + \eta \cdot e(\xi) = \xi + \sum_{p=1}^{q} \eta_p \, e_p(\xi)$ in $\Gamma(M,r_0)$. We denote the inner product of vectors in $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$. If $1 \leqq i \leqq m$,

$$g_{i,m+\ell}(\xi,\eta) = \langle \frac{\partial x}{\partial \xi_i}, \frac{\partial x}{\partial \eta_\ell} \rangle = \langle \frac{\partial x}{\partial \xi_i} + \sum_{p=1}^{q} \eta_p \frac{\partial e_p(\xi)}{\partial \xi_i}, e_\ell(\xi) \rangle = \sum_{p=1}^{q} \eta_p \langle \frac{\partial e_p(\xi)}{\partial \xi_i}, e_\ell(\xi) \rangle$$

The right hand side vanishes if $\eta = \mathbf{0}$ and this leads to (2.2). It follows that

$$\partial g_{i,m+\ell}/\partial \eta_\ell = \langle (\partial e_\ell(\xi)/\partial \xi_i), e_\ell(\xi) \rangle = 0$$

These calculation concludes (2.3) for the case $1 \leqq i \leqq m$. On the other hand if $m+1 \leqq i \leqq n$, it holds that $g_{i,m+\ell}(\xi,\eta) = \delta(i, m+\ell)$ and (2.3) is also true. The (2.3) is proved for all the case of $1 \leqq i \leqq n$. (2.4) is true if $m+1 \leqq i,j \leqq n$. Because $g_{ij} = \delta(i,j)$ in this case. For the case $1 \leqq i \leqq m$, $m+1 \leqq j \leqq n$, $1 \leqq \ell \leqq q$, we have

$$\frac{\partial g_{i,j}(\xi,\eta)}{\partial \xi_\ell} = \sum_{p=1}^{q} \eta_p \frac{\partial}{\partial \xi_\ell} \langle \frac{\partial e_p(\xi)}{\partial \xi_i}, e_\ell(\xi) \rangle$$

This term vanishes for $\eta = \mathbf{0}$. (2.4) is concluded. □

We also have the following properties for the inverse matrix $\{g^{ij}\}$ near M.

Lemma 2. *We have*

$$g^{ij}(\xi,\eta)_{|M} = g^{ji}(\xi,\eta)_{|M} = \delta(i,j) \quad (m+1 \leqq i,j \leqq n), \tag{2.5}$$

$$g^{ij}(\xi,\eta)_{|M} = g^{ji}(\xi,\eta)_{|M} = 0 \quad (1 \leqq i \leqq m, m+1 \leqq j \leqq n), \tag{2.6}$$

$$\left(\frac{\partial g^{i,m+\ell}(\xi,\eta)}{\partial \eta_\ell}\right)_{|M} = 0 \quad (1 \leqq \ell \leqq q, 1 \leqq i \leqq n), \tag{2.7}$$

$$\left(\frac{\partial g^{ij}(\xi,\eta)}{\partial \xi_\ell}\right)_{|M} = 0 \quad (1 \leqq i \leqq n, m+1 \leqq j \leqq n, 1 \leqq \ell \leqq m). \tag{2.8}$$

Proof. All these properties (2.5), (2.6), (2.7), (2.8) follows from Lemma 1 which are concerned with $\{g_{ij}\}$ and the formula for the derivative of the inverse matrix

$$\frac{\partial}{\partial \xi_\ell}(g^{ij})_{1 \leqq i,j \leqq n} = -(g^{ij})_{1 \leqq i,j \leqq n} \ (\frac{\partial g_{ij}}{\partial \xi_\ell})_{1 \leqq i,j \leqq n} \ (g^{ij})_{1 \leqq i,j \leqq n}.$$

□

[Mean curvature operator on M**]**

The second fundamental form $h_\xi(X,Y)$ of M is defined by the following formula

$$\nabla_Y X = \nabla_Y^M X + h_\xi(X,Y) \in T_\xi M \oplus N_\xi M \quad \text{(orthogonal decomposition)}$$

for any C^1 vector field on X, Y in a neighborhood of M at each $\xi \in M$. Here $\nabla_Y X$ is the covariant derivative of X with respect to Y in $\mathbb{R}^n$. The mean curvature vector H of M is defined by

$$H_\xi = \sum_{i=1}^m h_\xi(E_i, E_i)$$

for each $\xi \in M$. Here $\{E_1, E_2, \cdots, E_m\}$ is an orthonormal frame of $T_\xi M$. Note that H_ξ does not depend on choice of this frame (cf. Kobayasi-Nomizu [19]).

Lemma 3. *In this coordinate system* (ξ, η)*, the mean curvature operator of* M *is expressed as follows.*

$$\text{(2.9)} \quad H_\xi = -\sum_{\ell=1}^q \frac{1}{\sqrt{g(\xi,0)}} \left(\frac{\partial \sqrt{g(\xi,\eta)}}{\partial \eta_\ell} \right)_{|M} \frac{\partial}{\partial \eta_\ell} = -\sum_{\ell=1}^q \sum_{1 \leqq i,j \leqq m} \frac{g^{ij}(\xi,0)}{2} \frac{\partial g_{ij}}{\partial \eta_\ell}(\xi,0) \frac{\partial}{\partial \eta_\ell}$$

in this coordinate system $(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_q)$ *and it is expressed as a normal vector field*

$$\text{(2.10)} \qquad H_\xi = -\sum_{\ell=1}^q \frac{1}{\sqrt{g(\xi,\mathbf{0})}} \left(\frac{\partial \sqrt{g(\xi,\eta)}}{\partial \eta_\ell} \right)_{\eta=\mathbf{0}} e_\ell(\xi).$$

[Laplacian in terms of $(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_q)$**]**

In this (local) coordinate system in $\Gamma(M, r_0)$ and the metric tensor $(g_{ij}(\xi,\eta))_{ij}$, we can express the Laplacian Δ in terms of $(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_q)$ in the following form

$$\text{(2.11)} \qquad \Delta u = J_1(u) + J_2(u) + J_3(u) + J_4(u)$$

where

$$J_1(u) = \frac{1}{\sqrt{g(\xi,\eta)}} \sum_{1 \leqq i,j \leqq m} \frac{\partial}{\partial \xi_i} \left(\sqrt{g(\xi,\eta)} g^{ij}(\xi,\eta) \frac{\partial u}{\partial \xi_j} \right)$$

$$J_2(u) = \frac{1}{\sqrt{g(\xi,\eta)}} \sum_{1 \leqq i \leqq m, 1 \leqq \ell \leqq q} \frac{\partial}{\partial \xi_i} \left(\sqrt{g(\xi,\eta)} g^{i,m+\ell}(\xi,\eta) \frac{\partial u}{\partial \eta_\ell} \right)$$

$$J_3(u) = \frac{1}{\sqrt{g(\xi,\eta)}} \sum_{1 \leqq j \leqq m, 1 \leqq \ell \leqq q} \frac{\partial}{\partial \eta_\ell} \left(\sqrt{g(\xi,\eta)} g^{m+\ell,j}(\xi,\eta) \frac{\partial u}{\partial \xi_j} \right)$$

$$J_4(u) = \frac{1}{\sqrt{g(\xi,\eta)}} \sum_{1 \leqq \ell, s \leqq q} \frac{\partial}{\partial \eta_\ell} \left(\sqrt{g(\xi,\eta)} g^{m+\ell, m+s}(\xi,\eta) \frac{\partial u}{\partial \eta_s} \right).$$

Proposition 3. *For a C^2 function u which is defined in $B(M, r_0)$, we have*

$$(\Delta u)_{|M} = \Delta_M(u_{|M}) - H[u] + \sum_{\ell=1}^{q} \left(\frac{\partial^2 u}{\partial \eta_\ell^2} \right)_{|\eta = \mathbf{0}} \quad \text{on} \quad M. \tag{2.12}$$

Proof. We evaluate each term of the right hand side of (2.11) at $\eta = \mathbf{0}$. J_1 corresponds to the first term of (2.12). J_2 and J_3 vanish for $\eta = \mathbf{0}$ due to Lemma 2. From Lemma 3, J_4 corresponds to the second and the third term of (2.12).

□

[**Barrier function**]

We prepare some functions to construct a barrier function to constrol the behavior of solutions around M. We consider the following ODE,

$$\frac{d^2 K}{dr^2} + (\frac{q-1}{r} - m_1)\frac{dK}{dr} + m_2 K = 0 \quad (r > 0) \tag{2.13}$$

Here $m_1 > 0, m_2 > 0$ are constants. By applying the standard Frobenius method, we can construct a formal power series solution of the ODE (2.13) and we can discuss its convergence by the aid of the method of a majorant series. Consequently we get solutions $K_1 = K_1(r)$ and $K_2 = K_2(r)$ which are convergent for all $r > 0$ and linearly independent. We summarize the results in the following proposition.

Proposition 4. *The equation* (2.13) *has two (linearly independent) solutions $K_1(r), K_2(r)$ of the following form,*

$$\begin{cases} K_1(r) = \displaystyle\sum_{\ell=0}^{\infty} a_\ell \, r^\ell \quad (\textit{regular solution}), \\ K_2(r) = r^{-q+2} \left(\displaystyle\sum_{\ell=0}^{\infty} b_\ell \, r^\ell \right) + \left(\displaystyle\sum_{\ell=0}^{\infty} c_\ell \, r^\ell \right) \log r \quad (\textit{singular solution}), \end{cases} \tag{2.14}$$

and all the power series in (2.14) *are convergent for $r > 0$. We note*

$$\begin{cases} a_0 \neq 0 \quad \textit{for} \quad q \geqq 2, \\ b_0 \neq 0 \quad \textit{if} \quad q \geqq 3 \quad \textit{and} \quad c_0 \neq 0 \quad \textit{if} \quad q = 2. \end{cases}$$

Multiplying adequate constants to normalize the coefficient of the leading terms, we have the properties for some $r_1 \in (0, 3\zeta_0] \cap (0, r_0]$

$$\begin{cases} K_1 \text{ is regular at } r = 0 \text{ and } K_1(0) = 1,\ K_1'(0) = 0 \quad (q \geqq 2), \\ \lim_{r \downarrow 0} K_2(r)/r^{-q+2} = 1 \quad (q \geqq 3), \\ \lim_{r \downarrow 0} K_2(r)/\log(1/r) = 1 \quad (q = 2), \end{cases} \tag{2.15}$$

$$\begin{cases} K_1(r) > 0,\ K_1'(r) < 0,\ K_1''(r) < 0 \quad (0 < r \leqq r_1), \\ K_2(r) > 0,\ K_2'(r) < 0 \quad (0 < r \leqq r_1). \end{cases} \tag{2.16}$$

See Coddington-Levinson [6], Whittaker-Watson [38]. This is equivalent to the condition $a_0 = 1$ and $b_0 = 1$ if $q \geqq 3$, $c_0 = 1$ if $q = 2$. By a direct calculation we see $a_2 = -m_2/(2q)$. Later we need only K_1. The equation (2.3) is related with the Whittaker equation through a certain change of the variables.

Definition 3. We define a barrier function $\psi_\epsilon(x)$ as follows.

$$\psi_\epsilon(x) = 2K_1(|\eta|) \quad \text{for} \quad x = \xi + \eta \cdot e(\xi) \in B(M, r_0)$$

Recall $\eta \cdot e(\xi)$ is defined in (2.1). Note that $\psi_\epsilon(x)$ is well-defined ($\psi_\epsilon(x)$ does not depend on the choice of coordinate system). Remark that ψ_ϵ depends on two parameters $m_1 > 0, m_2 > 0$.

Lemma 4. *The barrier function $\psi_\epsilon(x)$ (defined from K_1) satisfies the following properties. For any $m_2 > 0$, there exist $m_1 > 0$, $r_1 \in (0, r_0]$ and $\epsilon_1 > 0$ such that*

$$\Delta\psi_\epsilon + m_2\psi_\epsilon \leqq 0 \quad \textit{in} \quad B(M, r_1) \setminus B(M, \epsilon), \tag{2.17}$$

$$\frac{\partial\psi_\epsilon}{\partial\nu} \geqq 0 \quad \textit{on} \quad \Gamma(M, \epsilon), \quad 1 \leqq \psi_\epsilon(x) \leqq 3 \quad \textit{in} \quad B(M, r_1) \setminus B(M, \epsilon), \tag{2.18}$$

for any $\epsilon \in (0, \epsilon_1)$.

Proof. We use the local coordinate system $(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_q)$ constructed previously in this section. We have $\partial\psi_\epsilon(\xi, \eta)/\partial\xi_i = 0$ $(1 \leqq i \leqq m)$ because $\psi_\epsilon(\xi, \eta)$ depends on $|\eta|$, but it does not depend on $\xi_1, \xi_2, \cdots, \xi_m$. We use the expression of the decomposition of Δ in (2.10). $J_1(\psi_\epsilon) = 0$, $J_3(\psi_\epsilon) = 0$ are clear. So we have

$$\Delta\psi_\epsilon + m_2\psi_\epsilon = J_2(\psi_\epsilon) + J_4(\psi_\epsilon) + m_2\psi_\epsilon. \tag{2.19}$$

We look into the sign of the right hand side of (2.19). The terms $J_2(\psi_\epsilon)$, $J_4(\psi_\epsilon)$ are calculated as follows.

$$J_2(\psi_\epsilon) = \frac{1}{\sqrt{g(\xi,\eta)}} \sum_{1 \leqq i \leqq m, 1 \leqq \ell \leqq q} \frac{\partial}{\partial\xi_i}\left(\sqrt{g(\xi,\eta)}g^{i,m+\ell}(\xi,\eta)\frac{\partial\psi_\epsilon}{\partial\eta_\ell}\right)$$

$$= \sum_{1 \leqq i \leqq m, 1 \leqq \ell \leqq q} \left(\frac{\partial g^{i,m+\ell}}{\partial\xi_i} + \frac{g^{i,m+\ell}}{\sqrt{g(\xi,\eta)}}\frac{\partial\sqrt{g(\xi,\eta)}}{\partial\xi_i}\right)\frac{\partial\psi_\epsilon}{\partial\eta_\ell}$$

$$J_4(\psi_\epsilon) = \frac{1}{\sqrt{g(\xi,\eta)}} \sum_{1 \leqq \ell, s \leqq q} \frac{\partial}{\partial\eta_\ell}\left(\sqrt{g(\xi,\eta)}g^{m+\ell,m+s}(\xi,\eta)\frac{\partial\psi_\epsilon}{\partial\eta_s}\right)$$

$$= \sum_{1 \leqq \ell, s \leqq q} \left(g^{m+\ell,m+s}(\xi,\eta)\frac{\partial^2\psi_\epsilon}{\partial\eta_s\partial\eta_\ell} + \frac{g^{m+\ell,m+s}(\xi,\eta)}{\sqrt{g(\xi,\eta)}}\frac{\partial\sqrt{g(\xi,\eta)}}{\partial\eta_\ell}\frac{\partial\psi_\epsilon}{\partial\eta_s}\right)$$

$$+\sum_{1\leqq \ell,s\leqq q}\frac{\partial g^{m+\ell,m+s}(\xi,\eta)}{\partial\eta_\ell}\frac{\partial\psi_\epsilon}{\partial\eta_s}$$

$$=\sum_{\ell=1}^{q}\frac{\partial^2\psi_\epsilon}{\partial\eta_\ell^2}+\sum_{1\leqq \ell,s\leqq q}(g^{m+\ell,m+s}(\xi,\eta)-\delta(\ell,s))\frac{\partial^2\psi_\epsilon}{\partial\eta_s\partial\eta_\ell}$$

$$+\sum_{1\leqq \ell,s\leqq q}\left(\frac{\partial g^{m+\ell,m+s}(\xi,\eta)}{\partial\eta_\ell}+\frac{g^{m+\ell,m+s}(\xi,\eta)}{\sqrt{g(\xi,\eta)}}\frac{\partial\sqrt{g(\xi,\eta)}}{\partial\eta_\ell}\right)\frac{\partial\psi_\epsilon}{\partial\eta_s}$$

From

$$\frac{\partial\psi_\epsilon}{\partial\eta_\ell}=2K_1'(|\eta|)\frac{\eta_\ell}{|\eta|},\quad \frac{\partial^2\psi_\epsilon}{\partial\eta_\ell\partial\eta_s}=2\left(K_1'(|\eta|)(\frac{\delta(\ell,s)}{|\eta|}-\frac{\eta_\ell\eta_s}{|\eta|^3})+K_1''(|\eta|)\frac{\eta_\ell\eta_s}{|\eta|^2}\right),$$

we have

$$J_2(\psi_\epsilon)=2\sum_{1\leqq i\leqq m,1\leqq \ell\leqq q}(\frac{\partial g^{i,m+\ell}}{\partial\xi_i}+\frac{g^{i,m+\ell}}{\sqrt{g(\xi,\eta)}}\frac{\partial\sqrt{g(\xi,\eta)}}{\partial\xi_i})K_1'(|\eta|)\frac{\eta_\ell}{|\eta|},$$

$$J_4(\psi_\epsilon)+m_2\psi_\epsilon=2\{K_1''(|\eta|)+\frac{q-1}{|\eta|}K_1'(|\eta|)+m_2K_1(|\eta|)\}$$

$$+2\sum_{1\leqq \ell,s\leqq q}(g^{m+\ell,m+s}(\xi,\eta)-\delta(\ell,s))\{K_1'(|\eta|)(\frac{\delta(\ell,s)}{|\eta|}-\frac{\eta_\ell\eta_s}{|\eta|^3})+K_1''(|\eta|)\frac{\eta_\ell\eta_s}{|\eta|^2}\}$$

$$+2\sum_{1\leqq \ell,s\leqq q}(\frac{\partial g^{m+\ell,m+s}(\xi,\eta)}{\partial\eta_\ell}+\frac{g^{m+\ell,m+s}(\xi,\eta)}{\sqrt{g(\xi,\eta)}}\frac{\partial\sqrt{g(\xi,\eta)}}{\partial\eta_\ell})K_1'(|\eta|)\frac{\eta_\ell}{|\eta|}.$$

