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ON THE MODULUS OF CONTINUITY OF SOLUTIONS TO THE n-LAPLACE EQUATION

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Abstract. Solutions to the n-Laplace equation with a right-hand side f are considered. We exhibit the largest rearrangement-invariant space to which f has to belong for every local weak solution to be continuous. Moreover, we find the optimal modulus of continuity of solutions when f ranges in classes of rearrangement-invariant spaces, including Lorentz, Lorentz-Zygmund and various standard Orlicz spaces.

1. Introduction

The aim of the present note is to announce some recent results dealing with continuity properties of local weak solutions to the n-Laplace equation in domains in $\mathbb{R}^n$, with $n \geq 2$.

Let Ω be an open subset of $\mathbb{R}^n$ having finite Lebesgue measure $|\Omega|$. Without loss of generality, we suppose that $|\Omega| = 1$ throughout. We deal with local weak solutions $u \in W^{1,n}(\Omega)$ to the n-Laplace equation

$$-\text{div}\,(|\nabla u|^{n-2}\nabla u) = f(x) \qquad \text{in } \Omega, \tag{1.1}$$

where f is a function from $\big(W^{1,n}(\Omega)\big)^*$, the dual of the Sobolev space $W^{1,n}(\Omega)$.

A function $u \in W^{1,n}(\Omega)$ is called a *local weak solution* to equation (1.1) if

$$\int_\Omega |\nabla u|^{n-2}\nabla u \cdot \nabla\phi\, dx = \int_\Omega f\phi\, dx \tag{1.2}$$

for every $\phi \in W^{1,n}_0(\Omega) \bigcap L^\infty(\Omega)$.

Precise regularity properties of local weak solutions to the n-Laplace equation (1.1) have been the object of a number of contributions over the last few years (see, e.g., [1, 4, 9, 11, 12, 17]). In particular, in dimension $n = 2$, a local weak solution u to (1.1) is known to be continuous if f belongs to the Zygmund space $L(\log L)(\Omega)$ [1, 4] (we refer to Section 2 below for the necessary background on the function space framework involved in our discussion). By contrast, if $n \geq 3$, the continuity of u is not guaranteed under the corresponding assumption $f \in L(\log L)^{n-1}(\Omega)$, as shown by counterexamples contained in [11]. A strengthening of this assumption has been shown in [11] to suffice for the continuity of local weak solutions to (1.1). Further refinements are contained in [12].

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Among other results in connection with solutions to (1.1), we are able to complement and enhance statements of [11] and [12]. In particular, we prove that a condition on f ensuring the continuity of solutions to (1.1), exhibited in [12], is in fact also necessary. Moreover, we improve a result on the modulus of continuity of solutions when f belongs to Zygmund spaces, established in [11], and find the optimal one.
With this regard, we prove that any local weak solution u to (1.1), with $f \in L(\log L)^{n-1+\epsilon}(\Omega)$, is locally uniformly continuous in Ω if $\epsilon > 0$, with a modulus of continuity not exceeding

$$\left(1+\log\tfrac{1}{s}\right)^{-\frac{\epsilon}{n-1}} \qquad \text{near } 0 \tag{1.3}$$

(see Theorem 9, Section 3). In [11], the weaker conclusion was derived that solutions u have a modulus of continuity bounded by

$$\left(1+\log\tfrac{1}{s}\right)^{-\frac{\epsilon}{n}} \qquad \text{near } 0. \tag{1.4}$$

Moreover, we show that the modulus of continuity φ given by (1.3) is optimal.

More generally, we investigate on continuity properties of local weak solutions to the n-Laplace equation (1.1) with f in various classes of function spaces. We provide conditions for the right-hand side f, in customary classes of rearrangement-invariant function spaces, for local weak solutions u to be continuous. The relevant conditions are also shown to be sharp in most cases. Moreover, we find the optimal modulus of continuity of local weak solutions to (1.1) when f belongs to customary families of rearrangement-invariant spaces, including the Lorentz-Zygmund spaces $L^{(r,q)}\left(\log L\right)^{\alpha}(\Omega)$, and various standard Orlicz spaces.

As a matter of fact, all the above-mentioned results for local weak solutions to the n-Laplace equation are special cases of more general theorems which hold for local weak solutions to the p-Laplace equation

$$-\text{div}\,(|\nabla u|^{p-2}\nabla u) = f(x) \qquad \text{in } \Omega, \tag{1.5}$$

with $2 \le p \le n$ and $f \in \left(W^{1,p}(\Omega)\right)^*$.
However, in order to simplify our exposition, here we limit ourselves to deal with the case when $p = n$. This is in fact a borderline case, which presents some interesting peculiar features.
We refer the reader to the forthcoming paper [3] for a detailed presentation of our contributions, and for the proofs of our results. Let us just mention here that our approach combines pointwise gradient estimates for the gradient of solutions to p-Laplace type equations established in [14], Sobolev type embeddings into spaces of continuous functions from [8], and one-dimensional Hardy type inequalities involving various Banach function norms which can be found in [7, 10].

2. Background

In this section we recall a few basic definitions and properties about functions and function spaces that we will take into account.

Let Ω be a measurable subset of $\mathbb{R}^n$ having finite measure (without loss of generality, we suppose that $|\Omega| = 1$), and let g be a real-valued measurable function in Ω. The *decreasing rearrangement* of g is the function $g^* : [0, +\infty) \to [0, +\infty]$ defined as

$$g^*(s) = \sup\{t \ge 0 : |\{x \in \Omega : |g(x)| > t\}| > s\} \qquad \text{for } s \ge 0.$$

In other words, g^* is the (unique) non increasing, right-continuous function in $[0,+\infty)$ equidistributed with g. By $g^{**}:(0,+\infty)\to[0,+\infty]$ we denote the function given by $g^{**}(s)=\frac{1}{s}\int_0^s g^*(r)\,dr$ for $s>0$.

A *quasi-normed function space* $X(\Omega)$ on $\Omega\subset\mathbb{R}^n$ is a linear space of measurable functions on Ω equipped with a quasi-norm $\|\cdot\|_{X(\Omega)}$ satisfying the following properties:

(*i*) $\|g\|_{X(\Omega)}>0$ if $g\neq 0$;
$\|\lambda g\|_{X(\Omega)}=|\lambda|\|g\|_{X(\Omega)}$ for every $\lambda\in\mathbb{R}$ and $g\in X(\Omega)$;
$\|g+h\|_{X(\Omega)}\le c(\|g\|_{X(\Omega)}+\|h\|_{X(\Omega)})$ for some constant $c\ge 1$ and for every $g,h\in X(\Omega)$;

(*ii*) $0\le|h|\le|g|$ a.e. in Ω implies $\|h\|_{X(\Omega)}\le\|g\|_{X(\Omega)}$;

(*iii*) $0\le g_k\nearrow g$ a.e. implies $\|g_k\|_{X(\Omega)}\nearrow\|g\|_{X(\Omega)}$ as $k\longrightarrow+\infty$;

(*iv*) if G is a measurable subset of Ω and $|G|<\infty$, then $\|\chi_G\|_{X(\Omega)}<\infty$;

(*v*) for every measurable subset G of Ω with $|G|<\infty$, there exits a constant C such that $\int_G|g|\,dx\le C\|g\|_{X(\Omega)}$ for every $g\in X(\Omega)$.

We denote by χ_G the characteristic function of a measurable subset G of Ω, and we define

$$\|g\|_{X(G)}=\|g\chi_G\|_{X(\Omega)}$$

for every measurable function g on Ω.
Moreover, we denote by $X_{loc}(\Omega)$ the space of measurable functions g in Ω such that $\|g\|_{X(G)}<\infty$ for every compact set $G\subset\Omega$. If the relation (i) holds with $c=1$ the functional $\|\cdot\|_{X(\Omega)}$ is a norm which makes $X(\Omega)$ a Banach space.

A quasi-normed function space (in particular, a Banach function space) $X(\Omega)$ is called *rearrangement-invariant* if there exists a quasi-normed function space $\overline{X}(0,|\Omega|)$, called the representation space of $X(\Omega)$, having the property that

$$\|g\|_{X(\Omega)}=\|g^*\|_{\overline{X}(0,|\Omega|)} \tag{2.1}$$

for every $g\in X(\Omega)$. Obviously, if $X(\Omega)$ is a rearrangement-invariant quasi-normed space, then

$$\|g\|_{X(\Omega)}=\|h\|_{X(\Omega)}\qquad\text{if}\qquad g^*=h^*. \tag{2.2}$$

We refer to [5] for a detailed exposition of the theory of rearrangement-invariant spaces. Note that, for customary spaces $X(\Omega)$, an expression for the norm $\|\cdot\|_{\overline{X}(0,|\Omega|)}$ is immediately derived from equation (2.1), via elementary properties of rearrangements.
If $X(\Omega)$ and $Y(\Omega)$ are rearrangement-invariant spaces, then

$$X(\Omega)\subset Y(\Omega)\quad\text{if and only if}\quad X(\Omega)\to Y(\Omega). \tag{2.3}$$

Let ν be a *weight*, namely a non-negative continuous function on $(0,+\infty)$ and let us denote

$$\mathcal{V}(t)=\int_0^t \nu(s)\,ds \qquad \text{for } t\in(0,\infty). \tag{2.4}$$