Therefore

$$J_2(\psi_\epsilon)+J_4(\psi_\epsilon)+m_2\psi_\epsilon=2K_1'(|\eta|)\times(m_1+I_1+I_2+I_3+I_4)$$

where

$$I_1=\sum_{1\leqq \ell,s\leqq q}(g^{m+\ell,m+s}(\xi,\eta)-\delta(\ell,s))\,\frac{K_1''(|\eta|)}{K_1'(|\eta|)}\,\frac{\eta_\ell\eta_s}{|\eta|^2}$$

$$=\sum_{1\leqq \ell,s\leqq q}\frac{(g^{m+\ell,m+s}(\xi,\eta)-\delta(\ell,s))}{|\eta|}\,\frac{|\eta|K_1''(|\eta|)}{K_1'(|\eta|)}\,\frac{\eta_\ell\eta_s}{|\eta|^2}$$

$$I_2=\sum_{1\leqq i\leqq m,1\leqq \ell\leqq q}(\frac{\partial g^{i,m+\ell}}{\partial\xi_i}+\frac{g^{i,m+\ell}}{\sqrt{g(\xi,\eta)}}\frac{\partial\sqrt{g(\xi,\eta)}}{\partial\xi_i})\frac{\eta_\ell}{|\eta|}$$

$$I_3=\sum_{1\leqq \ell,s\leqq q}(g^{m+\ell,m+s}(\xi,\eta)-\delta(\ell,s))(\frac{\delta(\ell,s)}{|\eta|}-\frac{\eta_\ell\eta_s}{|\eta|^3})$$

$$=\sum_{1\leqq \ell,s\leqq q}\frac{(g^{m+\ell,m+s}(\xi,\eta)-\delta(\ell,s))}{|\eta|}(\delta(\ell,s)-\frac{\eta_\ell\eta_s}{|\eta|^2})$$

$$I_4=\sum_{1\leqq \ell,s\leqq q}(\frac{\partial g^{m+\ell,m+s}(\xi,\eta)}{\partial\eta_\ell}+\frac{g^{m+\ell,m+s}(\xi,\eta)}{\sqrt{g(\xi,\eta)}}\frac{\partial\sqrt{g(\xi,\eta)}}{\partial\eta_\ell})\frac{\eta_\ell}{|\eta|}$$

It is easy to see that I_2 and I_4 are bounded in $B(M, r_0)$. From Lemma 2 there exists $c > 0$ such that

$$|g^{m+\ell,m+s}(\xi,\eta) - \delta(\ell,s)| \leqq c\,|\eta| \quad (0 < |\eta| \leqq r_0, 1 \leqq \ell, s \leqq q).$$

We also note that $r\,K_1''(r)/K_1'(r)$ is bounded in $0 < r \leqq r_1$ from (2.15)-(2.16), we see that I_1 and I_3 are bounded and

$$J_2(\psi_\epsilon) + J_4(\psi_\epsilon) + m_2\psi_\epsilon \leqq 2m_1 K_1'(|\eta|) + c\,|K_1'(|\eta|)|.$$

Since $K_1'(r) < 0$ in $(0, r_1)$, the right hand side of this expression is negative if m_1 is taken adequately large (depending on M, not on ϵ). We have discussed in one local neighborhood. Since M is compact and it is covered by a finite number of such neighorhoods. We can take the maximum of m_1 and the minimum of $r_1 > 0$ after discussion of each neighborhood.

□

3. Estimates for the eigenvalues and the eigenfunctions

The upper estimate of the behavior of is carried out in the variational formulation of the eigenvalue problem with the aid of the functional (Rayleigh quotient)

$$R_\epsilon(\Phi) = \left(\int_{\Omega(\epsilon)} |\nabla\Phi|^2 dx + \sigma\epsilon^\tau \int_{\Gamma(M,\epsilon)} \Phi^2 dS\right) / \|\Phi\|^2_{L^2(\Omega(\epsilon))}$$

$$\text{for} \quad \Phi \in H^1(\Omega(\epsilon)) \quad \text{such that} \quad \Phi(x) = 0 \quad \text{on} \quad \Gamma.$$

$\lambda_k^R(\epsilon)$ is characterized by the following formula.

Max-Min principle (Courant-Hilbert [7])

$$\lambda_k^R(\epsilon) = \sup_{E \subset L^2(\Omega(\epsilon)), \dim E \leqq k-1} \inf\{R_\epsilon(\Phi) \mid \Phi \in H^1(\Omega(\epsilon)), \Phi_{|\Gamma} \equiv 0, \Phi \perp E \,\text{in}\, L^2(\Omega(\epsilon))\}$$

Here E is a linear subspace of $L^2(\Omega(\epsilon))$.

Remark 5. For the max-min principle for $\lambda_k^N(\epsilon)$, we just put $\sigma = 0$ in $R_\epsilon(\Phi)$. For $\lambda_k^D(\epsilon)$, we impose the Dirichlet boundary condition $\Gamma(M,\epsilon)$ for the test function Φ (so the second term in $R_\epsilon(\Phi)$ vanishes).

By the aid of the above max-min principle, we can prove the upper estimate for $\lambda_k^R(\epsilon)$, $\lambda_k^N(\epsilon)$.

Lemma 5.

(3.1) $$\limsup_{\epsilon\to 0} \lambda_k^R(\epsilon) \leqq \lambda_k, \quad \limsup_{\epsilon\to 0} \lambda_k^N(\epsilon) \leqq \lambda_k \quad (k \in \mathbb{N}).$$

Proof. To construct a test function, we use the following cut-off function

$$\rho_\epsilon(\eta) = \frac{\log(|\eta|/\epsilon)}{\log(r_0/\epsilon)} \quad \text{for} \quad \eta \in \mathbb{R}^q, \epsilon \leqq |\eta| \leqq r_0.$$

A simple calculation gives

$$\int_{\epsilon\leqq|\eta|\leqq r_0} |\rho_\epsilon(\eta) - 1|^2 d\eta = O(\frac{1}{(\log\epsilon)^2}), \quad \int_{\epsilon\leqq|\eta|\leqq r_0} |\nabla_\eta \rho_\epsilon(\eta)|^2 d\eta = \begin{cases} O(\frac{1}{|\log\epsilon|}) & (q = 2) \\ O(\frac{1}{(\log\epsilon)^2}) & (q \geqq 3) \end{cases}$$

for $0 < \epsilon \leqq r_0$. We put

$$\widetilde{\phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) & \text{for} \quad x \in \Omega \setminus B(M, r_0) \\ \rho_\epsilon(\eta)\Phi_k(x) & \text{for} \quad x = \xi + \eta \cdot e(\xi) \in B(M, r_0) \setminus B(M, \epsilon) \end{cases}$$

and we see that $\widetilde{\phi}_{k,\epsilon} \in H^1(\Omega(\epsilon))$ and $\widetilde{\phi}_{k,\epsilon}$ vanishes on Γ. A simple calculation shows that

$$(\widetilde{\phi}_{p,\epsilon}, \widetilde{\phi}_{\ell,\epsilon})_{L^2(\Omega(\epsilon))} = \delta(p,\ell) + \kappa'(p,\ell,\epsilon), \quad (\nabla\widetilde{\phi}_{p,\epsilon}, \nabla\widetilde{\phi}_{\ell,\epsilon})_{L^2(\Omega(\epsilon))} = \lambda_p\delta(p,\ell) + \kappa(p,\ell,\epsilon).$$

where

$$\kappa'(p,\ell,\epsilon) = O(\frac{1}{|\log\epsilon|}), \quad \kappa(p,\ell,\epsilon) = O(\frac{1}{|\log\epsilon|^{1/2}}) \quad \text{for} \quad 1 \leqq p, \ell \leqq k.$$

With the aid of this function, we begin to estimate the eigenvalue $\lambda_k(\epsilon)$.

$$\widetilde{E}_\epsilon = \text{L.H.}[\widetilde{\phi}_{1,\epsilon}, \widetilde{\phi}_{2,\epsilon}, \cdots, \widetilde{\phi}_{k,\epsilon}]$$

then $\dim\widetilde{E}_\epsilon = k$ for small $\epsilon > 0$ since

$$(\widetilde{\phi}_{p,\epsilon}, \widetilde{\phi}_{\ell,\epsilon})_{L^2(\Omega(\epsilon))} = \delta(p,\ell) + O(|\log\epsilon|^{-1})$$

for $p, \ell \geqq 1$. Take any linear subspace E of $L^2(\Omega(\epsilon))$ such that $\dim E \leqq k-1$. Using the dimension theorem (in Linear Algebra), there exists non-zero function

$$\varphi(x) = \sum_{\ell=1}^{k} c_\ell\, \widetilde{\phi}_{\ell,\epsilon}(x) \in \widetilde{E}_\epsilon$$

such that $\varphi \perp E$ in $L^2(\Omega(\epsilon))$. Using this property of φ we have the estimate.

$$\inf\{R_\epsilon(\Phi) \mid \Phi \in H^1(\Omega(\epsilon)), \Phi_{|\Gamma} \equiv 0, \Phi \perp E \,\text{in}\, L^2(\Omega(\epsilon))\}$$

$$\leqq R_\epsilon(\varphi) = \left(\int_{\Omega(\epsilon)} |\nabla\varphi|^2 dx + \sigma\epsilon^\tau \int_{\Gamma(M,\epsilon)} \varphi^2 dS\right) / \|\varphi\|^2_{L^2(\Omega(\epsilon))}.$$

We can assume without loss of genrality $\sum_{\ell=1}^k |c_k|^2 = 1$ due to the homogeniety property (i.e. $R_\epsilon(t\phi) = R_\epsilon(\phi)$ $(t > 0)$). Note

$$(\nabla\Phi_{\ell_1}, \nabla\Phi_{\ell_2})_{L^2(\Omega)} = \lambda_{\ell_1}\delta(\ell_1,\ell_2), \quad (\Phi_{\ell_1}, \Phi_{\ell_2})_{L^2(\Omega))} = \delta(\ell_1,\ell_2)$$

and we have

$$\int_{\Omega(\epsilon)} |\nabla\varphi|^2 dx = \sum_{\ell_1,\ell_2=1}^{k} \int_{\Omega(\epsilon)} c_{\ell_1}c_{\ell_2}\nabla\Phi_{\ell_1}\nabla\Phi_{\ell_2} dx = \sum_{\ell_1,\ell_2=1}^{k} (\lambda_{\ell_1}\delta(\ell_1,\ell_2) + \kappa(\ell_1,\ell_2,\epsilon))c_{\ell_1}c_{\ell_2}$$

$$= \sum_{\ell=1}^{k} \lambda_\ell c_\ell^2 + \sum_{\ell_1,\ell_2=1}^{k} \kappa(\ell_1,\ell_2,\epsilon)c_{\ell_1}c_{\ell_2} \leqq \lambda_k \sum_{\ell=1}^{k} c_\ell^2 + \sum_{\ell_1,\ell_2=1}^{k} \kappa(\ell_1,\ell_2,\epsilon)c_{\ell_1}c_{\ell_2}$$

$$\int_{\Omega(\epsilon)} |\varphi|^2 dx = \sum_{\ell_1,\ell_2=1}^{k} \int_{\Omega(\epsilon)} c_{\ell_1}c_{\ell_2}\Phi_{\ell_1}\Phi_{\ell_2} dx = \sum_{\ell_1,\ell_2=1}^{k} (\delta(\ell_1,\ell_2) + \kappa'(\ell_1,\ell_2,\epsilon))c_{\ell_1}c_{\ell_2}$$

$$= \sum_{\ell=1}^{k} c_\ell^2 + \sum_{\ell_1,\ell_2=1}^{k} \kappa'(\ell_1,\ell_2,\epsilon)c_{\ell_1}c_{\ell_2}$$

Since $\sum_{\ell=1}^{k} c_\ell^2 = 1$ and $\varphi = 0$ on $\Gamma(M,\epsilon)$, we have

$$R_\epsilon(\varphi) \leqq \frac{\lambda_k + \sum_{\ell_1,\ell_2=1}^{k} |\kappa(\ell_1,\ell_2,\epsilon)|}{1 - \sum_{\ell_1,\ell_2=1}^{k} |\kappa'(\ell_1,\ell_2,\epsilon)|}$$

The right hand side the above expression does not depend on the choice of E with $\dim E \leqq k-1$ and hence we have

$$\lambda_k^R(\epsilon) \leqq \frac{\lambda_k + \sum_{\ell_1,\ell_2=1}^{k} |\kappa(\ell_1,\ell_2,\epsilon)|}{1 - \sum_{\ell_1,\ell_2=1}^{k} |\kappa'(\ell_1,\ell_2,\epsilon)|} = \lambda_k + O(|\log\epsilon|^{-1/2}). \tag{3.2}$$

This inequality gives an upper estimate for $\lambda_k^R(\epsilon)$ for small $\epsilon > 0$.

On the other hand, from the comparison principle of eigenvalues for the Robin B.C. and Neumann B.C on Γ, we have $\lambda_k^N(\epsilon) \leqq \lambda_k^R(\epsilon)$ (cf. Courant-Hilbert [7]) and obtain the same upper estimate for $\lambda_k^N(\epsilon)$.

□

Remark 6. The above argument is common in all cases of boundary conditions (Robin, Neumann, Dirichlet B.C) and so it is not a sharp estimate.

We prepare some auxiliary results for the estimates for the eigenfunctions, which will be used later in the proofs of the main results.

Lemma 6. *For each $k \in \mathbb{N}$ and any $r \in (0, r_0)$, there exists $c(k,r) > 0$ such that*

$$\sup_{x\in\overline{\Omega}\setminus B(M,r)} |\Phi_{k,\epsilon}^N(x)| \leqq c(k,r), \qquad \sup_{x\in\overline{\Omega}\setminus B(M,r)} |\Phi_{k,\epsilon}^R(x)| \leqq c(k,r), \tag{3.3}$$

for any $r \in (0, r_0)$ and $\epsilon \in (0, r_0/2]$.

Proof. This is proved by the cut-off and the interior and the boundary regularity estimates (cf. Evans [11](section 6)) with $\|\Phi_{k,\epsilon}^N\|_{L^2(\Omega(\epsilon))} = 1$ and $\|\Phi_{k,\epsilon}^R\|_{L^2(\Omega(\epsilon))} = 1$ (See also Jimbo-Kosugi [17] (Section 8)).

□

We also mention some results for basic estimates for the eigenvalue λ_k and its corresponding eigenfunction Φ_k for (1.1).

Proposition 5. *There exist positive constants c_1, c_1', c_2, c_2', c_3, c_3' such that*

$$c_1\, k^{2/n} - c_1' \leqq \lambda_k \leqq c_2\, k^{2/n} + c_2' \quad (k \in \mathbb{N}) \tag{3.4}$$

$$\|\Phi_k\|_{L^\infty(\Omega)} \leqq c_3\, \lambda_k^{n/2}, \quad \|\nabla^p \Phi_k\|_{L^\infty(\Omega)} \leqq c_3'\, \lambda_k^{(n/2)+p} \ (p, k \in \mathbb{N}). \tag{3.5}$$

These properties are easily deduced from the famous works in the spectral theory of the 2nd order elliptic operators. The first inequality for λ_k easily follows from the Weyl formula of the distribution of the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ (cf. Courant-Hilbert [7], Edmunds-Evans [10]). The second inequalities in (3.5) for Φ_k are essentially due to the work by Li [22]. Repeated application of the estimates of solutions of the Poisson equations (cf. Evans [11]), the higher order estimates can be deduced.