Let $\sigma\in(0,+\infty)$ and let ν be a weight. We define the so-called *classical Lorentz spaces* $\Lambda^\sigma(\nu)$ and $\Gamma^\sigma(\nu)$ and the *weak Lorentz spaces* $\Lambda^{\sigma,\infty}(\nu)$ and $\Gamma^{\sigma,\infty}(\nu)$ as a space endowed with the quantities

$$\begin{cases} \|g\|_{\Lambda^\sigma(\nu)} = \left(\int_0^1 (g^*(t))^\sigma\,\nu(t)\,dt\right)^{1/\sigma} \\ \\ \|g\|_{\Lambda^{\sigma,\infty}(\nu)} = \sup\limits_{0<t<1} g^*(t)\mathcal{V}(t)^{1/\sigma} \\ \\ \|g\|_{\Gamma^\sigma(\nu)} = \left(\int_0^1 (g^{**}(t))^\sigma\,\nu(t)\,dt\right)^{1/\sigma} \\ \\ \|g\|_{\Gamma^{\sigma,\infty}(\nu)} = \sup\limits_{0<t<1} g^{**}(t)\mathcal{V}(t)^{1/\sigma} \end{cases} \tag{2.5}$$

for a measurable function g in Ω. Note that, by Fubini's theorem,

$$\|\cdot\|_{\Gamma^1(\nu)(\Omega)} = \|\cdot\|_{\Lambda^1(\tilde\nu)(\Omega)} \qquad \text{where } \tilde\nu(t)=\textstyle\int_t^1 \frac{\nu(s)}{s}\,ds. \tag{2.6}$$

Likewise, for $\sigma\in(0,\infty)$,

$$\begin{cases} \|\cdot\|_{\Lambda^{\sigma,\infty}(\nu)(\Omega)} = \|\cdot\|_{\Lambda^{1,\infty}(\hat\nu)(\Omega)} \\ \\ \|\cdot\|_{\Gamma^{\sigma,\infty}(\nu)(\Omega)} = \|\cdot\|_{\Gamma^{1,\infty}(\hat\nu)(\Omega)} \end{cases} \quad \text{where} \quad \hat\nu(t)=\tfrac{1}{\sigma}\mathcal{V}(t)^{\frac{1}{\sigma}-1}\nu(t). \tag{2.7}$$

The quantities defined in (2.5) are not always norms. For example, for $\sigma\ge 1$, $\|g\|_{\Lambda^\sigma(\nu)(\Omega)}$ is a norm if and only if ν is non-increasing. For $\sigma\in(1,\infty)$, $\Lambda^\sigma(\nu)(\Omega)$ and $\Lambda^{\sigma,\infty}(\nu)(\Omega)$ are equivalent to a Banach space, respectively, if and only if

$$t^\sigma\int_t^1 s^{-\sigma}\nu(s)\,ds \le C\int_0^t \nu(s)\,ds \tag{2.8}$$

for some C and all $t>0$.
Furthermore, for $\sigma=1$, $\Lambda^1(\nu)(\Omega)$ and $\Lambda^{1,\infty}(\nu)(\Omega)$ are equivalent to a Banach space, respectively, if and only if

$$\frac{1}{t}\int_t^1 \nu(s)\,ds \le \frac{C}{s}\int_0^s \nu(\tau)\,d\tau \qquad \text{for } 0<s\le t \tag{2.9}$$

for some C and all $t>0$.
Whereas $\Gamma^\sigma(\nu)(\Omega)$ and $\Gamma^{\sigma,\infty}(\nu)(\Omega)$ are equivalent to a Banach space, respectively, if and only if $\sigma\ge 1$.

The most familiar examples of classical Lorentz spaces are the standard Lorentz spaces $L^{(r,q)}(\Omega)$ with $r,q\in(0,\infty]$, more in general, the *Lorentz-Zygmund spaces* $L^{(r,q)}(\log L)^\alpha(\Omega)$.

Given $r, q \in (0,\infty]$ and $\alpha \in \mathbb{R}$, define

$$\|g\|_{L^{(r,q)}\,(\log L)^{\alpha}(\Omega)} = \|s^{\frac{1}{r}-\frac{1}{q}}\left(1+\log\tfrac{1}{s}\right)^{\alpha} g^{**}(s)\|_{L^{q}(0,|\Omega|)} \tag{2.10}$$

for a measurable function g in Ω. If $r, q \geq 1$ and $\alpha \in \mathbb{R}$, the space $L^{(r,q)}(\log L)^{\alpha}(\Omega)$ is a rearrangement-invariant space endowed with the norm (2.10). In particular, the Lorentz-Zygmund space $L^{(r,q)}(\log L)^{0}(\Omega)$ reduces to the standard Lorentz space $L^{(r,q)}(\Omega)$, and, if $r > 1$, the Lorentz space $L^{(r,r)}(\Omega)$ agrees with the Lebesgue space $L^{r}(\Omega)$ (up to equivalent norms). Observe that, instead, $L^{(1,1)}(\Omega) \neq L^{1}(\Omega)$. Actually, $L^{(1,1)}(\Omega)$ is equivalent to $L(\log L)(\Omega)$.

A function $A : [0,\infty) \to [0,\infty]$ is called a *Young function* if it has the form

$$A(t) = \int_0^t a(\tau)d\tau \qquad \text{for } t \geq 0, \tag{2.11}$$

for some non-decreasing, left-continuous function $a : [0,\infty) \to [0,\infty]$ which is neither identically equal to 0 nor to ∞. Clearly, any convex (non trivial) function from $[0,\infty)$ into $[0,\infty]$, which is left-continuous and vanishes at 0, is a Young function.
The *Orlicz space* $L^{A}(\Omega)$, associated with a Young function A, is the Banach function space of those real-valued measurable functions g in Ω for which the Luxemburg norm

$$\|g\|_{L^{A}(\Omega)} = \inf\left\{\lambda > 0 : \int_{\Omega} A\left(\frac{|g(x)|}{\lambda}\right)dx \leq 1\right\}$$

is finite.
In particular, $L^{A}(\Omega) = L^{r}(\Omega)$ if $A(t) = t^{r}$ for some $r \in [1,\infty)$, and $L^{A}(\Omega) = L^{\infty}(\Omega)$ if $A(t) = 0$ for $t \in [0,1]$ and $A(t) = \infty$ for $t > 0$.
Moreover, if $A(t)$ is equivalent to $t^{r}\left(1+\log\frac{1}{t}\right)^{\alpha}$ near infinity, where either $r > 1$ and $\alpha \in \mathbb{R}$, or $r = 1$ and $\alpha \geq 0$, then $L^{A}(\Omega)$ agrees with the *Zygmund space* $L^{r}\left(\log L\right)^{\alpha}(\Omega)$. Note that $L^{r}\left(\log L\right)^{\alpha}(\Omega)$ agrees with $L^{(r,r)}\left(\log L\right)^{\alpha+1}(\Omega)$, up to equivalent norms.

3. Main results

We begin our discussion on exhibiting a sharp assumption on the right-hand side f ensuring the continuity of the local weak solutions to the n-Laplace equation

$$-\text{div}\left(|\nabla u|^{n-2}\nabla u\right) = f(x) \qquad \text{in } \Omega.$$

The relevant assumption is that f belongs to the quasi-normed space $L^{(1,\frac{1}{n-1})}(\Omega)$ and this condition is sharp in the sense that $L^{(1,\frac{1}{n-1})}(\Omega)$ is the largest rearrangement-invariant class to which f has to belong for every local weak solution u to (1.1) to be continuous in Ω. Here we state the following result.

Theorem 1. *Let $n \geq 2$. Then $L^{(1,\frac{1}{n-1})}(\Omega)$ is the largest rearrangement-invariant class $X(\Omega)$ contained in $\left(W^{1,n}(\Omega)\right)^{*}$ such that for every $f \in X(\Omega)$, any local weak solution to equation* (1.1) *is continuous.*

Remark 1. A theorem from [12] yields the continuity of local weak solutions to (1.1) when $f \in L^{(1,\frac{1}{n-1})}(\Omega)$. Theorem 1 complements the result from [12] by showing the optimality of the space $L^{(1,\frac{1}{n-1})}(\Omega)$.

Now we focus on the necessary and sufficient conditions for the right-hand side of the n-Laplace equation (1.1), for local weak solution u to be continuous. The relevant results involving the classical Lorentz spaces $\Gamma^\sigma(\nu)(\Omega)$, $\Lambda^\sigma(\nu)(\Omega)$, $\Gamma^{\sigma,\infty}(\nu)(\Omega)$ and $\Lambda^{\sigma,\infty}(\nu)(\Omega)$, associated with a weight function ν and a power $\sigma \in (0,\infty)$ are contained in the next theorems.
In what follows, we make use of the notation

$$V_\sigma(t) = \int_0^t \nu(s)\, ds + t^\sigma \int_t^1 s^{-\sigma}\nu(s)ds \qquad \text{for } t \in (0,1).$$

Theorem 2. *Let $0 < \sigma < \infty$ and let ν be a weight. Assume that*

$$\text{(3.1)} \quad \begin{cases} \sup\limits_{0<t<1} \dfrac{t\left(1+\log\frac{1}{t}\right)^{n-1}}{V_\sigma(t)^{\frac{1}{\sigma}}} < \infty & \text{if } 0<\sigma\le\frac{1}{n-1},\ n\ge 2; \\[2ex] \displaystyle\int_0 \frac{t^{\frac{\sigma}{\sigma(n-1)-1}-1}}{V_\sigma(t)^{\frac{1}{\sigma(n-1)-1}}} dt < \infty & \text{if } \frac{1}{n-1}<\sigma\le 1,\ n>2; \\[2ex] \displaystyle\int_0 \frac{t^{\frac{\sigma}{\sigma(n-1)-1}-1}\left(1+\log\frac{1}{t}\right)^{\frac{1}{\sigma(n-1)-1}}}{V_\sigma(t)^{\frac{1}{\sigma(n-1)-1}}} dt < \infty & \text{if } \sigma\ge 1, n>2; \\[2ex] \displaystyle\int_0 \frac{t\left(1+\log\frac{1}{t}\right)^{\frac{1}{\sigma-1}}}{V_\sigma(t)^{\frac{1}{\sigma-1}}}\, dt < \infty & \text{if } \sigma>1, n=2. \end{cases}$$

If $f \in \Gamma^\sigma(\nu)(\Omega)$, then every local weak solution $u \in W^{1,n}(\Omega)$ to (1.1) is continuous. Conversely, if $\sigma \ge 1$, $\Gamma^\sigma(\nu)(\Omega) \subset \left(W^{1,n}(\Omega)\right)^*$, and for every $f \in \Gamma^\sigma(\nu)(\Omega)$ any local weak solution to (1.1) is continuous, then (3.1) holds.