Proof of Proposition 1 and Proposition 2

We first prove that there is a uniform bound for $\Phi_{k,\epsilon}^R$ in $\Omega(\epsilon)$. From Lemma 5-(3.1), there exists $\epsilon_0(k) > 0$ such that

$$0 \leqq \lambda_k^R(\epsilon) \leqq \lambda_k + 1 \quad \text{for} \quad 0 < \epsilon \leqq \epsilon_0(k). \tag{3.6}$$

Put $m_2 = \lambda_k + 2$ and apply Lemma 4 for this m_2. Then we have $m_1 > 0, r_1 > 0$, $\epsilon_1 > 0$ and a function $\psi_\epsilon(x)$ in $B(M, r_1) \setminus B(M, \epsilon)$ such that (2.17)-(2.18) hold. On the other hand we have the estimate (3.3) in Lemma 6 for $r = r_1$. Combining these estimates, we have

$$-(c(k,r_1)+1)\psi_\epsilon(x) < \Phi_{k,\epsilon}(x) < (c(k,r_1)+1)\psi_\epsilon(x) \quad (x \in \Gamma(M,r_1),\ 0 < \epsilon \leqq \epsilon_1). \tag{3.7}$$

Using the conditions for $\Phi_{k,\epsilon}^R$ and ψ_ϵ, we have

$$\Delta \Phi_{k,\epsilon}^R + \lambda_k^R(\epsilon)\Phi_{k,\epsilon}^R = 0 \quad \text{in} \quad B(M,r_1) \setminus B(M,\epsilon), \quad \frac{\partial \Phi_{k,\epsilon}^R}{\partial \nu} + \sigma\epsilon^\tau \Phi_{k,\epsilon}^R = 0 \quad \text{on} \quad \Gamma(M,\epsilon)$$

$$\Delta\psi_\epsilon + m_2\psi_\epsilon \leqq 0, \quad 1 \leqq \psi_\epsilon(x) \leqq 3 \quad \text{in} \quad B(M,r_1)\setminus B(M,\epsilon), \quad \frac{\partial \psi_\epsilon}{\partial \nu} \geqq 0 \quad \text{on} \quad \Gamma(M,\epsilon),$$

we can claim that

$$-(c(k,r_1)+1)\,\psi_\epsilon(x) \leqq \Phi_{k,\epsilon}^R(x) \leqq (c(k,r_1)+1)\psi_\epsilon(x) \text{ in } \quad B(M,r_1)\setminus B(M,\epsilon) \tag{3.8}$$

for $0 < \epsilon \leqq \epsilon_1$. Actually we first note that

$$t|\Phi_{k,\epsilon}^R(x)| \leqq (c(k,r_1)+1)\psi_\epsilon(x) \quad \text{in} \quad B(M,r_1)\setminus B(M,\epsilon) \tag{3.9}$$

holds if the positive parameter t is small for each ϵ. Then define

$$\alpha_* = \sup\{\alpha \in (0,\infty) \mid (3.9) \text{ holds for any } t \in [0,\alpha]\}$$

which is positive. We note that α_* may depend on ϵ. If $\alpha_* \geqq 1$, the claim (3.8) holds uniformly for $0 < \epsilon \leqq \epsilon_1$. Assume the contrary, that is $0 < \alpha_* < 1$. Put non-negative functions

$$\Psi^{\pm}(x) = (c(k,r_1)+1)\psi_\epsilon(x) \pm \alpha_*\Phi_{k,\epsilon}^R(x) \quad \text{in} \quad B(M,r_1)\setminus B(M,\epsilon),$$

and then we have, from the definition of α_* that the following (i) or (ii) holds.

(i) $\Psi^+(x)$ attains 0 in $\overline{B(M,r_1)} \setminus B(M,\epsilon)$,

(ii) $\Psi^-(x)$ attains 0 in $\overline{B(M,r_1)} \setminus B(M,\epsilon)$.

A simple calculation gives

$$\Delta\Psi^{\pm} + \lambda_k^R(\epsilon)\Psi^{\pm} \leqq -(m_2 - \lambda_k^R(\epsilon))(c(k,r_1)+1)\psi_\epsilon < 0 \quad \text{in} \quad B(M,r_1)\setminus B(M,\epsilon),$$

$$\frac{\partial \Psi^{\pm}}{\partial \nu} + \sigma\epsilon^\tau\Psi^{\pm} \geqq \sigma\epsilon^\tau(c(k,r_1)+1)\psi_\epsilon \geqq 0 \quad \text{on} \quad \Gamma(M,\epsilon), \quad \Psi^{\pm} > 0 \quad \text{on} \quad \Gamma(M,r_1).$$

We used the property of ψ_ϵ in Lemma 4 and (3.6). From the maximum principle or the Hopf maximum principle (cf. Gilbarg-Trudinger [13], Protter-Weinberger [32]), both cases (i) and (ii) are contradiction. Therefore the assumption $0 < \alpha_* < 1$ is false. Thus we conclude

$\alpha_* \geqq 1$ and we have established the estimate (3.8) for $0 < \epsilon \leqq \epsilon_1$. Using the property of ψ_ϵ with Lemma 4, we conclude Proposition 2-(1.6). The completely same argument applies to $\Phi^N_{k,\epsilon}$.

Next we deal with the convergence of $\Phi^R_{k,\epsilon}$. From the conditions

$$\|\Phi^R_{k,\epsilon}\|_{L^2(\Omega(\epsilon))} = 1, \quad \|\nabla\Phi^R_{k,\epsilon}\|^2_{L^2(\Omega(\epsilon))} + \sigma\epsilon^\tau \int_{\Gamma(M,\epsilon)} (\Phi^R_{k,\epsilon})^2 dS = \lambda^R_k(\epsilon) \leqq \lambda_k + 1 \quad (0 < \epsilon \leqq \epsilon_1),$$

$\{\Phi^R_{k,\epsilon}\}_{0<\epsilon\leqq r}$ are bounded in $H^1(\Omega \setminus B(M,r))$ for any $r \in (0,\epsilon_1]$. We use the Rellich theorem and the weak relative compactness of bounded sequence in a Hilbert space with the diagonal argument. We can conclude the followings.

For any sequence of positive values $\{\epsilon_s\}_{s=1}$, there exist a subsequence $\{\zeta_s\}^\infty_{s=1}$, values $\{\lambda'_k\}^\infty_{k=1}$ and functions $\{\Phi'_k\}^\infty_{k=1}$ in $\Omega \setminus M$ such that

$$\lim_{s\to\infty} \lambda^R_k(\zeta_s) = \lambda'_k, \quad \lim_{s\to\infty} \|\Phi^R_{k,\zeta_s} - \Phi'_k\|_{L^2(\Omega\setminus B(M,r))} = 0 \quad \text{for} \quad r > 0,$$

$$\lim_{s\to\infty} \Phi^R_{k,\zeta_s} = \Phi'_k \quad \text{weakly in} \quad H^1(\Omega \setminus B(M,r)) \quad \text{for} \quad r > 0,$$

$$\lambda'_k = \liminf_{s\to\infty} \lambda^R_k(\zeta_s) \geqq \liminf_{s\to\infty} \int_{\Omega\setminus B(M,r)} |\nabla\Phi^R_{k,\zeta_s}|^2 dx \geqq \int_{\Omega\setminus B(M,r)} |\nabla\Phi'_k|^2 dx \text{ for } r > 0, \tag{3.10}$$

$$\Delta\Phi'_k + \lambda'_k\Phi'_k = 0 \text{ in } \Omega \setminus M, \quad \Phi'_k = 0 \text{ on } \Gamma. \tag{3.11}$$

From (1.6),

$$1 \geqq \int_{\Omega\setminus B(M,r)} (\Phi^R_{k,\zeta_s})^2 dx = 1 - \int_{B(M,r)\setminus B(M,\epsilon)} (\Phi^R_{k,\zeta_s})^2 dx \geqq 1 - |B(M,r)|\, c(k)^2$$

Put $\epsilon = \zeta_s$ and take $s \to \infty$ and get

$$1 \geqq \int_{\Omega\setminus B(M,r)} (\Phi'_k)^2 dx \geqq 1 - |B(M,r)|\, c(k)^2$$

for any $r > 0$. Since $\mathrm{codim}(M) \geqq 2$ and the measure of M is zero, we get $\int_\Omega (\Phi'_k)^2 dx = 1$. From a similar arguments gives $(\Phi'_{k_1}, \Phi'_{k_2})_{L^2(\Omega)} = \delta(k_1,k_2)$ for $k_1, k_2 \in \mathbb{N}$. From (1.6), each Φ'_k is bounded in $\Omega \setminus M$. So we apply the removable singularity argument to the equation (3.11) (cf. Jimbo-Kosugi [17](Section 8)) and obtain that Φ'_k is smooth in Ω. Taking $r \to 0$ in (3.10), we have

$$\int_\Omega |\nabla\Phi'_k|^2 dx \leqq \lambda'_k.$$

On the other hand, in view of the orthonormal property of $\{\Phi'_k\}^\infty_{k=1}$ and the Max-Min principle characterization for λ_k,

$$\int_\Omega |\nabla\Phi'_k|^2 dx \geqq \lambda_k.$$

we obtain $\lambda'_k \geqq \lambda_k$ for $k \in \mathbb{N}$. Eventually we get $\lambda'_k = \lambda_k$ for $k \in \mathbb{N}$, by Lemma 5. Since the choice of $\{\epsilon_s\}^\infty_{s=1}$ (satisfying $\lim_{s\to\infty}\epsilon_s = 0$) is arbitrary, We obtain the conclusion of Proposition 1 for $\lambda^R_k(\epsilon)$. A same argument also applies to $\lambda^N_k(\epsilon)$ and $\Phi^N_{k,\epsilon}$. □

4. Proof of Theorem 1

In this section we deal with the case $q \geqq 3$ to prove Theorem 1. We first construct an approximate eigenfunction $\widetilde{\Phi}_{k,\epsilon}$, which is given by modifying Φ_k in $B(M, r_0) \setminus B(M, \epsilon)$ with taking account of the Robin boundary condition. For the approximation, we need a correction term. We consider

$$\left\{ \begin{array}{l} \Delta_\eta \phi = 0 \quad \text{for} \quad \epsilon < |\eta| < r_0, \quad \phi = 0 \quad \text{for} \quad |\eta| = r_0, \\ \left(\dfrac{\partial \phi}{\partial \nu_\eta} + \sigma \epsilon^\tau \phi \right)_{|\eta|=\epsilon} = \left(\dfrac{\partial}{\partial \nu_\eta} \Phi_k(\xi + \eta \cdot \boldsymbol{e}(\xi)) + \sigma \epsilon^\tau \Phi_k(\xi + \eta \cdot \boldsymbol{e}(\xi)) \right)_{|\eta|=\epsilon} \end{array} \right. \tag{4.1}$$

for each $\xi \in M$. Here $\Delta_\eta = \partial^2/\partial \eta_1^2 + \cdots + \partial^2/\partial \eta_q^2$.

This is the Laplace equation in a $q-$dimensional spherical shell domain and it can be solved by a kind of the Fourier series expansion method. We use the polar coordinate $\eta = r\,\omega$ in $\mathbb{R}^q$. Each of $r^\ell \varphi_{\ell,p}(\omega)$, $r^{-\ell-q+2}\varphi_{\ell,p}(\omega)$ $(\ell \geqq 0, 1 \leqq p \leqq \iota(\ell))$ is a harmonic function in $\mathbb{R}^q \setminus \{\mathbf{0}\}$. Here $\{\varphi_{\ell,p}\}$ is a system of the spherical harmonic functions in S^{q-1} and it is the complete system of the eigenfunctions of the Laplace-Beltrami operator in S^{q-1} and in other words it is given through the following eigenvalue problem (cf. Shimakura [35]).

$$\Delta_\omega \varphi + \gamma\, \varphi = 0 \quad \text{in} \quad S^{q-1} \tag{4.2}$$

Actually it is known that the eigenvalue $\gamma(\ell)$ with its multiplicity $\iota(\ell)$ are given as follows

$$\gamma(\ell) = \ell(\ell + q - 2),\ \iota(\ell) = \frac{(2\ell + q - 2)(q + \ell - 3)!}{(q-2)!\ell!}) \quad (\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)) \tag{4.3}$$

and the corresponding eigenfunctions $\{\varphi_{\ell,p}\}_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)}$ are obtained by restricting all the harmonic polynomials to S^{q-1}. It is easy to see that $\iota(\ell)$ is a polynomial of order $q-2$.

All of these functions $r^\ell \varphi_{\ell,p}(\omega)$, $r^{-\ell-q+2}\varphi_{\ell,p}(\omega)$ $(\ell \geqq 0, 1 \leqq p \leqq \iota(\ell))$ form a complete basis of harmonic functions in the spherical shell region $\epsilon < |\eta| < r_0$. So the solution $\phi(\eta)$ of (4.1) is expressed by

$$\phi(\eta) = \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} (a_{l,p} r^\ell + b_{\ell,p} r^{-\ell-q+2}) \varphi_{\ell,p}(\omega) \quad (\epsilon < r < r_0, \omega \in S^{q-1}).$$

The coefficients $a_{\ell,p}$, $b_{\ell,p}$ can be calculated by the infinite series of relations determined by the boundary condition. From the boundary condition on $|\eta| = r_0$, we have

$$\sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} (a_{\ell,p} r_0^\ell + b_{\ell,p} r_0^{-\ell-q+2}) \varphi_{\ell,p}(\omega) = 0 \quad (\omega \in S^{q-1})$$

which gives

$$a_{\ell,p} r_0^\ell + b_{\ell,p} r_0^{-\ell-q+2} = 0 \quad \text{for} \quad \ell \geqq 0, 1 \leqq p \leqq \iota(\ell).$$

ϕ is written by

$$\phi(\eta) = \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} b_{\ell,p} (r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^\ell) \varphi_{\ell,p}(\omega) \quad (\epsilon < r < r_0, \omega \in S^{q-1}).$$

We calculate the Robin condition on $|\eta| = \epsilon$. Noting

$$\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial r} = -\sum_{i=1}^{q} \frac{\eta_i}{|\eta|} \frac{\partial}{\partial \eta_i} \quad \text{on} \quad \Gamma(M, \epsilon) = \{x = \xi + \eta \cdot e(\xi) \mid \xi \in M, |\eta| = \epsilon\}$$

we have the equations for the coefficients $a_{\ell,p}, b_{\ell,p}$ as follows.

$$-\sum_{\ell\geqq 0,1\leqq p\leqq\iota(\ell)} b_{\ell,p}\left((-\ell-q+2)r^{-\ell-q+1}-\ell r_0^{-2\ell-q+2}r^{\ell-1}\right)_{r=\epsilon}\varphi_{\ell,p}(\omega)$$

$$+\sum_{\ell\geqq 0,1\leqq p\leqq\iota(\ell)} b_{\ell,p}\sigma\epsilon^{\tau}\left(r^{-\ell-q+2}-r_0^{-2\ell-q+2}r^{\ell}\right)_{r=\epsilon}\varphi_{\ell,p}(\omega)$$

$$=-\sum_{i=1}^{q}\langle\nabla\Phi_k(\xi+(\epsilon\omega)\cdot e(\xi)),e_i(\xi)\rangle\eta_i/|\eta|+\sigma\epsilon^{\tau}\Phi_k(\xi+(\epsilon\omega)\cdot e(\xi))$$

for $\omega\in S^{q-1}$. Multiply both sides by $\varphi_{p,\ell}$ and integrate on S^{q-1} and we get

$$b_{\ell,p}\left\{(\ell+q-2)\epsilon^{-\ell-q+1}+\ell r_0^{-2\ell-q+2}\epsilon^{\ell-1}+\sigma(\epsilon^{-\ell-q+2+\tau}-r_0^{-2\ell-q+2}\epsilon^{\ell+\tau})\right\}$$

$$=\int_{S^{q-1}}\left\{-\sum_{i=1}^{q}\{(\nabla\Phi_k(\xi+(\epsilon\omega)\cdot e(\xi)),e_i(\xi))\omega_i\}+\sigma\epsilon^{\tau}\Phi_k(\xi+(\epsilon\omega)\cdot e(\xi))\right\}\varphi_{\ell,p}(\omega)d\omega$$

We used $\omega_i=\eta_i/|\eta|$. From these equations we get $a_{\ell,p}, b_{\ell,p}$ as follows

$$a_{\ell,p}=-r_0^{-2\ell-q+2}b_{\ell,p}$$

$$b_{\ell,p}=\frac{1}{(\ell+q-2)\epsilon^{-\ell-q+1}+\ell r_0^{-2\ell-q+2}\epsilon^{\ell-1}+\sigma(\epsilon^{-\ell-q+2+\tau}-r_0^{-2\ell-q+2}\epsilon^{\ell+\tau})} \tag{4.4}$$

$$\times\int_{S^{q-1}}\left\{-\sum_{i=1}^{q}\{(\nabla\Phi_k(\xi+(\epsilon\omega)\cdot e(\xi)),e_i(\xi))\omega_i\}+\sigma\epsilon^{\tau}\Phi_k(\xi+(\epsilon\omega)\cdot e(\xi))\right\}\varphi_{\ell,p}(\omega)d\omega$$

We remark that these ($\epsilon-$dependent) coefficients $a_{\ell,p}$, $b_{\ell,p}$ are smoothly dependent on $\xi\in M$ since Φ_k is smooth. So we denote this function $\phi(x)$ in $B(M,r_0)\setminus B(M,\epsilon)$ by $G_{k,\epsilon}(x)$. That is

$$G_{k,\epsilon}(x)=\sum_{\ell\geqq 0,1\leqq p\leqq\iota(\ell)} b_{\ell,p}(r^{-\ell-q+2}-r_0^{-2\ell-q+2}r^{\ell})\varphi_{\ell,p}(\omega)\quad (x=\xi+(r\omega)\cdot e(\xi)).$$

Definition 4. The approximate eigenfunction is defined by

$$\widetilde{\Phi}_{k,\epsilon}(x)=\begin{cases}\Phi_k(x) & \text{for}\quad x\in\Omega\setminus B(M,r_0),\\ \Phi_k(x)-G_{k,\epsilon}(x) & \text{for}\quad x=\xi+\eta\cdot e(\xi)\in B(M,r_0)\setminus B(M,\epsilon).\end{cases}$$

For later use, we need to look into the detailed behavior of the correction term $G_{k,\epsilon}$. Decompose $G_{k,\epsilon}$ into two parts.

$$G_{k,\epsilon}^{(1)}(x)=b_{0,1}(r^{-q+2}-r_0^{-q+2})\varphi_{0,1}(\omega)+\sum_{p=1}^{q}b_{1,p}(r^{-q+1}-r_0^{-q}r)\varphi_{1,p}(\omega), \tag{4.5}$$

$$G_{k,\epsilon}^{(2)}(x)=\sum_{\ell\geqq 2,\ 1\leqq p\leqq\iota(\ell)} b_{\ell,p}(r^{-\ell-q+2}-r_0^{-2\ell-q+2}r^{\ell})\varphi_{\ell,p}(\omega). \tag{4.6}$$

Note that $\iota(0)=1,\iota(1)=q$ with

$$\varphi_{0,1}(\omega)=1/|S^{q-1}|^{1/2},\varphi_{1,p}(\omega)=(q^{1/2}/|S^{q-1}|^{1/2})\omega_p\quad (1\leqq p\leqq q). \tag{4.7}$$

It is also known that there exists constant $c_4(s) > 0$ such that

$$\|\nabla_\omega^s \varphi_{\ell,p}\|_{L^\infty(S^{q-1})} \leqq c_4\,\gamma(\ell)^{(q-1)/2+s}. \quad (\ell \geqq 0, 1 \leqq p \leqq \iota(\ell), s \geqq 0). \tag{4.8}$$

See Li [22] for the proof (cf. Jimbo-Morita [18](Appendix)).

For the perturbation of the eigenvalue, the first several terms are most important and we need to evaluate and estimate these terms. In the calculation below we use

$$\int_{S^{q-1}} \omega_j\, d\omega = 0, \quad \int_{S^{q-1}} \omega_i\omega_j d\omega = \delta(i,j)|S^{q-1}|/q \quad (1 \leqq i,j \leqq q).$$

The case $\ell = 0$:

$$\int_{S^{q-1}} \left\{ -\sum_{i=1}^{q} \{(\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi))\omega_i\}\varphi_{0,1}(\omega) \right\} d\omega \tag{4.9}$$

$$= -\sum_{i=1}^{q} \int_{S^{q-1}} \{(\nabla\Phi_k(\xi), e_i(\xi))\omega_i + \sum_{j=1}^{q}((\nabla^2\Phi_k(\xi)e_j(\xi), e_i(\xi))\omega_i\omega_j\epsilon + O(\epsilon^2))\}\frac{1}{|S^{q-1}|^{1/2}} d\omega$$

$$= -\frac{|S^{q-1}|^{1/2}}{q}\sum_{i=1}^{q}(\nabla^2\Phi_k(\xi)e_i(\xi), e_i(\xi))\,\epsilon + O(\epsilon^2)$$

$$\sigma\epsilon^\tau \int_{S^{q-1}} \Phi_k(\xi + (\epsilon\omega)\cdot e(\xi))\varphi_{0,1}(\omega)d\omega = \sigma\epsilon^\tau|S^{q-1}|^{1/2}(\Phi_k(\xi) + O(\epsilon)) \tag{4.10}$$

The case $\ell = 1$:

$$\int_{S^{q-1}} \left\{ -\sum_{i=1}^{q} \{(\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi))\omega_i\}\varphi_{1,p}(\omega) \right\} d\omega \tag{4.11}$$

$$= -\frac{q^{1/2}}{|S^{q-1}|^{1/2}} \sum_{1\leqq i,p\leqq q} \int_{S^{q-1}} \{(\nabla\Phi_k(\xi), e_i(\xi))\omega_i\omega_p + O(\epsilon)\}d\omega$$

$$= -\frac{|S^{q-1}|^{1/2}}{q^{1/2}}(\nabla\Phi_k(\xi), e_p(\xi)) + O(\epsilon)$$

$$\sigma\epsilon^\tau \int_{S^{q-1}} \Phi_k(\xi + (\epsilon\omega)\cdot e(\xi))\varphi_{1,p}(\omega)d\omega \tag{4.12}$$

$$= \sigma\epsilon^\tau \frac{q^{1/2}}{|S^{q-1}|^{1/2}} \int_{S^{q-1}} (\Phi_k(\xi) + \sum_{i=1}^{q}\langle\nabla\Phi_k(\xi), e_i(\xi)\rangle\omega_i\epsilon + O(\epsilon^2))\omega_p\, d\omega$$

$$= \sigma\epsilon^{\tau+1}\frac{|S^{q-1}|^{1/2}}{q^{1/2}}(\langle\nabla\Phi_k(\xi), e_p(\xi)\rangle + O(\epsilon))$$

Lemma 7. *(i)* $\ell = 0$

$$b_{0,1} = |S^{q-1}|^{1/2} \begin{cases} \frac{-\epsilon^q}{q(q-2)}\{\sum_{i=1}^q \langle e_i(\xi), \nabla^2\Phi_k(\xi)e_i(\xi)\rangle + O(\epsilon)\} & (\tau > 1) \\ \frac{\epsilon^q}{q-2}\{(-1/q)\sum_{i=1}^q \langle e_i(\xi), \nabla^2\Phi_k(\xi)e_i(\xi)\rangle + \sigma\Phi_k(\xi) + O(\epsilon)\} & (\tau = 1) \\ \frac{\sigma\epsilon^{q-1+\tau}}{q-2}(\Phi_k(\xi) + O(\epsilon)) & (-1 < \tau < 1) \\ \frac{\sigma\epsilon^{q-2}}{q-2+\sigma}(\Phi_k(\xi) + O(\epsilon)) & (\tau = -1) \\ \epsilon^{q-2}(\Phi_k(\xi) + O(\epsilon)) & (\tau < -1) \end{cases}$$

(ii) $\ell = 1$

$$b_{1,p} = \frac{|S^{q-1}|^{1/2}}{q^{1/2}}\langle \nabla\Phi_k(\xi), e_p(\xi)\rangle \epsilon^q(1 + O(\epsilon)) \times \begin{cases} -1/(q-1) & (\tau > -1) \\ (\sigma - 1)/(q - 1 + \sigma) & (\tau = -1) \\ 1 & (\tau < -1) \end{cases}$$

Remark 7. Since Φ_k is smooth, the estimates in $b_{0,1}$, $b_{1,p}$ are uniform in $\xi \in M$. Moreover we easily see that

$$|\nabla_\xi^s b_{0,1}| = \begin{cases} O(\epsilon^q) & (\tau \geqq 1), \\ O(\epsilon^{q+\tau-1}) & (-1 < \tau < 1), \\ O(\epsilon^{q-2}) & (\tau \leqq -1), \end{cases}, \quad |\nabla_\xi^s b_{1,p}| = O(\epsilon^q) \quad (-\infty < \tau < \infty).$$

Notation. In the above lemma, we used the notation

$$\langle e(\xi), \nabla^2\Phi_k(\xi)e(\xi)\rangle = \sum_{i=1}^q \langle e_i(\xi), \nabla^2\Phi_k(\xi)e_i(\xi)\rangle.$$

(Proof of Lemma 7) (i)

$$b_{0,1} = \frac{1}{(q-2)\epsilon^{-q+1} + \sigma(\epsilon^{-q+2+\tau} - r_0^{-q+2}\epsilon^\tau)} \times \frac{|S^{q-1}|^{1/2}}{q}$$
$$\times \left(-\sum_{i=1}^q (\nabla^2\Phi_k(\xi)e_i(\xi), e_i(\xi))\,\epsilon + O(\epsilon^2) + q\sigma\epsilon^\tau(\Phi_k(\xi) + O(\epsilon)) \right)$$

From this expression, the asymptotics of $b_{0,1}$ in (i) follows depending on the ranges $\tau > 1$, $\tau = 1$, $-1 < \tau < 1$, $\tau = -1$, $\tau < -1$.

(ii)

$$b_{1,p} = \frac{1}{(q-1)\epsilon^{-q} + r_0^{-q} + \sigma(\epsilon^{-q+1+\tau} - r_0^{-q}\epsilon^{\tau+1})} \times \frac{|S^{q-1}|^{1/2}}{q^{1/2}}$$
$$\times \left(-(\nabla\Phi_k(\xi), e_p(\xi)) + O(\epsilon) + \sigma\epsilon^{\tau+1}(\langle\nabla\Phi_k(\xi), e_p(\xi)\rangle + O(\epsilon)) \right)$$

From this expression, the asymptotics of $b_{1,p}$ in (ii) follows depending on the ranges $\tau > -1$, $\tau = -1$, $\tau < -1$.

Lemma 8. *For any $N \in \mathbb{N}$, there exists $d_N > 0$ (independent of $\xi \in M$) such that*

$$|b_{\ell,p}| \leqq \frac{d_N}{\gamma(\ell)^N} \begin{cases} \epsilon^{\ell+q} & (\tau \geqq 0) \\ \epsilon^{\ell+q+\tau} & (-1 < \tau < 0) \\ \epsilon^{\ell+q-1} & (\tau \leqq -1) \end{cases}, \quad (1 \leqq p \leqq \iota(\ell), \ell \geqq 2). \tag{4.13}$$

We also have the following estimates.

Remark 8. For any $N, s \in \mathbb{N}$, there exists $d_{N,s}$ such that

$$|\nabla_\xi^s b_{\ell,p}| \leqq \frac{d_{N,s}}{\gamma(\ell)^N} \begin{cases} \epsilon^{\ell+q} & (\tau \geqq 0) \\ \epsilon^{\ell+q+\tau} & (-1 < \tau < 0) \\ \epsilon^{\ell+q-1} & (\tau \leqq -1) \end{cases} .$$

(Proof of Lemma 8) Substituting $\varphi_{\ell,p} = (-\Delta_\omega)^N \varphi_{\ell,p}/\gamma(\ell)^N$ into the right hand side of (4.4)

$$b_{\ell,p} = \frac{1}{(\ell+q-2)\epsilon^{-\ell-q+1} + \ell r_0^{-2\ell-q+2}\epsilon^{\ell-1} + \sigma(\epsilon^{-\ell-q+2+\tau} - r_0^{-2\ell-q+2}\epsilon^{\ell+\tau})} \times \frac{1}{\gamma(\ell)^N} \times I(\ell,p)$$

where

$$I(\ell,p) = \int_{S^{q-1}} \left\{ -\sum_{i=1}^{q} \{(\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi))\omega_i\} + \sigma\epsilon^\tau \Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)) \right\} (-\Delta_\omega)^N \varphi_{\ell,p} d\omega$$

which is calculated through partial integration

$$I(\ell,p) = \int_{S^{q-1}} \varphi_{\ell,p}\, (-\Delta_\omega)^N \left\{ -\sum_{i=1}^{q} \{(\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi))\omega_i\} + \sigma\epsilon^\tau \Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)) \right\} d\omega.$$

Note that ω_i is the constant multiple of the eigenfunction $\varphi_{1,i}(\omega)$ and so $\int_{S^{q-1}} \varphi_{\ell,p}(\omega)\,\omega_i\, d\omega = 0$ for $\ell \geqq 2$, $1 \leqq p \leqq \iota(\ell)$. We use the Taylor expansion at $\epsilon = 0$ and see that the integration of the zero-th order term vanishes in $I(\ell,p)$. So we get

$$\int_{S^{q-1}} \varphi_{\ell,p}(\omega)(-\Delta_\omega)^N \left\{ -\sum_{i=1}^{q} \{(\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi))\omega_i\} \right\} d\omega = O(\epsilon),$$

$$\int_{S^{q-1}} \varphi_{\ell,p}(\omega)(-\Delta_\omega)^N \{\sigma\epsilon^\tau \Phi_k(\xi + (\epsilon\omega)\cdot e(\xi))\}\, d\omega = O(\epsilon^{\tau+1}).$$

Using these estimates in the right hand side of $b_{\ell,p}$, we get the conclusion of Lemma 8. □

With the estimates in Lemma 8 and $\gamma(\ell) = \ell(\ell+q-2)$ (since $q \geqq 3$), with (4.8), $G_{k,\epsilon}^{(2)}(x)$ is uniformly convergent in $B(M, r_0) \setminus B(M, \epsilon)$ for each $\epsilon > 0$. This property is also true for $\nabla^p G_\epsilon^{(2)}(x)$ for any $p \in \mathbb{N}$.

Remark 9. We note that the estimates in Lemma 7, Lemma 8 are uniform in $\xi \in M$.

[Calculation for the detailed asymptotic behavior for $\lambda_k^R(\epsilon)$]

Here in this proof, we denote $\lambda_k^R(\epsilon)$, $\Phi_{k,\epsilon}^R$ by $\lambda_k(\epsilon)$, $\Phi_{k,\epsilon}$ for simplicity of notation. First we apply Proposition 2. Take any positive sequence $\{\epsilon_p\}_{p=1}^\infty$ with $\epsilon_p \to 0$ (for $p \to \infty$). There exist a subsequence $\{\zeta_s\}_{s=1}^\infty$ $\{\epsilon_p\}_{p=1}^\infty$ and an orthonormal system of eigenfunctions $\{\Phi_k'\}_{k=1}^\infty$ of (1.1) corresponding to $\{\lambda_k\}_{k=1}^\infty$ with (1.7)-(1.8). From now we take any $k \in \mathbb{N}$ and fix it and assume that λ_k is simple eigenvalue of (1.1) (as in the assumption of Theorem 1). So we should note that

$$\Phi_k'(x) = \Phi_k(x) \quad \text{or} \quad \Phi_k'(x) = -\Phi_k(x).$$

Now we begin the calculation of $\lambda_k(\epsilon)$ with the equation (1.3), we have

$$\int_{\Omega(\epsilon)} (\Delta\Phi_{k,\epsilon} + \lambda_k(\epsilon)\Phi_{k,\epsilon})\Phi\, dx = 0 \quad (\forall \Phi \in H^1(\Omega(\epsilon)))$$

which leads to

$$\int_{\partial\Omega(\epsilon)} \frac{\partial \Phi_{k,\epsilon}}{\partial \nu}\Phi\, dS - \int_{\Omega(\epsilon)} \nabla\Phi_{k,\epsilon}\nabla\Phi\, dx + \lambda_k(\epsilon)\int_{\Omega(\epsilon)} \Phi_{k,\epsilon}\Phi\, dx = 0.$$

Then by putting $\Phi = \widetilde{\Phi}_{k,\epsilon}$ and multiplying -1, we have

$$\sigma\epsilon^\tau \int_{\partial\Omega(\epsilon)} \Phi_{k,\epsilon}\widetilde{\Phi}_{k,\epsilon}\, dS + \int_{\Omega(\epsilon)} \nabla\Phi_{k,\epsilon}\nabla\widetilde{\Phi}_{k,\epsilon}\, dx - \lambda_k(\epsilon)\int_{\Omega(\epsilon)} \Phi_{k,\epsilon}\widetilde{\Phi}_{k,\epsilon}\, dx = 0$$

$$\sigma\epsilon^\tau \int_{\Gamma(M,\epsilon)} \Phi_{k,\epsilon}\widetilde{\Phi}_{k,\epsilon}\, dS + \int_{B(M,r_0)\setminus B(M,\epsilon)} \nabla\Phi_{k,\epsilon}\nabla\widetilde{\Phi}_{k,\epsilon}\, dx$$
$$+ \int_{\Omega(\epsilon)\setminus B(M,r_0)} \nabla\Phi_{k,\epsilon}\nabla\widetilde{\Phi}_{k,\epsilon}\, dx - \lambda_k(\epsilon)\int_{\Omega(\epsilon)} \Phi_{k,\epsilon}\widetilde{\Phi}_{k,\epsilon}\, dx = 0$$

$$\sigma\epsilon^\tau \int_{\Gamma(M,\epsilon)} \Phi_{k,\epsilon}\widetilde{\Phi}_{k,\epsilon}\, dS + \int_{\Gamma(M,\epsilon)\cup\Gamma(M,r_0)} \Phi_{k,\epsilon}\frac{\partial\widetilde{\Phi}_{k,\epsilon}}{\partial\nu_1} dS - \int_{B(M,r_0)\setminus B(M,\epsilon)} \Phi_{k,\epsilon}\Delta\widetilde{\Phi}_{k,\epsilon}\, dx$$
$$+ \int_{\Gamma(M,r_0)} \Phi_{k,\epsilon}\frac{\partial\widetilde{\Phi}_{k,\epsilon}}{\partial\nu_2} dS - \int_{\Omega(\epsilon)\setminus B(M,r_0)} \Phi_{k,\epsilon}\Delta\widetilde{\Phi}_{k,\epsilon}\, dx - \lambda_k(\epsilon)\int_{\Omega(\epsilon)} \Phi_{k,\epsilon}\widetilde{\Phi}_{k,\epsilon}\, dx = 0$$

In the above, we denoted the unit outward normal vector on $\partial(B(M,r_0)\setminus B(M,\epsilon))$ by ν_1.