Theorem 3. *Let $0 < \sigma < \infty$ and let ν be a weight. Assume that*

$$\text{(3.2)} \qquad \int_0 \left(\sup_{s\ge t} \frac{1}{s} \left(\int_0^s \nu(\tau)\, d\tau \right)^{\frac{1}{\sigma}} \right)^{-1} \frac{dt}{t} < \infty.$$

If $f \in \Gamma^{\sigma,\infty}(\nu)(\Omega)$, then every local weak solution $u \in W^{1,n}(\Omega)$ to (1.1) is continuous. Conversely, if $\sigma \ge 1$, $\Gamma^{\sigma,\infty}(\nu)(\Omega) \subset \left(W^{1,n}(\Omega)\right)^*$, and for every $f \in \Gamma^{\sigma,\infty}(\nu)(\Omega)$ any local weak solution to (1.1) is continuous, then (3.2) holds.

Theorem 4. *Let $0 < \sigma < \infty$ and let ν be a weight. Assume that*

$$\begin{cases} \sup\limits_{0<t<1} \dfrac{t\left(1+\log\frac{1}{t}\right)^{n-1}}{\mathcal{V}(t)^{\frac{1}{\sigma}}} < \infty & \text{if } 0<\sigma\le\frac{1}{n-1},\ n\ge 2; \\ \displaystyle\int_0 \sup_{0<s\le t}\left[\left(\frac{s^\sigma}{\mathcal{V}(s)}\right)^{\frac{1}{\sigma(n-1)-1}}\right]\left(1+\log\frac{1}{t}\right)^{\frac{1}{\sigma(n-1)-1}}\frac{dt}{t}<\infty & \text{if } \frac{1}{n-1}<\sigma\le 1,\ n>2; \\ \displaystyle\int_0 \left(\frac{t}{\mathcal{V}(t)}\right)^{\frac{1}{\sigma-1}}\left(1+\log\frac{1}{t}\right)^{\frac{\sigma}{\sigma-1}} dt<\infty & \text{if } \sigma>1,\ n=2; \\ \displaystyle\int_0 \left(\int_0^t\left(\frac{s}{\mathcal{V}(s)}\right)^{\sigma'-1} ds\right)^{\frac{\sigma-1}{\sigma(n-1)-1}}\left(1+\log\frac{1}{t}\right)^{\frac{1}{\sigma(n-1)-1}}\frac{dt}{t}<\infty & \text{if } \sigma>1,\ n>2. \end{cases} \tag{3.3}$$

If $f \in \Lambda^\sigma(\nu)(\Omega)$, then every local weak solution $u \in W^{1,n}(\Omega)$ to (1.1) is continuous. Conversely, if either $\sigma > 1$ and (2.8) holds, or $\sigma = 1$ and (2.9) holds, $\Lambda^\sigma(\nu)(\Omega) \subset \left(W^{1,n}(\Omega)\right)^*$, and for every $f \in \Lambda^\sigma(\nu)(\Omega)$ any local weak solution to (1.1) is continuous, then (3.3) holds.

Remark 2. Note that if ν is (equivalent to) a non-increasing weight, condition in (3.3) for $\sigma = 1$ and $n > 2$ simply reduces to

$$\int_0 \left[\frac{(1+\log\frac{1}{t})}{\frac{1}{t}\int_0^t \nu(s)\,ds}\right]^{\frac{1}{n-2}}\frac{dt}{t}<\infty\,,$$

and assumption (2.9) is certainly fulfilled since $\Lambda^1(\nu)(\Omega)$ is a Banach space.

Remark 3. The continuity of u when f belongs to some space $\Lambda^1(\nu)(\Omega)$ has also been studied in [12]. The condition exhibited in [12] has a more implicit form, and is only sufficient for the continuity of u.