Thus we obtain the following basic relation about $\lambda_k(\epsilon) - \lambda_k$.

$$(\lambda_k(\epsilon) - \lambda_k)\int_{\Omega(\epsilon)} \Phi_k(x)\Phi_{k,\epsilon}(x)dx = I_1(\epsilon) + I_2(\epsilon) + I_3(\epsilon) + I_4(\epsilon), \tag{4.14}$$

where

$$I_1(\epsilon) = -\int_{\Gamma(M,r_0)} \frac{\partial G_{k,\epsilon}}{\partial\nu_1}(\Phi_{k,\epsilon}(x) - \Phi_k'(x))dS, \tag{4.15}$$

$$I_2(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} G_{k,\epsilon}(x)(\Delta\Phi_k'(x) + \lambda_k(\epsilon)\Phi_{k,\epsilon}(x))dx, \tag{4.16}$$

$$I_3(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} (\Delta G_{k,\epsilon}(x))(\Phi_{k,\epsilon}(x) - \Phi_k'(x))dx, \tag{4.17}$$

$$I_4(\epsilon) = \int_{\Gamma(M,\epsilon)} \left(\frac{\partial G_{k,\epsilon}}{\partial\nu_1}\Phi_k' - G_{k,\epsilon}\frac{\partial\Phi_k'}{\partial\nu_1}\right) dS. \tag{4.18}$$

From the boundary condition

$$(\partial G_{k,\epsilon}/\partial\nu_1) + \sigma\epsilon^\tau G_{k,\epsilon} = (\partial\Phi_k/\partial\nu_1) + \sigma\epsilon^\tau\Phi_k \quad \text{on} \quad \Gamma(M,\epsilon),$$

$I_4(\epsilon)$ is also written as

$$I_4(\epsilon) = \int_{\Gamma(M,\epsilon)} \left((\frac{\partial\Phi_k}{\partial\nu_1} + \sigma\epsilon^\tau\Phi_k)\Phi_k' - G_{k,\epsilon}(\frac{\partial\Phi_k'}{\partial\nu_1} + \sigma\epsilon^\tau\Phi_k')\right) dS. \tag{4.19}$$

In the left hand side of (4.14), we note from (1.6), (1.8),

$$\lim_{s\to\infty}\int_{\Omega(\zeta_s)}\Phi_k(x)\Phi_{k,\zeta_s}(x)dx=\int_\Omega\Phi_k(x)\Phi_k'(x)dx\neq 0.$$

From now we estimate and evaluate the behaviors of the terms $I_1(\epsilon)$, $I_2(\epsilon)$, $I_3(\epsilon)$, $I_4(\epsilon)$ of the right hand side of (4.14) for $\epsilon=\zeta_s$ $(s\to\infty)$. For those purposes we carry out discussions through several lemmas.

Lemma 9. *There exists $c_k>0$ (independet of $\xi\in M$) such that*

$$|G^{(2)}_{k,\epsilon}(x)|\leqq c_k\times\begin{cases}\epsilon^2 & (\tau\geqq 0)\\ \epsilon^{2+\tau} & (-1<\tau<0)\\ \epsilon & (\tau\leqq -1)\end{cases}\quad (x\in\Gamma(M,\epsilon))\tag{4.20}$$

$$\left|\frac{\partial G^{(2)}_{k,\epsilon}}{\partial\nu_1}(x)\right|\leqq c_k\times\begin{cases}\epsilon & (\tau\geqq 0)\\ \epsilon^{\tau+1} & (-1<\tau<0)\\ 1 & (\tau\leqq -1)\end{cases}\quad (x\in\Gamma(M,\epsilon))\tag{4.21}$$

$$\left|\frac{\partial G^{(2)}_{k,\epsilon}}{\partial\nu_1}(x)\right|\leqq c_k\times\begin{cases}\epsilon^{q+2} & (\tau\geqq 0)\\ \epsilon^{q+\tau+2} & (-1<\tau<0)\\ \epsilon^{q+1} & (\tau\leqq -1)\end{cases}\quad (x\in\Gamma(M,r_0))\tag{4.22}$$

(Proof of Lemma 9) Proof for (4.20): To see the behavior of $G^{(2)}_{k,\epsilon}(x)$ for $x\in\Gamma(M,\epsilon)$, we substitute $r=\epsilon$ and we have

$$G^{(2)}_{k,\epsilon}(x)=\sum_{\ell\geqq 2,1\leqq p\leqq\iota(\ell)}b_{\ell,p}(\epsilon^{-\ell-q+2}-r_0^{-2\ell-q+2}\epsilon^\ell)\varphi_{\ell,p}(\omega).$$

Since $0\leqq\epsilon^{-\ell-q+2}-r_0^{-2\ell-q+2}\epsilon^\ell\leqq\epsilon^{-\ell-q+2}$ (because $0<\epsilon<r_0$), we have

$$|G^{(2)}_{k,\epsilon}(x)|\leqq\sum_{\ell\geqq 2,1\leqq p\leqq\iota(\ell)}|b_{\ell,p}|\epsilon^{-\ell-q+2}|\varphi_{\ell,p}(\omega)|\leqq\sum_{\ell\geqq 2,1\leqq p\leqq\iota(\ell)}|b_{\ell,p}|\epsilon^{-\ell-q+2}c_4\gamma(\ell)^{(q-1)/2}$$

$$\leqq\sum_{\ell\geqq 2}c_4\gamma(\ell)^{(q-1)/2}\iota(\ell)\frac{d_N}{\gamma(\ell)^N}\epsilon^{-\ell-q+2}\begin{cases}\epsilon^{\ell+q} & (\tau\geqq 0)\\ \epsilon^{\ell+q+\tau} & (-1<\tau<0)\ ,\\ \epsilon^{\ell+q-1} & (\tau\leqq -1)\end{cases}$$

$$\leqq\sum_{\ell\geqq 2}c_4d_N\frac{\iota(\ell)}{\gamma(\ell)^{N-(q-1)/2}}\begin{cases}\epsilon^2 & (\tau\geqq 0)\\ \epsilon^{2+\tau} & (-1<\tau<0)\\ \epsilon & (\tau\leqq -1)\end{cases}\quad (x\in\Gamma(M,\epsilon)).$$

We used Lemma 8-(4.13). Fix N adequately large so that $\sum_{\ell\geqq 2}\iota(\ell)/\gamma(\ell)^{N-(q-1)/2}<\infty$. This estimate gives (4.20).

Proof for (4.21): For $x\in\Gamma(M,\epsilon)$, we calculate $\partial G^{(2)}_{k,\epsilon}/\partial\nu_1$ by putting $r=\epsilon$.

$$\left|\frac{\partial G_{k,\epsilon}}{\partial\nu_1}(x)\right|=\left|-\sum_{\ell\geqq 2,\ 1\leqq p\leqq\iota(\ell)}b_{\ell,p}((-\ell-q+2)\epsilon^{-\ell-q+1}-\ell r_0^{-2\ell-q+2}\epsilon^{\ell-1})\varphi_{\ell,p}(\omega)\right|$$

$$\leqq \sum_{\ell\geqq 2,\ 1\leqq p\leqq\iota(\ell)} |b_{\ell,p}|((\ell+q-2)\epsilon^{-\ell-q+1}+\ell r_0^{-2\ell-q+2}\epsilon^{\ell-1})|\varphi_{\ell,p}(\omega)|$$

$$\leqq \sum_{\ell\geqq 2}\iota(\ell)c_4\gamma(\ell)^{(q-1)/2}((\ell+q-2)\epsilon^{-\ell-q+1}+\ell r_0^{-2\ell-q+2}\epsilon^{\ell-1})\frac{d_N}{\gamma(\ell)^N}\begin{cases}\epsilon^{\ell+q} & (\tau\geqq 0)\\ \epsilon^{\ell+q+\tau} & (-1<\tau<0)\\ \epsilon^{\ell+q-1} & (\tau\leqq -1)\end{cases},$$

$$\leqq \sum_{\ell\geqq 2} c_4 d_N((\ell+q-2)+\ell r_0^{-2\ell-q+2}\epsilon^{q-1})\frac{\iota(\ell)}{\gamma(\ell)^{N-(q-1)/2}}\begin{cases}\epsilon & (\tau\geqq 0)\\ \epsilon^{1+\tau} & (-1<\tau<0)\\ 1 & (\tau\leqq -1)\end{cases},$$

Since $0<\epsilon<r_0$, if we fix N adequately large so that

$$\sum_{\ell\geqq 2}((\ell+q-2)+\ell r_0^{-2\ell-q+2}r_0^{q-1})\frac{\iota(\ell)}{\gamma(\ell)^{N-(q-1)/2}}<\infty,$$

this estimate gives (4.21).

Proof for (4.22): For $x\in\Gamma(M,r_0)$, we calculate $\partial G^{(2)}_{k,\epsilon}/\partial\nu_1$ by putting $r=r_0$.

$$\left|\frac{\partial G^{(2)}_{k,\epsilon}}{\partial\nu_1}(x)\right| = \left|\sum_{\ell\geqq 2,\ 1\leqq p\leqq\iota(\ell)} b_{\ell,p}((-\ell-q+2)r_0^{-\ell-q+1}-\ell r_0^{-2\ell-q+2}r_0^{\ell-1})\varphi_{\ell,p}(\omega)\right|$$

$$\leqq \sum_{\ell\geqq 2,\ 1\leqq p\leqq\iota(\ell)} |b_{\ell,p}|(2\ell+q-2)r_0^{-\ell-q+1})|\varphi_{\ell,p}(\omega)|$$

$$\leqq \sum_{\ell\geqq 2} d_N c_4(2\ell+q-2)r_0^{-\ell-q+1}\frac{\iota(\ell)}{\gamma(\ell)^{N-(q-1)/2}}\begin{cases}\epsilon^{\ell+q} & (\tau\geqq 0)\\ \epsilon^{\ell+q+\tau} & (-1<\tau<0)\\ \epsilon^{\ell+q-1} & (\tau\leqq -1)\end{cases}$$

$$=\sum_{\ell\geqq 2} d_N c_4(\epsilon/r_0)^{\ell-2}r_0^{-q-1}\frac{(2\ell+q-2)\iota(\ell)}{\gamma(\ell)^{N-(q-1)/2}}\begin{cases}\epsilon^{q+2} & (\tau\geqq 0)\\ \epsilon^{q+\tau+2} & (-1<\tau<0)\\ \epsilon^{q+1} & (\tau\leqq -1)\end{cases}.$$

Since $0<\epsilon<r_0$, if we fix N adequately large so that

$$\sum_{\ell\geqq 2}\frac{(2\ell+q-2)\iota(\ell)}{\gamma(\ell)^{N-(q-1)/2}}<\infty,$$

this estimate gives (4.22). □

Lemma 10. *For $j=1,2,3$, we have*

$$I_j(\epsilon)=\begin{cases} o(\epsilon^q) & (\tau\geqq 1)\\ o(\epsilon^{q+\tau-1}) & (-1<\tau<1) \quad \textit{for} \quad \epsilon=\zeta_s.\\ o(\epsilon^{q-2}) & (\tau\leqq -1)\end{cases} \tag{4.23}$$

(Proof of Lemma 10) Proof for $I_1(\epsilon)$:

$$|I_1(\epsilon)| \leqq |\Gamma(M,r_0)| \left(\sup_{x\in\Gamma(M,r_0)} |\frac{\partial G_{k,\epsilon}}{\partial \nu_1}| \right) \left(\sup_{x\in\Gamma(M,r_0)} |\Phi_{k,\epsilon}(x) - \Phi'_k(x)| \right)$$

Using the estimates about $b_{0,1}, b_{1,p}$ in Lemma 4.1, we have

$$|(\partial G^{(1)}_{k,\epsilon}/\partial\nu_1)_{|\Gamma(M,r_0)}| \leqq c \begin{cases} \epsilon^q & (\tau \geqq 1) \\ \epsilon^{q+\tau-1} & (-1<\tau<1) \\ \epsilon^{q-2} & (\tau \leqq -1) \end{cases} \tag{4.24}$$

where c is a constant independent of ϵ and ξ. From the above estimate and Lemma 9-(4.22), $(\partial G_{k,\epsilon}/\partial \nu_1)_{|\Gamma(M,r_0)}$ satisfies the same estimate as (4.24). Then we get the estimate for $I_1(\epsilon)$ by using

$$\lim_{s\to\infty} \sup_{x\in\Gamma(M,r_0)} |\Phi_{k,\zeta_s}(x) - \Phi'_k(x)| = 0.$$

Proof for $I_2(\epsilon)$: From $\Delta\Phi'_k = -\lambda_k\Phi'_k$, $I_2(\epsilon)$ is rewritten as

$$\begin{aligned} I_2(\epsilon) &= \int_{B(M,r_0)\setminus B(M,\epsilon)} G_{k,\epsilon}(x)(-\lambda_k\Phi'_k(x)+\lambda_k(\epsilon)\Phi_{k,\epsilon}(x))dx \\ &= \sum_{\ell\geqq 0, 1\leqq p\leqq \iota(\ell)} \int_M\int_\epsilon^{r_0}\int_{S^{q-1}} b_{\ell,p}(r^{-\ell-q+2} - r_0^{-2\ell-q+2}r^\ell)\varphi_{\ell,p}(\omega) \\ &\quad \times(-\lambda_k\Phi'_k(x)+\lambda_k(\epsilon)\Phi_{k,\epsilon}(x)) \times \rho_2(\xi,r,\omega)r^{q-1}d\omega dr ds(\xi) \end{aligned} \tag{4.25}$$

Here we put $\overline{\rho} = \sup_{x\in B(M,r_0)}\rho_2(x)$. Since $0 \leqq (r^{-\ell-q+2} - r_0^{-2\ell-q+2}r^\ell)r^{q-1} \leqq r^{-\ell+1}$ for $0<r\leqq r_0$, we have

$$\begin{aligned} |I_2(\epsilon)| &\leqq \overline{\rho} \sum_{\ell\geqq 0,1\leqq p\leqq\iota(\ell)} \int_M\int_\epsilon^{r_0}\int_{S^{q-1}} |b_{\ell,p}|\,|\varphi_{\ell,p}(\omega)||-\lambda_k\Phi'_k(x)+\lambda_k(\epsilon)\Phi_{k,\epsilon}(x)|r^{-\ell+1}d\omega dr ds(\xi) \\ &\leqq \overline{\rho} \sum_{\ell\geqq 0,1\leqq p\leqq\iota(\ell)} c_4\gamma(\ell)^{(q-1)/2}\int_M\int_\epsilon^{r_0}\int_{S^{q-1}} |b_{\ell,p}|\,|-\lambda_k\Phi'_k(x)+\lambda_k(\epsilon)\Phi_{k,\epsilon}(x)|r^{-\ell+1}d\omega dr ds(\xi) \\ &= I_{2,1}(\epsilon)+I_{2,2}(\epsilon)+I_{2,3}(\epsilon). \end{aligned}$$

The above terms $I_{2,1}(\epsilon)$, $I_{2,2}(\epsilon)$, $I_{2,3}(\epsilon)$ are given below.

$$I_{2,1}(\epsilon) = \overline{\rho}c_4\gamma(0)^{(q-1)/2}\int_M\int_\epsilon^{r_0}\int_{S^{q-1}} r\,|b_{0,1}|\,|-\lambda_k\Phi'_k(x)+\lambda_k(\epsilon)\Phi_{k,\epsilon}(x)|d\omega dr ds(\xi)$$

$$I_{2,2}(\epsilon) = \overline{\rho}c_4\gamma(1)^{(q-1)/2}\sum_{1\leqq p\leqq q}\int_M\int_\epsilon^{r_0}\int_{S^{q-1}} |b_{1,p}|\,|-\lambda_k\Phi'_k(x)+\lambda_k(\epsilon)\Phi_{k,\epsilon}(x)|d\omega dr ds(\xi)$$

$$I_{2,3}(\epsilon) = \overline{\rho}c_4\sum_{\ell\geqq 2,1\leqq p\leqq\iota(\ell)}\gamma(\ell)^{(q-1)/2}\int_M\int_\epsilon^{r_0}\int_{S^{q-1}} |b_{\ell,p}|\,|-\lambda_k\Phi'_k(x)+\lambda_k(\epsilon)\Phi_{k,\epsilon}(x)|r^{-\ell+1}d\omega dr ds(\xi).$$

[The estimate for $I_{2,1}(\epsilon)$] Using $b_{0,1}$ in Lemma 7, we have, for $\tau\geqq 1$,

$$I_{2,1}(\epsilon)/\epsilon^q = \overline{\rho}c_4\gamma(0)^{(q-1)/2}\int_M\int_\epsilon^{r_0}\int_{S^{q-1}} r\,|(b_{0,1}/\epsilon^q)|\,|-\lambda_k\Phi'_k(x)+\lambda_k(\epsilon)\Phi_{k,\epsilon}(x)|d\omega dr ds(\xi).$$

Put $\epsilon = \zeta_s$ and take $s \to \infty$ and apply Lebesgue bounded convergence theorem and we get

$$\lim_{s\to\infty} I_{2,1}(\zeta_s)/\zeta_s^q = 0 \quad \text{for} \quad \tau \geqq 1.$$