Example 1. Assume that $n > 2$ and $f \in \Lambda^1\left(\left(1+\log\frac{1}{t}\right)^{n-1}\omega(t)\right)(\Omega)$ for some slowly varying non-increasing function ω in the sense of [6] (this is certainly the case if $\lim_{t\to 0^+}\frac{t\,\omega'(t)}{\omega(t)} = 0$). By [6, Theorem 1.5.11]

$$\int_0^t \left(1+\log\tfrac{1}{s}\right)^{n-1}\omega(s)\,ds \approx t\left(1+\log\tfrac{1}{t}\right)^{n-1}\omega(t) \qquad \text{as } t\to 0.$$

By Theorem 4, any local weak solution $u \in W^{1,n}(\Omega)$ to equation (1.1) is continuous, provided that condition (3.3) is satisfied, namely

$$\int_0 \frac{dt}{t\left(1+\log\frac{1}{t}\right)\omega(t)^{\frac{1}{n-2}}}<\infty. \tag{3.4}$$

For instance, the choice

$$\omega(t) = \ell_2\left(\tfrac{1}{t}\right)^{n-2}\cdots\ell_k\left(\tfrac{1}{t}\right)^{n-2+\varepsilon} \tag{3.5}$$

for $k \geq 2$ and for any $\varepsilon > 0$ is admissible, where ℓ_k is inductively defined as

$$\ell_1(s) = \max\{1, \log s\}, \qquad \ell_k(s) = \max\{1, \ell_{k-1}(s)\} \qquad k \geq 2, \tag{3.6}$$

for $s > 0$.

This recovers a result of [12]. On the other hand, discontinuous (in fact, locally unbounded) solutions to equation (1.1) may exist if ω is defined as in (3.5), with $\varepsilon = 0$. Let us emphasize that discontinuous solutions may exist with this choice of ε even if, on the right-hand side of (3.5), an infinite product extended to all $k \geq 2$ appears, namely if

$$\omega(t) = \prod_{k=2}^{\infty} \ell_k\left(\tfrac{1}{t}\right)^{n-2} \tag{3.7}$$

for $t > 0$. Indeed, condition (3.4) can be shown to fail for such ω. This last example is closely related to a question raised in [12, Remark 5.2] about the case when the right-hand side of (1.1) belongs to an Orlicz space associated with a Young function defined in terms of an infinite product of logarithms.

Theorem 5. *Let $0 < \sigma < \infty$ and let ν be a weight. Assume that*

$$\int_0 \left(\frac{1}{t} \int_0^t \left(\int_0^s \nu(\tau)\, d\tau \right)^{-\frac{1}{\sigma}} ds \right)^{\frac{1}{n-1}} t^{\frac{1}{(n-1)}-1}\, dt < \infty. \tag{3.8}$$

If $f \in \Lambda^{\sigma,\infty}(\nu)(\Omega)$, then every local weak solution $u \in W^{1,n}(\Omega)$ to (1.1) is continuous. Conversely, if either $\sigma > 1$ and (2.8) holds, or $\sigma = 1$ and (2.9) holds, $\Lambda^{\sigma,\infty}(\nu) \subset \left(W^{1,n}(\Omega)\right)^*$, and for every $f \in \Lambda^{\sigma,\infty}(\nu)(\Omega)$ any local weak solution to (1.1) is continuous, then (3.8) holds.

We now address the problem of the continuity of solutions to the n-Laplace equation in the case when f belongs to an Orlicz space.

Theorem 6. *Let A be a Young function. Assume that*

$$\begin{cases} A(t) \geq Ct\log(1+t) & \text{for } t \geq 1 \quad \text{if } n = 2 \\[2ex] \displaystyle\int^{\infty} \left(\frac{t}{tA'(t) - A(t)} \right)^{\frac{1}{n-2}} \frac{dt}{t} < \infty & \text{if } n > 2. \end{cases}$$

If $f \in L^A(\Omega)$, then any local weak solution $u \in W^{1,n}(\Omega)$ to equation (1.1) is continuous.

Let us now discuss the question of the modulus of continuity of solutions to (1.1). With this regard, we find the optimal modulus of continuity of solutions to (1.1) when f belongs to a wide class of rearrangement-invariant spaces, including Lorentz-Zygmund spaces $L^{(r,q)}(\log L)^{\alpha}(\Omega)$, and various standard Orlicz spaces. Specifically, we obtain estimates of the form

$$\|u\|_{\mathcal{C}^{0,\varphi}(B_\rho)} \leq C \left(\|f\|_{X(B_{2\rho})}^{\frac{1}{n-1}} + \|\nabla u\|_{L^1(B_{2\rho})} \right),$$

for every ball $B_{2\rho} \subset\subset \Omega$, where $\mathcal{C}^{0,\varphi}(B_\rho)$ denotes the space of uniformly continuous functions with modulus of continuity φ, $X(\Omega)$ is a rearrangement-invariant space, and C is a positive constant.

Some definitions about the space $\mathcal{C}^{0,\varphi}(\Omega)$ are needed in the statement below. A *modulus of continuity* $\varphi : (0,\infty) \to (0,\infty)$ is a function equivalent (up to multiplicative constants) near 0 to a non-decreasing function and such that $\lim_{s\to 0^+} \varphi(s) = 0$ and $\limsup_{s\to 0^+} \dfrac{s}{\varphi(s)} < \infty$. The Banach space $\mathcal{C}^{0,\varphi}(\Omega)$ is the set of the measurable functions g on Ω for which the semi-norm

$$\|g\|_{\mathcal{C}^{0,\varphi}(\Omega)} = \sup_{\substack{x,\, y\, \in\, \Omega \\ x \neq y}} \frac{|g(x) - g(y)|}{\varphi(|x-y|)} \tag{3.9}$$

is finite. Note that moduli of continuity which are equivalent (up to multiplicative constants) near 0 yield the same spaces (up to equivalent norms).

In the case when $\varphi(s) = s^a$ with $a \in (0,1]$, the space $\mathcal{C}^{0,\varphi}(\Omega)$ coincides with the classical Hölder space $\mathcal{C}^{0,a}(\Omega)$. In particular, $\mathcal{C}^{0,1}(\Omega)$ is the space of Lipschitz continuous functions in Ω. Moreover, $\mathcal{C}^{0,\varphi}_{loc}(\Omega)$ denotes the space of those functions which belong to $\mathcal{C}^{0,\varphi}(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.
Henceforth, a norm $\||\nabla u|\|_{X(\Omega)}$ will simply be denoted by $\|\nabla u\|_{X(\Omega)}$.

Theorem 7. *Let $f \in \Gamma^\sigma(\nu)(\Omega)$ with $0 < \sigma < \infty$. Let $u \in W^{1,n}(\Omega)$ be a local weak solution to equation* (1.1).
(j) Assume that the function φ defined as

$$\varphi(s) \simeq \begin{cases} \left(\displaystyle\int_0^{s^n} t^{\frac{(\sigma(n-1))'}{n}-1} \left(\int_t^1 \tau^{-\frac{\sigma'}{n'}} \nu(\tau)^{-\frac{1}{\sigma-1}}\, d\tau \right)^{\frac{\sigma-1}{\sigma(n-1)-1}} dt \right)^{\frac{1}{(\sigma(n-1))'}} & \text{if } \sigma > 1, n \geq 2; \\ \left(\displaystyle\int_0^{s^n} t^{\frac{(n-1)'}{n}-1} \frac{1}{\left[\inf_{t<\tau<1} \left(\tau^{1/n'}\nu(\tau)\right)\right]^{(n-1)'-1}} dt \right)^{\frac{1}{(n-1)'}} & \text{if } \sigma = 1, n > 2; \\ \displaystyle\sup_{0<t<1} \frac{\min\left\{t^{1/2}, s\right\}}{\inf_{t<\tau<1}\left(\tau^{1/2}\nu(\tau)\right)} & \text{if } \sigma = 1, n = 2, \end{cases} \tag{3.10}$$

for s near 0, is finite, and, in the last case, it also satisfies $\lim_{s\to 0} \varphi(s) = 0$. Then $u \in \mathcal{C}^{0,\varphi}_{loc}(\Omega)$, and there exists a constant C such that

$$\|u\|_{\mathcal{C}^{0,\varphi}(B_\rho)} \leq C \left(\|f\|^{\frac{1}{n-1}}_{\Gamma^\sigma(\nu)(B_{2\rho})} + \|\nabla u\|_{L^1(B_{2\rho})} \right) \tag{3.11}$$

for every $B_{2\rho} \subset\subset \Omega$.
(jj) In particular, if

$$\|t^{-1/n'}\nu(t)^{-1/\sigma}\|_{L^{\sigma'}(0,1)} < \infty, \tag{3.12}$$

then $u \in \mathrm{Lip}_{loc}(\Omega)$, and there exists a constant C such that

$$\|u\|_{\mathcal{C}^{0,1}(B_\rho)} \leq C \left(\|f\|^{\frac{1}{n-1}}_{\Gamma^\sigma(\nu)(B_{2\rho})} + \|\nabla u\|_{L^1(B_{2\rho})} \right) \tag{3.13}$$

for every $B_{2\rho} \subset\subset \Omega$.

Theorem 8. *Let $f \in \Gamma^{\sigma,\infty}(\nu)(\Omega)$ with $0 < \sigma < \infty$. Let $u \in W^{1,n}(\Omega)$ be a local weak solution to equation (1.1). Assume that the function φ defined as*

$$\varphi(s) = \int_0^{s^n} t^{-\frac{1}{n'}} \left(\int_t^1 \tau^{-\frac{1}{n'}} \mathcal{V}(\tau)^{-\frac{1}{\sigma}} d\tau \right)^{\frac{1}{n-1}} dt \tag{3.14}$$

for s near 0, is finite. Then $u \in \mathcal{C}^{0,\varphi}_{loc}(\Omega)$, and there exists a constant C such that

$$\|u\|_{\mathcal{C}^{0,\varphi}(B_\rho)} \le C \left(\|f\|^{\frac{1}{n-1}}_{\Gamma^{\sigma,\infty}(\nu)(B_{2\rho})} + \|\nabla u\|_{L^1(B_{2\rho})} \right) \tag{3.15}$$

for every $B_{2\rho} \subset\subset \Omega$.