Similarly we have

$$\lim_{s\to\infty} I_{2,1}(\zeta_s)/\zeta_s^{q+\tau-1} = 0 \quad \text{for} \quad -1 < \tau < 1, \quad \lim_{s\to\infty} I_{2,1}(\zeta_s)/\zeta_s^{q-2} = 0 \quad \text{for} \quad \tau \leqq -1.$$

[The estimate for $I_{2,2}(\epsilon)$] Applying the same argument we get

$$\lim_{s\to\infty} I_{2,2}(\zeta_s)/\zeta_s^q = 0 \quad \text{for any} \quad \tau \in (-\infty, \infty).$$

[The estimate for $I_{2,3}(\epsilon)$]

$$\frac{I_{2,3}(\epsilon)}{\epsilon^q} \leqq \left(\overline{\rho} c_4 d_N \sum_{\ell \geqq 2} \iota(\ell)\gamma(\ell)^{-N+(q-1)/2} \right)$$

$$\times \int_M \int_\epsilon^{r_0} \int_{S^{q-1}} |-\lambda_k \Phi_k'(x) + \lambda_k(\epsilon)\Phi_{k,\epsilon}(x)| d\omega dr ds(\xi) \times \epsilon \text{ for } \tau \geqq 1.$$

Take N adequately large so that $\sum_{\ell\geqq 2} \iota(\ell)\gamma(\ell)^{-N+(q-1)/2} < \infty$ and get $\lim_{\epsilon\to 0} I_{2,3}(\epsilon)/\epsilon^q = 0$ for $\tau \geqq 1$. Similarly we get

$$\lim_{\epsilon\to 0} \frac{I_{2,3}(\epsilon)}{\epsilon^{q+\tau-1}} = 0 \quad \text{for} \quad -1 < \tau < 1, \quad \lim_{\epsilon\to 0} \frac{I_{2,3}(\epsilon)}{\epsilon^{q-2}} = 0 \quad \text{for} \quad \tau \leqq -1.$$

Summing up these estimates we get the estimates for $I_2(\epsilon)$ in Lemma 10.
Proof for $I_3(\epsilon)$: Using the expression of the Laplacian in (2.10) in the local coordinate system $(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_q)$, we have

$$\Delta G_{k,\epsilon}(x) = J_1(G_{k,\epsilon}) + J_2(G_{k,\epsilon}) + J_3(G_{k,\epsilon}) + J_4(G_{k,\epsilon})$$

where

$$J_1(G_{k,\epsilon}) = \frac{1}{\sqrt{g(\xi,\eta)}} \sum_{1\leqq i,j\leqq m} \frac{\partial}{\partial \xi_i} \left(\sqrt{g(\xi,\eta)} g^{ij}(\xi,\eta) \frac{\partial G_{k,\epsilon}}{\partial \xi_j} \right) \tag{4.26}$$

$$= \frac{1}{\sqrt{g}} \sum_{1\leqq i,j\leqq m} \frac{\partial}{\partial \xi_i} \left(\sqrt{g} g^{ij} \right) \frac{\partial G_{k,\epsilon}}{\partial \xi_j} + \sum_{1\leqq i,j\leqq m} g^{ij} \frac{\partial^2 G_{k,\epsilon}}{\partial \xi_i \partial \xi_j}$$

$$J_2(G_{k,\epsilon}) = \frac{1}{\sqrt{g(\xi,\eta)}} \sum_{1\leqq i\leqq m, 1\leqq \ell\leqq q} \frac{\partial}{\partial \xi_i} \left(\sqrt{g(\xi,\eta)} g^{i,m+\ell}(\xi,\eta) \frac{\partial G_{k,\epsilon}}{\partial \eta_\ell} \right) \tag{4.27}$$

$$= \frac{1}{\sqrt{g}} \sum_{1\leqq i\leqq m, 1\leqq \ell\leqq q} \frac{\partial}{\partial \xi_i} \left(\sqrt{g} g^{i,m+\ell} \right) \frac{\partial G_{k,\epsilon}}{\partial \eta_\ell} + \sum_{1\leqq i\leqq m, 1\leqq \ell\leqq q} g^{i,m+\ell} \frac{\partial}{\partial \eta_\ell} \left(\frac{\partial G_{k,\epsilon}}{\partial \xi_i} \right)$$

$$J_3(G_{k,\epsilon}) = \frac{1}{\sqrt{g(\xi,\eta)}} \sum_{1\leqq j\leqq m, 1\leqq \ell\leqq q} \frac{\partial}{\partial \eta_\ell} \left(\sqrt{g(\xi,\eta)} g^{m+\ell,j}(\xi,\eta) \frac{\partial G_{k,\epsilon}}{\partial \xi_j} \right) \tag{4.28}$$

$$= \frac{1}{\sqrt{g}} \sum_{1\leqq j\leqq m, 1\leqq \ell\leqq q} \frac{\partial}{\partial \eta_\ell} \left(\sqrt{g} g^{m+\ell,j} \right) \frac{\partial G_{k,\epsilon}}{\partial \xi_j} + \sum_{1\leqq j\leqq m, 1\leqq \ell\leqq q} g^{m+\ell,j} \frac{\partial}{\partial \eta_\ell} \left(\frac{\partial G_{k,\epsilon}}{\partial \xi_j} \right)$$

$$J_4(G_{k,\epsilon}) = \frac{1}{\sqrt{g(\xi,\eta)}} \sum_{1\leqq \ell_1,\ell_2\leqq q} \frac{\partial}{\partial \eta_{\ell_1}} \left(\sqrt{g(\xi,\eta)} g^{m+\ell_1,m+\ell_2}(\xi,\eta) \frac{\partial G_{k,\epsilon}}{\partial \eta_{\ell_2}} \right) \tag{4.29}$$

$$= \frac{1}{\sqrt{g}} \sum_{1\leqq \ell_1,\ell_2\leqq q} \frac{\partial}{\partial \eta_\ell} \left(\sqrt{g} g^{m+\ell_1,m+\ell_2}\right) \frac{\partial G_{k,\epsilon}}{\partial \eta_{\ell_2}} + \sum_{1\leqq \ell_1,\ell_2\leqq q} (g^{m+\ell_1,m+\ell_2}(\xi,\eta) - \delta(\ell_1,\ell_2)) \frac{\partial^2 G_{k,\epsilon}}{\partial \eta_{\ell_2}\partial \eta_{\ell_1}}$$

We used $\sum_{i=1}^q \partial^2 G_{k,\epsilon}/\partial \eta_i^2 = 0$ in the last line of J_4.

$$I_3(\epsilon) = I_{3,1}(\epsilon) + I_{3,2}(\epsilon) + I_{3,3}(\epsilon) + I_{3,4}(\epsilon) \tag{4.30}$$

where

$$I_{3,1}(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} J_1(G_{k,\epsilon})(\Phi_{k,\epsilon} - \Phi'_k)dx$$
$$I_{3,2}(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} J_2(G_{k,\epsilon})(\Phi_{k,\epsilon} - \Phi'_k)dx$$
$$I_{3,3}(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} J_3(G_{k,\epsilon})(\Phi_{k,\epsilon} - \Phi'_k)dx$$
$$I_{3,4}(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} J_4(G_{k,\epsilon})(\Phi_{k,\epsilon} - \Phi'_k)dx.$$

To deal with these four terms in (4.30), we need to look at several terms of first and second derivatives of $G_{k,\epsilon}$. We consider the law of the change of the variable as follows.

$$\frac{\partial r}{\partial \eta_i} = \frac{\eta_i}{r}, \quad \frac{\partial \omega}{\partial \eta_i} = \frac{1}{r}(a_i - \omega_i\,\omega)$$

Here a_i is the unit vector in $\mathbb{R}^q$ whose $i-$th component is 1 while other components are all zero. Note that $\eta_i/r = \omega_i$.

$$\frac{\partial^2 r}{\partial \eta_i \partial \eta_j} = \frac{1}{r}(\delta(i,j) - \omega_i\omega_j), \quad \frac{\partial^2 \omega}{\partial \eta_i \partial \eta_j} = \frac{1}{r^2}(\delta(i,j)\omega + \omega_i\omega_j\omega - \omega_i a_j - \omega a_i)$$

By using this transformation of the variables, we get the expression of the higher order derivatives of $G_{k,\epsilon}$ w.r.t. ξ_i, η_j $(1 \leqq i \leqq m,\ 1 \leqq j \leqq q)$.

$$\frac{\partial G_{k,\epsilon}}{\partial \eta_i} = \frac{\partial G_{k,\epsilon}}{\partial r}\omega_i + \frac{1}{r}\langle \nabla_\omega G_{k,\epsilon}, (a_i - \omega_i\omega)\rangle \tag{4.31}$$

$$\frac{\partial^2 G_{k,\epsilon}}{\partial \eta_i \partial \eta_j} = \frac{\partial^2 G_{k,\epsilon}}{\partial r^2}\omega_i\omega_j + \frac{1}{r}\frac{\partial G_{k,\epsilon}}{\partial r}(\delta(i,j) - \omega_i\omega_j) + \frac{1}{r}\langle \frac{\partial}{\partial r}\nabla_\omega G_{k,\epsilon}, (a_i - \omega_i\omega)\rangle \tag{4.32}$$
$$+\frac{1}{r^2}\langle \nabla^2_\omega G_{k,\epsilon}(a_j - \omega_j\omega), (a_i - \omega_i\omega)\rangle + \frac{1}{r^2}\langle \nabla_\omega G_{k,\epsilon},\ \delta(i,j)\omega + \omega_i\omega_j\omega - \omega_i a_j - \omega_j a_i\rangle$$

Here

$$\nabla_\omega G_{k,\epsilon} = \sum_{\ell\geqq 0, 1\leqq p\leqq \iota(\ell)} b_{\ell,p}(r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^\ell)\nabla_\omega \varphi_{\ell,p}(\omega) \tag{4.33}$$

$$\nabla^2_\omega G_{k,\epsilon} = \sum_{\ell\geqq 0, 1\leqq p\leqq \iota(\ell)} b_{\ell,p}(r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^\ell)\nabla^2_\omega \varphi_{\ell,p}(\omega), \tag{4.34}$$

$$(4.35)\qquad \frac{\partial G_{k,\epsilon}}{\partial r} = \sum_{\ell\geqq 0, 1\leqq p\leqq \iota(\ell)} b_{\ell,p}((-\ell-q+2)r^{-\ell-q+1} - \ell r_0^{-2\ell-q+2} r^{\ell-1})\varphi_{\ell,p}(\omega),$$

$$(4.36)\qquad \frac{\partial^2 G_{k,\epsilon}}{\partial r^2} = \sum_{\ell\geqq 0, 1\leqq p\leqq \iota(\ell)} b_{\ell,p}((-\ell-q+2)(-\ell-q+1)r^{-\ell-q} - \ell(\ell-1) r_0^{-2\ell-q+2} r^{\ell-2})\varphi_{\ell,p}(\omega),$$

On the other hand, we have

$$(4.37)\qquad \frac{\partial G_{k,\epsilon}}{\partial \xi_i}(x) = \sum_{\ell\geqq 0, 1\leqq p\leqq \iota(\ell)} (\frac{\partial b_{\ell,p}}{\partial \xi_i})(r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^{\ell})\varphi_{\ell,p}(\omega),$$

$$(4.38)\qquad \frac{\partial^2 G_{k,\epsilon}}{\partial \xi_i \partial \xi_j}(x) = \sum_{\ell\geqq 0, 1\leqq p\leqq \iota(\ell)} (\frac{\partial^2 b_{\ell,p}}{\partial \xi_i \partial \xi_j})(r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^{\ell})\varphi_{\ell,p}(\omega).$$

From now, we estimate each term of $I_3(\epsilon)$ in (4.30). $I_{3,1}(\epsilon)$ can be dealt with similarly as in $I_2(\epsilon)$ because $\nabla_\xi G_{k,\epsilon}$ and $\nabla_\xi^2 G_{k,\epsilon}$ satisfy the same property (i.e. Lemma 7 and Remark 7).

Next we consider $|I_{3,2}(\epsilon)|$, $|I_{3,3}(\epsilon)|$,

$$|I_{3,2}(\epsilon)| \leqq \int_{B(M,r_0)\setminus B(M,\epsilon)} |J_2(G_{k,\epsilon})|\,|\Phi_{k,\epsilon} - \Phi_k'|dx,$$

$$|I_{3,3}(\epsilon)| \leqq \int_{B(M,r_0)\setminus B(M,\epsilon)} |J_3(G_{k,\epsilon})|\,|\Phi_{k,\epsilon} - \Phi_k'|dx$$

Both of $|J_2(G_{k,\epsilon})|$, $|J_3(G_{k,\epsilon})|$ are bounded by a fintite linear combination of $|\nabla_\eta G_{k,\epsilon}|$, $|\nabla_\eta(\nabla_\xi G_{k,\epsilon})|$. Moreover, $|\nabla_\eta G_{k,\epsilon}|$ is bounded by a finite linear combination of $|\partial_r G_{k,\epsilon}|$ and $(1/r)|\nabla_\omega G_{k,\epsilon}|$. $|\nabla_\eta(\nabla_\xi G_{k,\epsilon})|$ is bounded by a finite linear combination of $|\partial_r(\nabla_\xi G_{k,\epsilon})|$ and $(1/r)|\nabla_\xi G_{k,\epsilon}|$. So we estimate the following four quantities,

$$\int_{B(M,r_0)\setminus B(M,\epsilon)} |\partial_r G_{k,\epsilon}|\,|\Phi_{k,\epsilon} - \Phi_k'|dx,\ \int_{B(M,r_0)\setminus B(M,\epsilon)} (1/r)|\nabla_\omega G_{k,\epsilon}|\,|\Phi_{k,\epsilon} - \Phi_k'|dx$$

$$\int_{B(M,r_0)\setminus B(M,\epsilon)} |\partial_r(\nabla_\xi G_{k,\epsilon})|\,|\Phi_{k,\epsilon} - \Phi_k'|dx,\ \int_{B(M,r_0)\setminus B(M,\epsilon)} (1/r)|\nabla_\omega(\nabla_\xi G_{k,\epsilon})|\,|\Phi_{k,\epsilon} - \Phi_k'|dx$$

Using (4.33), (4.35), we can estimate these quantities (as in I_2) and get the same estimate. Each of this term has the asymptotic behavior

$$\begin{cases} o(\epsilon^q) & \text{for} \quad \tau \geqq 1 \\ o(\epsilon^{q+\tau-1}) & \text{for} \quad -1 < \tau < 1 \\ o(\epsilon^{q-2}) & \text{for} \quad \tau \leqq -1 \end{cases}$$

Accordingly we obtain

$$I_{3,2}(\epsilon) + I_{3,3}(\epsilon) = \begin{cases} o(\epsilon^q) & \text{for} \quad \tau \geqq 1 \\ o(\epsilon^{q+\tau-1}) & \text{for} \quad -1 < \tau < 1\ . \\ o(\epsilon^{q-2}) & \text{for} \quad \tau \leqq -1 \end{cases}$$

$$|I_{3,4}(\epsilon)| \leqq \int_{B(M,r_0)\setminus B(M,\epsilon)} |J_4(G_{k,\epsilon})|\,|\Phi_{k,\epsilon} - \Phi_k'|dx.$$

In the expression of J_4, the following inequality

$$|g^{m+\ell_1,m+\ell_2}(\xi,\eta)-\delta(\ell_1,\ell_2)| \leqq c|\eta| = c\,r.$$

holds where c is a constant which depends on M and r_0 (cf. Lemma 2). From (4.29), $|J_4(G_{k,\epsilon})|$ is bounded by a finite linear combination of $|\nabla_\eta G_{k,\epsilon}|$ and $r\,|\nabla^2_\eta G_{k,\epsilon}|$. Moreover

We need to estimate $\int_{B(M,r_0)\setminus B(M,\epsilon)} r\,|h_\epsilon(x)||\Phi_{k,\epsilon}-\Phi'_k|dx$ where $h_\epsilon(x)$ is one of the following terms

$$\frac{\partial^2 G_{k,\epsilon}}{\partial r^2},\quad \frac{1}{r}\frac{\partial G_{k,\epsilon}}{\partial r},\quad \frac{1}{r}\frac{\partial}{\partial r}(\nabla_\omega G_{k,\epsilon}),\quad \frac{1}{r^2}\nabla^2_\omega G_{k,\epsilon},\quad \frac{1}{r^2}\nabla_\omega G_{k,\epsilon}.$$

Eventually we have obtained

$$I_{3,4}(\epsilon)=\begin{cases} o(\epsilon^q) & \text{for} \quad \tau \geqq 1 \\ o(\epsilon^{q+\tau-1}) & \text{for} \quad -1<\tau<1\ . \\ o(\epsilon^{q-2}) & \text{for} \quad \tau\leqq -1 \end{cases}$$

Summing up these estimate we have completed the proof of Lemma 10. □

This lemma asserts that $I_1(\epsilon), I_2(\epsilon), I_3(\epsilon)$ can be absorbed in higher order tems and are negligible in the Theorem 1. So we will see that all the important terms the theorem come from $I_4(\epsilon)$.