In particular, for right-hand sides f from Lorentz-Zygmund spaces, we have the following result. Here, we agree upon the notation "$\frac{1}{\infty} = 0$".

Theorem 9. *Let $f \in L^{(r,q)}(\log L)^\alpha(\Omega)$ with $1 \le r \le n$, $1 \le q \le \infty$, $\alpha \in \mathbb{R}$ and $n \ge 2$. Let $u \in W^{1,n}(\Omega)$ be a local weak solution to equation (1.1).*
Part I: Assume that the function φ defined as

$$\varphi(s) \simeq \begin{cases} \left(1 + \log \frac{1}{s}\right)^{\frac{-\alpha+n-1-\frac{1}{q}}{n-1}} & \text{if } r = 1, 1 \le q \le \infty, \alpha > n-1-\frac{1}{q}; \\ s^{\frac{n(r-1)}{r(n-1)}} \left(1 + \log \frac{1}{s}\right)^{-\frac{\alpha}{n-1}} & \text{if } 1 < r < n, 1 \le q \le \infty, \alpha \in \mathbb{R}; \\ s \left(1 + \log \frac{1}{s}\right)^{\frac{-\alpha+1-\frac{1}{q}}{n-1}} & \text{if } r = n, 1 \le q \le \infty, \alpha < \frac{1}{q'}; \\ s \left(1 + \log \left(1 + \log \frac{1}{s}\right)\right)^{\frac{1-\frac{1}{q}}{n-1}} & \text{if } r = n, 1 < q \le \infty, \alpha = \frac{1}{q'} \end{cases} \tag{3.16}$$

for s near 0, is finite. Then, $u \in \mathcal{C}^{0,\varphi}_{loc}(\Omega)$ and there exists a constant C such that

$$\|u\|_{\mathcal{C}^{0,\varphi}(B_\rho)} \le C \left(\|f\|^{\frac{1}{n-1}}_{L^{(r,q)}(\log L)^\alpha(B_{2\rho})} + \|\nabla u\|_{L^1(B_{2\rho})} \right) \tag{3.17}$$

for every $B_{2\rho} \subset\subset \Omega$.
Part II: In particular, if one of the following conditions is satisfied:

$$\begin{cases} r = n, q = 1, \alpha \ge 0\,; \\ r = n, q > 1, \alpha > \frac{1}{q'}; \\ r > n, 1 \le q \le \infty, \alpha \in \mathbb{R}, \end{cases} \tag{3.18}$$

then $u \in \mathrm{Lip}_{loc}(\Omega)$, and there exists a constant C such that

$$\|u\|_{\mathcal{C}^{0,1}(B_\rho)} \le C \left(\|f\|^{\frac{1}{n-1}}_{L^{(r,q)}(\log L)^\alpha(B_{2\rho})} + \|\nabla u\|_{L^1(B_{2\rho})} \right) \tag{3.19}$$

for every $B_{2\rho} \subset\subset \Omega$.

Proposition 1. *Let* $f \in L^{(r,q)}(\log L)^\alpha(\Omega)$ *with* $1 \leq r \leq n$, $1 \leq q \leq \infty$, $\alpha \in \mathbb{R}$ *and* $n \geq 2$. *The modulus of continuity* φ *defined as in* (3.16) *is optimal, provided that one of the following alternatives holds:*

$$\left\{\begin{array}{ll} r=1, 1 \leq q \leq \infty, \alpha > n-1-\frac{1}{q} & (r,q,n) \neq (1,1,2); \\ 1 < r < n, 1 \leq q \leq \infty, \alpha \in \mathbb{R}; & \\ r=n, q=1, \alpha < 0, . & \end{array}\right. \tag{3.20}$$

This means that, if ψ *is another modulus of continuity such that* (3.17) *holds for every* $f \in L^{(r,q)}(\log L)^\alpha(\Omega)$, *and any local weak solution* u *to* (1.1), *then* $\mathcal{C}^{0,\varphi}_{loc}(\Omega) \subseteq \mathcal{C}^{0,\psi}_{loc}(\Omega)$.

Remark 4. Theorem 9 overlaps with results from [13] and [17]. Specifically, [13, Corollary 8.1] deals with the case when f belongs to a Marcinkiewicz space, corresponding to the choice $q = \infty$ and $\alpha = 0$ in Theorem 9. The proof in [13] is based on decay estimates for solutions and on nonlinear potential techniques. Data f from Marcinkiewicz spaces are also the object of [17, Theorem 4.1]. Moreover, [17, Theorem 4.2] is concerned with right-hand sides f from Lebesgue spaces, which is included as the special case when $r = q$ and $\alpha = 0$ in Theorem 9. The approach of [17] makes use of a compactness argument, which enables to relate the decay of the solutions to the equations in question to that of solutions to homogeneous problems with vanishing right-hand side.

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