Lemma 11.

$$\int_{\Gamma(M,\epsilon)} \frac{\partial \Phi_k}{\partial \nu_1}\Phi'_k dS = -\frac{|S^{q-1}|\epsilon^q}{q}\int_M (\langle \nabla^2\Phi_k(\xi)e(\xi), e(\xi)\rangle \Phi'_k + \langle \nabla_N \Phi_k(\xi), \nabla_N\Phi'_k(\xi)\rangle)ds(\xi) + O(\epsilon^{q+1})$$

$$\int_{\Gamma(M,\epsilon)} \sigma\epsilon^\tau \Phi_k(x)\Phi'_k(x)dS = |S^{q-1}|\sigma\epsilon^{\tau+q-1}\int_M \Phi_k(\xi)\Phi'_k(\xi)ds(\xi) + O(\epsilon^{\tau+q})$$

Proof. These two asymptotics are deduced with the aid of the Taylor expansion around $\xi\in M$. Actually we expand two functions $\nabla\Phi_k(x+(\epsilon\omega)\cdot e(\xi))$ and $\Phi'_k(x+(\epsilon\omega)\cdot e(\xi))$ at $\epsilon = 0$ and carry out integration on $\Gamma(M,\epsilon)$.

□

Lemma 12.

$$\int_{\Gamma(M,\epsilon)} G^{(2)}_{k,\epsilon}(x)\frac{\partial \Phi'_k}{\partial \nu_1}dS = \begin{cases} o(\epsilon^q) & (\tau\geqq 1) \\ o(\epsilon^{q+\tau-1}) & (-1<\tau<1)\ , \\ o(\epsilon^{q-2}) & (\tau \leqq -1) \end{cases}$$

$$\int_{\Gamma(M,\epsilon)} G^{(2)}_{k,\epsilon}(x)\sigma\epsilon^\tau\Phi'_k dS = \begin{cases} o(\epsilon^q) & (\tau\geqq 1) \\ o(\epsilon^{q+\tau-1}) & (-1<\tau<1)\ . \\ o(\epsilon^{q-2}) & (\tau \leqq -1) \end{cases}$$

Proof. The estimates directly follow from Lemma 9-(4.20).

□

We begin to evaluate $I_4(\epsilon)$. Since $G_{k,\epsilon}(x) = G^{(1)}_{k,\epsilon}(x) + G^{(2)}_{k,\epsilon}(x)$, we have

$$I_4(\epsilon) = \int_{\Gamma(M,\epsilon)} (\frac{\partial \Phi_k}{\partial \nu_1} + \sigma\epsilon^\tau \Phi_k)\Phi'_k dS - \int_{\Gamma(M,\epsilon)} G^{(1)}_{k,\epsilon}(\frac{\partial \Phi'_k}{\partial \nu_1} + \sigma\epsilon^\tau \Phi'_k)dS - \int_{\Gamma(M,\epsilon)} G^{(2)}_{k,\epsilon}(\frac{\partial \Phi'_k}{\partial \nu_1} + \sigma\epsilon^\tau \Phi'_k)dS. \tag{4.39}$$

From Lemma 12, the last term in (4.39) is included in higher order terms. The first term, second term and third term are denoted by $I_{4,1}(\epsilon)$, $I_{4,2}(\epsilon)$ and $I_{4.3}(\epsilon)$, respectively. That is,

$$I_4(\epsilon) = I_{4.1}(\epsilon) + I_{4.2}(\epsilon) + I_{4,3}(\epsilon) \tag{4.40}$$

and $I_{4,3}(\epsilon)$ is a higher order term.

We calculate $I_{4,1}(\epsilon)$ with the aid of Lemma 11 in five different cases.

(i) $\tau > 1$, (ii) $\tau = 1$, (iii) $-1 < \tau < 1$, (iv) $\tau = -1$, (v) $\tau < -1$.

(i) $\tau > 1$:

$$\lim_{\epsilon \to 0} \frac{I_{4,1}(\epsilon)}{\epsilon^q} = -\frac{|S^{q-1}|}{q} \int_M (\langle \nabla^2 \Phi_k(\xi) e(\xi), e(\xi)\rangle \Phi'_k(\xi) + \langle \nabla_N \Phi_k(\xi), \nabla_N \Phi'_k(\xi)\rangle) ds(\xi)$$

(ii) $\tau = 1$:

$$\lim_{\epsilon \to 0} \frac{I_{4,1}(\epsilon)}{\epsilon^q} = -\frac{|S^{q-1}|}{q} \int_M (\langle \nabla^2 \Phi_k(\xi) e(\xi), e(\xi)\rangle \Phi'_k(\xi) + \langle \nabla_N \Phi_k(\xi), \nabla_N \Phi'_k(\xi)\rangle) ds(\xi) + |S^{q-1}|\sigma \int_M \Phi_k(\xi)\Phi'_k(\xi) ds(\xi)$$

(iii) $-1 < \tau < 1$:

$$\lim_{\epsilon \to 0} \frac{I_{4,1}(\epsilon)}{\epsilon^{q+\tau-1}} = |S^{q-1}|\sigma \int_M \Phi_k(\xi)\Phi'_k(\xi) ds(\xi)$$

(iv) $\tau = -1$:

$$\lim_{\epsilon \to 0} \frac{I_{4,1}(\epsilon)}{\epsilon^{q-2}} = |S^{q-1}|\sigma \int_M \Phi_k(\xi)\Phi'_k(\xi) ds(\xi)$$

(v) The case $\tau < -1$ is dealt with seperately later.

Next we calculate the second term $I_{4,2}(\epsilon)$ in $I_4(\epsilon)$.

$$\begin{aligned} -I_{4,2}(\epsilon) &= \int_{\Gamma(M,\epsilon)} G^{(1)}_{k,\epsilon}(\frac{\partial \Phi'_k}{\partial \nu_1} + \sigma\epsilon^\tau \Phi'_k)dS \\ &= \int_{\Gamma(M,\epsilon)} b_{0,1}(r^{-q+2} - r_0^{-q+2})\varphi_{0,1}(\omega)(\frac{\partial \Phi'_k}{\partial \nu_1} + \sigma\epsilon^\tau \Phi'_k)dS \\ &\quad + \sum_{p=1}^q \int_{\Gamma(M,\epsilon)} b_{1,p}(r^{-q+1} - r_0^{-q} r)\varphi_{1,p}(\omega)(\frac{\partial \Phi'_k}{\partial \nu_1} + \sigma\epsilon^\tau \Phi'_k)dS \\ &= \frac{1}{|S^{q-1}|^{1/2}} \int_{\Gamma(M,\epsilon)} b_{0,1}(\epsilon^{-q+2} - r_0^{-q+2})(\frac{\partial \Phi'_k}{\partial \nu_1} + \sigma\epsilon^\tau \Phi'_k)dS \\ &\quad + \frac{q^{1/2}}{|S^{q-1}|^{1/2}} \sum_{p=1}^q \int_{\Gamma(M,\epsilon)} b_{1,p}(\epsilon^{-q+1} - r_0^{-q}\epsilon)\omega_p(\frac{\partial \Phi'_k}{\partial \nu_1} + \sigma\epsilon^\tau \Phi'_k)dS \end{aligned} \tag{4.41}$$

$$= \frac{(\epsilon^{-q+2} - r_0^{-q+2})}{|S^{q-1}|^{1/2}} \int_{\Gamma(M,\epsilon)} b_{0,1}(\frac{\partial \Phi'_k}{\partial \nu_1} + \sigma\epsilon^\tau \Phi'_k) dS$$

$$+ \frac{q^{1/2}}{|S^{q-1}|^{1/2}} (\epsilon^{-q+1} - r_0^{-q}\epsilon) \sum_{p=1}^{q} \int_{\Gamma(M,\epsilon)} b_{1,p}\omega_p(\frac{\partial \Phi'_k}{\partial \nu_1} + \sigma\epsilon^\tau \Phi'_k) dS$$

By the way, we have

$$\int_{|\eta|=\epsilon} \frac{\partial \Phi'_k}{\partial \nu_1} dS'_\eta = -\int_{S^{q-1}} \sum_{i=1}^{q} \omega_i \langle \nabla \Phi'_k(\xi + (\epsilon\omega \cdot e(\xi))), e_i(\xi)\rangle \epsilon^{q-1} d\omega \tag{4.42}$$

$$= -\epsilon^q \sum_{i=1}^{q} \langle \nabla^2 \Phi'_k(\xi) e_i(\xi), e_i(\xi)\rangle \frac{|S^{q-1}|}{q} + O(\epsilon^{q+1}).$$

$$\int_{|\eta|=\epsilon} \omega_p \frac{\partial \Phi'_k}{\partial \nu_1} dS'_\eta = -\int_{S^{q-1}} \sum_{i=1}^{q} \omega_p \langle \nabla \Phi'_k(\xi + (\epsilon\omega \cdot e(\xi))), e_i(\xi)\rangle \omega_i \epsilon^{q-1} d\omega \tag{4.43}$$

$$= -\epsilon^{q-1} \langle \nabla \Phi'_k(\xi), e_p(\xi)\rangle \frac{|S^{q-1}|}{q} + O(\epsilon^q).$$

$$\int_{|\eta|=\epsilon} \sigma\epsilon^\tau \Phi'_k dS'_\eta = \sigma\epsilon^{\tau+q-1} |S^{q-1}| (\Phi'_k(\xi) + O(\epsilon)). \tag{4.44}$$

$$\int_{|\eta|=\epsilon} \sigma\epsilon^\tau \omega_p \Phi'_k dS'_\eta = \sigma\epsilon^{\tau+q} \frac{|S^{q-1}|}{q} (\langle \nabla \Phi'_k(\xi), e_p(\xi)\rangle + O(\epsilon)). \tag{4.45}$$

Substituting (4.42)-(4.45) into $I_{4,2}(\epsilon)$, we have

$$-I_{4,2}(\epsilon) =$$

$$\frac{(\epsilon^{-q+2} - r_0^{-q+2})}{|S^{q-1}|^{1/2}} \{\int_M b_{0,1}(-\epsilon^q \sum_{i=1}^{q} \langle \nabla^2 \Phi'_k(\xi) e_i(\xi), e_i(\xi)\rangle \frac{|S^{q-1}|}{q} + O(\epsilon^{q+1}))(1 + O(\epsilon)) ds(\xi)$$

$$+ \int_M \sigma\epsilon^{\tau+q-1} |S^{q-1}| b_{0,1}(\Phi'_k(\xi) + O(\epsilon)) ds(\xi)\}$$

$$+ \frac{q^{1/2}}{|S^{q-1}|^{1/2}} (\epsilon^{-q+1} - r_0^{-q}\epsilon) \sum_{p=1}^{q} \{\int_M b_{1,p}(-\epsilon^{q-1} \langle \nabla \Phi'_k(\xi), e_p(\xi)\rangle \frac{|S^{q-1}|}{q} + O(\epsilon^q)) ds(\xi)$$

$$+ \int_M \sigma\epsilon^{\tau+q} \frac{|S^{q-1}|}{q} b_{1,p}(\langle \nabla \Phi'_k(\xi), e_p(\xi)\rangle + O(\epsilon)) ds(\xi)\}$$

$$= \frac{(\epsilon^{-q+2} - r_0^{-q+2})|S^{q-1}|^{1/2}}{q} \{\int_M b_{0,1}(-\epsilon^q \sum_{i=1}^{q} \langle \nabla^2 \Phi'_k(\xi) e_i(\xi), e_i(\xi)\rangle + O(\epsilon^{q+1}))(1 + O(\epsilon)) ds(\xi)$$

$$+ \int_M q\sigma\epsilon^{\tau+q-1} b_{0,1}(\Phi'_k(\xi) + O(\epsilon)) ds(\xi)\}$$

$$+ \frac{|S^{q-1}|^{1/2}}{q^{1/2}} (\epsilon^{-q+1} - r_0^{-q}\epsilon) \sum_{p=1}^{q} \{\int_M b_{1,p}(-\epsilon^{q-1} \langle \nabla \Phi'_k(\xi), e_p(\xi)\rangle + O(\epsilon^q)) ds(\xi)$$

$$+\int_M \sigma\epsilon^{\tau+q} b_{1,p}(\langle\nabla\Phi_k'(\xi), e_p(\xi)\rangle + O(\epsilon))ds(\xi)\}.$$

Using Lemma 7, we can calculate the asymptotics of $I_{4,2}(\epsilon)$ seperately in five different cases:

$$\text{(i) } \tau > 1, \quad \text{(ii) } \tau = 1, \quad \text{(iii) } -1 < \tau < 1, \quad \text{(iv) } \tau = -1, \quad \text{(v) } \tau < -1.$$

(1) The case $\tau > 1$:

$$\lim_{\epsilon\to 0} \frac{-I_{4,2}(\epsilon)}{\epsilon^q} = \frac{|S^{q-1}|}{q(q-1)} \sum_{p=1}^{q} \int_M \langle\nabla\Phi_k(\xi), e_p(\xi)\rangle\langle\nabla\Phi_k'(\xi), e_p(\xi)\rangle ds(\xi)$$

$$= \frac{|S^{q-1}|}{q(q-1)} \int_M \langle\nabla_N\Phi_k(\xi), \nabla_N\Phi_k'(\xi)\rangle ds(\xi)$$

(2) The case $\tau = 1$:

$$\lim_{\epsilon\to 0} \frac{-I_{4,2}(\epsilon)}{\epsilon^q} = \frac{|S^{q-1}|}{q(q-1)} \sum_{p=1}^{q} \int_M \langle\nabla\Phi_k(\xi), e_p(\xi)\rangle\langle\nabla\Phi_k'(\xi), e_p(\xi)\rangle ds(\xi)$$

$$= \frac{|S^{q-1}|}{q(q-1)} \int_M \langle\nabla_N\Phi_k(\xi), \nabla_N\Phi_k'(\xi)\rangle ds(\xi)$$

(3) The case $-1 < \tau < 1$:

$$\lim_{\epsilon\to 0} \frac{-I_{4,2}(\epsilon)}{\epsilon^{q+\tau-1}} = 0$$

(4) The case $\tau = -1$:

$$\lim_{\epsilon\to 0} \frac{-I_{4,2}(\epsilon)}{\epsilon^{q-2}} = |S^{q-1}|\sigma \int_M \frac{\sigma}{q-2+\sigma} \Phi_k(\xi)\Phi_k'(\xi) ds(\xi)$$

(5) The case $\tau < -1$ is dealt with seperately.

We calculate the asymptotics of $I_4(\epsilon)$ by $I_{4,1}(\epsilon)$ and $I_{4,2}(\epsilon)$, we have

$$\lim_{\epsilon\to 0} \frac{I_4(\epsilon)}{\epsilon^q} = \lim_{\epsilon\to 0} \frac{I_{4,1}(\epsilon) + I_{4,2}(\epsilon) + I_{4,3}(\epsilon)}{\epsilon^q} \tag{4.46}$$

$$= -\frac{|S^{q-1}|}{q} \int_M (\langle\nabla^2\Phi_k(\xi)e(\xi), e(\xi)\rangle\Phi_k'(\xi) + \langle\nabla_N\Phi_k(\xi), \nabla_N\Phi_k'(\xi)\rangle) ds(\xi)$$

$$- \frac{|S^{q-1}|}{q(q-1)} \int_M \langle\nabla_N\Phi_k(\xi), \nabla_N\Phi_k'(\xi)\rangle ds(\xi)$$

$$= -\frac{|S^{q-1}|}{q} \int_M (\langle\nabla^2\Phi_k(\xi)e(\xi), e(\xi)\rangle\Phi_k'(\xi) + \frac{q}{q-1}\langle\nabla_N\Phi_k(\xi), \nabla_N\Phi_k'(\xi)\rangle) ds(\xi)$$

Using Proposition 3, we have

$$\langle\nabla^2\Phi_k(\xi)e(\xi), e(\xi)\rangle = \Delta\Phi_k - \Delta_M\Phi_k + H[\Phi_k] \quad \text{on} \quad M.$$

Substituting this relation into the above with the aid of

$$\Delta\Phi_k = -\lambda_k\Phi_k, \qquad \int_M (\Delta_M\Phi_k)\Phi_k' ds(\xi) = -\int_M \langle\nabla_M\Phi_k, \nabla_M\Phi_k'\rangle ds(\xi)$$

we obtain the formula

$$\lim_{\epsilon\to 0}\frac{I_4(\epsilon)}{\epsilon^q}=\frac{|S^{q-1}|}{q}\int_M\left\{-\frac{q}{q-1}\langle\nabla_N\Phi_k,\nabla_N\Phi'_k\rangle-\langle\nabla_M\Phi_k,\nabla_M\Phi'_k\rangle+\lambda_k\Phi_k\Phi'_k-\Phi'_kH[\Phi_k]\right\}ds(\xi).$$

which gives Theorem 1-(1).

For the case $\tau=1$, we can similarly calculate

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{I_4(\epsilon)}{\epsilon^q}&=-\frac{|S^{q-1}|}{q}\int_M(\langle\nabla^2\Phi_k(\xi)e(\xi),e(\xi)\rangle\Phi'_k(\xi)+\frac{q}{q-1}\langle\nabla_N\Phi_k(\xi),\nabla_N\Phi'_k(\xi)\rangle)ds(\xi)\\&\qquad+|S^{q-1}|\sigma\int_M\Phi_k(\xi)\Phi'_k(\xi)ds(\xi)\\&=\frac{|S^{q-1}|}{q}\int_M\left\{-\frac{q}{q-1}\langle\nabla_N\Phi_k,\nabla_N\Phi'_k\rangle-\langle\nabla_M\Phi_k,\nabla_M\Phi'_k\rangle-\Phi'_kH[\Phi_k]\right\}ds(\xi)\\&\qquad+|S^{q-1}|(\sigma+(\lambda_k/q))\int_M\Phi_k(\xi)\Phi'_k(\xi)ds(\xi)\end{aligned}$$

which gives Theorem 1-(2).

For the case $-1<\tau<1$.

$$\lim_{\epsilon\to 0}\frac{I_4(\epsilon)}{\epsilon^{q+\tau-1}}=\sigma|S^{q-1}|\int_M\Phi_k(\xi)\Phi'_k(\xi)ds(\xi)$$

which gives Theorem 1-(3).

For the case $\tau=-1$.

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{I_4(\epsilon)}{\epsilon^{q-2}}&=\sigma|S^{q-1}|\int_M\Phi_k(\xi)\Phi'_k(\xi)ds(\xi)-\sigma|S^{q-1}|\int_M\frac{\sigma}{q-2+\sigma}\Phi_k(\xi)\Phi'_k(\xi)ds(\xi)\\&=|S^{q-1}|\int_M\frac{\sigma(q-2)}{q-2+\sigma}\Phi_k(\xi)\Phi'_k(\xi)ds(\xi)\end{aligned}$$

which gives Theorem 1-(4).

For the case $\tau<-1$, we use (4.18).

$$I_4(\epsilon)=\int_{\Gamma(M,\epsilon)}\left(\frac{\partial G^{(1)}_{k,\epsilon}}{\partial\nu_1}\Phi'_k-G^{(1)}_{k,\epsilon}\frac{\partial\Phi'_k}{\partial\nu_1}\right)dS+\int_{\Gamma(M,\epsilon)}\left(\frac{\partial G^{(2)}_{k,\epsilon}}{\partial\nu_1}\Phi'_k-G^{(2)}_{k,\epsilon}\frac{\partial\Phi'_k}{\partial\nu_1}\right)dS\tag{4.47}$$

Due to Lemma 9, the second term is $O(\epsilon^{q-1})$. Evaluating the first term with

$$G^{(1)}_{k,\epsilon}(x)=b_{0,1}(\epsilon^{-q+2}-r_0^{-q+2})\varphi_{0,1}(\omega)$$

$$\left(\frac{\partial G_{k,\epsilon^{(1)}}}{\partial\nu_1}\right)_{r=\epsilon}=-b_{0,1}(-q+2)\epsilon^{-q+1}\varphi_{0,1}(\omega)-\sum_{p=1}^{q}b_{1,p}((-q+1)\epsilon^{-q}-r_0^{-q})\varphi_{1,p}(\omega)$$

$$b_{0,1}=|S^{q-1}|^{1/2}\epsilon^{q-2}(\Phi_k(\xi)+O(\epsilon)),\quad b_{1,p}=\frac{|S^{q-1}|^{1/2}}{q^{1/2}}\epsilon^q\langle\nabla\Phi_k(\xi),e_p(\xi)\rangle(1+O(\epsilon))$$

for $\tau<-1$, we have

$$\lim_{\epsilon\to 0}\frac{I_4(\epsilon)}{\epsilon^{q-2}}=(q-2)|S^{q-1}|\int_M\Phi_k(\xi)\Phi'_k(\xi)ds(\xi)$$

which leads to Theorem 1-(5).

We investigated the asymptotics of $(\lambda_k(\epsilon)-\lambda_k)$ for $\epsilon=\zeta_s$ and $s\to\infty$ through (4.14) and saw that the result does not depend on the choice of the original sequence of $\{\epsilon_p\}_{p=1}^{\infty}$. So the limit holds for $\epsilon\to 0$ in each case of Theorem 1.

The case of Neumann condition on $\Gamma(M,\epsilon)$ can be dealt with similarly as the case $\tau>1$ of the Robin condition on $\Gamma(M,\epsilon)$. Actually we can put $\sigma=0$ and rewrite the proof and deduce the formula of Theorem 1-(0). □

5. Proof of Theorem 2

In this section we deal with the case $q=2$ to prove Theorem 2. All the process of the proof of Theorem 2 is quite similar to the case of Theorem 1 ($q\geqq 3$). So we do not give a complete proof for Theorem 2, but give only the scketch.

First we construct an approximate eigenfunction $\widetilde{\Phi}_{k,\epsilon}$.

For the approximate $k-$th eigenfunction, we need to add a correction term to Φ_k (similarly as the case $q\geqq 3$). In the local coordinate $(\xi_1,\xi_2,\cdots,\xi_m,\eta_1,\eta_2)$ we consider the following Laplace equation for each $\xi\in M$.

$$\Delta_\eta\phi=0\quad\text{for}\quad \epsilon<|\eta|<r_0,\qquad \phi=0\quad\text{for}\quad |\eta|=r_0,$$

$$\left(\frac{\partial\phi}{\partial\nu_\eta}+\sigma\epsilon^\tau\phi\right)_{|\eta|=\epsilon}=\left(\frac{\partial}{\partial\nu_\eta}\Phi_k(\xi+\eta\cdot e(\xi))++\sigma\epsilon^\tau\Phi_k(\xi+\eta\cdot e(\xi))\right)_{|\eta|=\epsilon}$$

for each $\xi\in M$.

This is the Laplace equation in a $2-$dimensional annulus and it can be solved by a kind of the Fourier series expansion method. We use the polar coordinate $\eta=r\,\omega\in\mathbb{R}^2$, $\omega=\omega(\theta)=(\cos\theta,\sin\theta)$ in $\mathbb{R}^2$. $\omega_1=\cos\theta$, $\omega_2=\sin\theta$. The solution ϕ is expressed by

$$\phi(r,\theta)=a_0+b_0\log r+\sum_{\ell\geqq 1,p=1,2}(a_{\ell,p}r^\ell+b_{\ell,p}r^{-\ell})\varphi_{\ell,p}(\theta)\quad(\epsilon<r<r_0,\theta\in S^1).\tag{5.1}$$

Here $\varphi_{\ell,1}(\theta)=(1/\sqrt{\pi})\cos\ell\theta$, $\varphi_{\ell,2}(\theta)=(1/\sqrt{\pi})\sin\ell\theta$. Note that $\iota(\ell)=2$ for $\ell\geqq 1$, $\iota(0)=1$.

From the boundary condition on $\Gamma(M,r_0)$, we have

$$a_0+b_0\log r_0+\sum_{\ell\geqq 1,p=1,2}(a_{\ell,p}r_0^\ell+b_{\ell,p}r_0^{-\ell})\varphi_{\ell,p}(\theta)=0$$

which gives

$$a_0+b_0\log r_0=0,\quad a_{\ell,p}r_0^\ell+b_{\ell,p}r_0^{-\ell}=0\quad(\ell\geqq 1,p=1,2).$$
$$a_0=-b_0\log r_0,\,a_{\ell,p}=-b_{\ell,p}r_0^{-2\ell}$$
$$\phi(r,\theta)=b_0\log(r/r_0)+\sum_{\ell\geqq 1,\,p=1,2}b_{\ell,p}(r^{-\ell}-r_0^{-2\ell}r^\ell)\varphi_{\ell,p}(\theta)$$

From the Robin boundary condition on $|\eta|=\epsilon$,

$$-\left(\frac{b_0}{r}+\sum_{\ell\geqq 1,p=1,2}b_{\ell,p}(-\ell r^{-\ell-1}-\ell r_0^{-2\ell_0}r^{\ell-1})\varphi_{\ell,p}(\theta)\right)_{r=\epsilon}$$
$$+\sigma\epsilon^\tau\left(b_0\log(r/r_0)+\sum_{\ell\geqq 1,p=1,2}b_{\ell,p}(r^{-\ell}-r_0^{-2\ell}r^\ell)\varphi_{\ell,p}(\theta)\right)_{r=\epsilon}$$

$$= -\sum_{i=1}^{2}\langle \nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi)\rangle\omega_i + \sigma\epsilon^\tau\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi))$$

which is rewritten as

$$-\frac{b_0}{\epsilon} + \sum_{\ell \geqq 1, p=1,2} b_{\ell,p}(\ell\epsilon^{-\ell-1} + \ell r_0^{-2\ell_0}\epsilon^{\ell-1})\varphi_{\ell,p}(\omega)$$

$$+\sigma\epsilon^\tau\left(b_0\log(\epsilon/r_0) + \sum_{\ell\geqq 1, p=1,2} b_{\ell,p}(\epsilon^{-\ell} - r_0^{-2\ell}\epsilon^\ell)\varphi_{\ell,p}(\omega)\right)$$

$$= -\sum_{i=1}^{2}\langle \nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi)\omega + \sigma\epsilon^\tau\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi))$$

$$((-1/\epsilon) + \sigma\epsilon^\tau\log(\epsilon/r_0))b_0$$

$$= \frac{1}{2\pi}\int_{S^1}(-\sum_{i=1}^{2}\langle\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi)\rangle\omega_i + \sigma\epsilon^\tau\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)))d\theta$$

$$(\ell\epsilon^{-\ell-1} + \ell r_0^{-2\ell_0}\epsilon^{\ell-1} + \sigma\epsilon^\tau(\epsilon^{-\ell} - r_0^{-2\ell}\epsilon^\ell))b_{\ell,p}$$

$$= \int_{S^1}(-\sum_{i=1}^{2}\langle\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi)\rangle\omega_i + \sigma\epsilon^\tau\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)))\varphi_{\ell,p}(\theta)\,d\theta$$

$$b_0 = \frac{1}{(-1/\epsilon) + \sigma\epsilon^\tau\log(\epsilon/r_0)} \tag{5.2}$$

$$\times\frac{1}{2\pi}\int_{S^1}(-\sum_{i=1}^{2}\langle\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi)\rangle\omega_i + \sigma\epsilon^\tau\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)))d\theta$$

$$b_{\ell,p} = \frac{1}{\ell\epsilon^{-\ell-1} + \ell r_0^{-2\ell_0}\epsilon^{\ell-1} + \sigma\epsilon^\tau(\epsilon^{-\ell} - r_0^{-2\ell}\epsilon^\ell)} \tag{5.3}$$

$$\times\int_{S^1}(-\sum_{i=1}^{2}\langle\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi)\rangle\omega_i + \sigma\epsilon^\tau\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)))\varphi_{\ell,p}(\theta)\,d\theta$$

Here $\omega_1 = \cos\theta, \omega_2 = \sin\theta$.

The asymptotic behaviors of $b_{0,1}$, $b_{0,p}$ and the estimates of $b_{\ell,p}$ ($\ell \geqq 2$) in Lemma 13, Lemma 14 are proved similarly as in Lemma 7 and Lemma 8.

Lemma 13. *(i)* $\ell = 0$.

$$b_0 = \begin{cases} \pi\epsilon^2\{\langle e(\xi), \nabla^2\Phi_k(\xi)e(\xi)\rangle + O(\epsilon)\} & (\tau > 1) \\ \pi\epsilon^2\{\langle e(\xi), \nabla^2\Phi_k(\xi)e(\xi)\rangle - 2\sigma\Phi_k(\xi) + O(\epsilon)\} & (\tau = 1) \\ -2\sigma\pi\epsilon^{\tau+1}(\Phi_k(\xi) + O(\epsilon)) & (-1 < \tau < 1) \\ (2\pi/\log(1/\epsilon))(\Phi_k(\xi) + O(\epsilon)) & (\tau \leqq -1) \end{cases}$$

(ii) $\ell = 1$

$$b_{1,p} = \pi^{1/2}\langle \nabla\Phi_k(\xi), e_p(\xi)\rangle \epsilon^2 \begin{cases} -1 & (\tau > -1) \\ (\sigma-1)/(1+\sigma) & (\tau = -1) \\ 1 & (\tau < -1) \end{cases}$$

Lemma 14. $\ell \geqq 2$.

$$|b_{\ell,p}| \leqq \frac{d_N}{\gamma(\ell)^N} \begin{cases} \epsilon^{\ell+2} & \text{for} \quad \tau \geqq 0 \\ \epsilon^{\ell+\tau+2} & \text{for} \quad -1 < \tau < 0 \\ \epsilon^{\ell+1} & \text{for} \quad \tau \leqq -1 \end{cases} \quad (p = 1, 2)$$

For $q = 2$, $\gamma(\ell) = \ell^2$. $G_{k,\epsilon}(x)$ is uniformly convergent in $B(M, r_0) \setminus B(M, \epsilon)$ for each $\epsilon > 0$. This property is also true for $\nabla^p G_\epsilon(x)$ for any $p \in \mathbb{N}$.

Similarly as in section 4, divide $G_{k,\epsilon}$ as $G_{k,\epsilon}(x) = G^{(1)}_{k,\epsilon}(x) + G^{(2)}_{k,\epsilon}(x)$ where

$$G^{(1)}_{k,\epsilon}(x) = b_{0,1}\log(r/r_0) + \sum_{p=1}^{2} b_{1,p}(r^{-1} - r_0^{-2}r)\varphi_{1,p}(\theta),$$

$$G^{(2)}_{k,\epsilon}(x) = \sum_{\ell \geqq 2, p=1,2} b_{\ell,p}(r^{-\ell} - r_0^{-2\ell}r^{\ell})\varphi_{\ell,p}(\theta).$$

Lemma 15. *There exists $c_k > 0$ such that*

$$|G^{(2)}_{k,\epsilon}(x)| \leqq c_k \times \begin{cases} \epsilon^2 & \text{for} \quad \tau \geqq 0 \\ \epsilon^{\tau+2} & \text{for} \quad -1 < \tau < 0 \\ \epsilon & \text{for} \quad \tau \leqq -1 \end{cases} \quad (x \in \Gamma(M, \epsilon)$$

$$|\frac{\partial G^{(2)}_{k,\epsilon}}{\partial \nu_1}(x)| \leqq c_k \times \begin{cases} \epsilon & \text{for} \quad \tau \geqq 0 \\ \epsilon^{\tau+1} & \text{for} \quad -1 < \tau < 0 \\ 1 & \text{for} \quad \tau \leqq -1 \end{cases} \quad (x \in \Gamma(M, \epsilon))$$

$$|G^{(2)}_{k,\epsilon}(x)| \leqq c_k \times \begin{cases} \epsilon^4 & \text{for} \quad \tau \geqq 0 \\ \epsilon^{\tau+4} & \text{for} \quad -1 < \tau < 0 \\ \epsilon^3 & \text{for} \quad \tau \leqq -1 \end{cases} \quad (x \in \Gamma(M, \epsilon)$$

for $0 < \epsilon \leqq \epsilon_0$.

[Calculation for the detailed asymptotic behavior for $\lambda^R_k(\epsilon)$]

We put $\lambda_k(\epsilon) = \lambda^R_{k,\epsilon}$ and $\Phi_{k,\epsilon}(x) = \Phi^R_{k,\epsilon}(x)$ for brevity. A similar calculation as in the section 4 gives

$$(\lambda_k(\epsilon) - \lambda_k)\int_{\Omega(\epsilon)} \Phi_k(x)\Phi_{k,\epsilon}(x)dx = I_1(\epsilon) + I_2(\epsilon) + I_3(\epsilon) + I_4(\epsilon) \tag{5.4}$$

where

$$I_1(\epsilon) = -\int_{\Gamma(M,r_0)} \frac{\partial G_{k,\epsilon}}{\partial \nu_1}(\Phi_{k,\epsilon}(x) - \Phi'_k(x))dS$$

$$I_2(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} (\Delta G_{k,\epsilon}(x))(\Phi_{k,\epsilon}(x) - \Phi'_k(x))dx$$

$$I_3(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} G_{k,\epsilon}(x)(\Delta\Phi'_k(x) + \lambda_k(\epsilon)\Phi_{k,\epsilon}(x))dx$$

$$I_4(\epsilon) = \int_{\Gamma(M,\epsilon)} \left((\frac{\partial \Phi_k}{\partial \nu_1} + \sigma\epsilon^\tau \Phi_k)\Phi'_k - G_{k,\epsilon}(\frac{\partial \Phi'_k}{\partial \nu_1} + \sigma\epsilon^\tau \Phi'_k) \right) dS.$$

$I_4(\epsilon)$ is also written as

$$I_4(\epsilon) = \int_{\Gamma(M,\epsilon)} \left(\frac{\partial G_{k,\epsilon}}{\partial \nu_1}\Phi'_k - G_{k,\epsilon}\frac{\partial \Phi'_k}{\partial \nu_1} \right) dS.$$

We carry out the similar estimates for $I_1(\epsilon)$, $I_2(\epsilon)$, $I_3(\epsilon)$ for $\epsilon = \zeta_s$ and see the behaviors for $s \to \infty$ and we can conclude that thiese terms are negligible in the asymptotics in Theorem 2-(1), (2), (3), (4). So the main term of pertubation comes from $I_4(\epsilon)$. We calculate $I_4(\epsilon)$ with the aid of Lemma 13, Lemma 14, Lemma 15 with the similar estimates like in the section 4. and get Theorem 2-(1),(2),(3),(4). Theorem 2-(0) is proved just as in Theorem 2-(1) by putting $\sigma = 0$.

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