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MULTISCALE WEAK COMPACTNESS IN METRIC SPACES

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Abstract. The aim of this paper is to provide a useful tool for a better understanding of the approach proposed in [11] to extend to the setting of metric spaces profile decomposition theorems. To this aim we shall deal with a less general context which has the advantage of making the analogies with the linear case more evident.

1. Introduction

The aim of this brief survey is to link some notions appeared in some recent papers dealing with Profile Decomposition (see for instance [9], [11] to which we refer, from now on, for further details) with the final purpose of showing the main ideas which let these notions, which, at first glance, heavily depend on the linear structure of the space, to be formulated even in the much more general setting of metric spaces.

This result has been proved in [11], but in this note we prefer to deal with a less general case which has the advantage to make the analogies with the linear case more evident. This note, which contains the exposition presented by the author at the "9^{th} European conference on Elliptic and Parabolic Problems " held in May 2016 in Gaeta (Italy), is organized as follows. In Section 2 we give a brief sketch of the main papers which led to the so called profile decomposition theorems and point out some differences between the main statements. In Section 3 we put into evidence the main difficulties met in [25] (see also [26]) in the generalization process from Hilbert spaces or some concrete functional spaces to the functional analytic setting of Banach spaces. In sections 4 and 5 we describe the main tools used to overcome such difficulties. In Section 6 we recall some definitions given in [9] in the setting of L^p spaces to point out some differences between the main statements about profile decomposition in literature. In Section 7 we propose suitable surrogates for the group G of L^p invariant scalings and for the algebraic zero, which allow to generalize to metric spaces the definition of all the notions which are basic for profile decomposition and profile reconstruction, and to recover the main ingredients (see (NB) and (EB) below) which are necessary to get the multiscale polar compactness property. In Section 8 we give a brief sketch of profile decomposition theorems by means of a (polar) profile reconstruction defined by means of a characterizing formula (similar to [9, Formula (4.18)]) which does not require any linear structure.

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2. Origins of profile decompositions theorems

Let us briefly mention the main papers which led to the so called profile decomposition theorems which are a "quantitative" counterpart of the concentration-compactness principle of P.L. Lions (see [17] and [18]). Indeed, while from one side concentration-compactness principle was originally introduced for studying defects of compactness for minimizing sequences of functionals arising in the field of Calculus of Variations, profile decomposition theorems are *structural statements* about the behavior of bounded sequences at the light of concentration-compactness phenomena, which have been applied to a great number of evolution equations leading to striking global existence and scattering results in critical contexts.

The first profile decomposition result has been given in 1984 by M. Struwe [28, Proposition 2.1] for Palais-Smale (P.S. for short) sequences related to the functional $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx$, corresponding to the elliptic problem

$$\begin{cases} -\Delta u &= |u|^{2^*-2}u + \lambda u \qquad \text{in } \Omega\,, \\ \quad u &= 0 \qquad\qquad\qquad\quad \text{in } \partial\Omega\,, \end{cases} \tag{2.1}$$

settled in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N > 2$, where $\lambda \in \mathbb{R}$ and, for $p < N$ $p^* = \frac{Np}{N-p}$ is, as usual, the critical exponent for the Sobolev embedding of $H_0^{1,p}(\Omega)$ into $L^p(\Omega)$. In such a case the sequence is approximated trough a finite sum of suitably scaled solutions to the limit problem corresponding to (2.1) (i.e. when Ω and λ are replaced by $\mathbb{R}^N$ and 0 respectively).

In 1995 S. Solimini proved a result [24, Theorem 2] in the spirit of Struwe Theorem, where the sum of "deflated profiles" is possibly infinite, but which holds true for any bounded sequence (in the Sobolev Spaces $H^{1,p}$ with respect to the Lebesgue norm of index p^* corresponding to the critical embedding).

The statement of [24, Theorem 2] guarantees that the embedding of $H^{1,p}$ into L^{p^*} is in some sense "multiscale compact" (nowadays the terminology is *cocompact*, i.e. compact with respect to a "scalings group") and quantifies the remarks given by P.L. Lions in the description of the concentration-compactness phenomena. Indeed, any bounded sequence in the homogeneous Sobolev space $H^{1,p}(\mathbb{R}^N)$ for $1 < p < N$ admits a subsequence whose elements are approximated by the sums $\sum_{i\in\mathbb{N}} \rho_n^i(\varphi_i)$ (which are convergent unconditionally with respect to i and uniformly with respect to n) of fixed "profiles" φ_i each one modulated by a sequence of scalings $(\rho_n^i)_{n\in\mathbb{N}}$ (which leave the Sobolev norm invariant) and of a reminder term which converges to zero in the critical Lebesgue space $L^{p^*}(\mathbb{R}^N)$.

One of the main purposes of [24, Theorem 2] was to show how the concentration - compactness phenomena are a consequence of the nonoptimality of L^{p^*} for the embedding of Sobolev spaces in the wider class of Lorentz spaces. Indeed these phenomena disappear in the case of the optimal Lorentz space $L^{(p^*,p)}$ as well as Rellich Theorem fails in L^{p^*}. In addition, Struwe Theorem can be very easily deduced from [24, Theorem 2] by just noticing that profiles of a P.S. sequence are solutions to a limit problem (and therefore their norm is bounded from below), so they are in finite number (see the energy estimate (EB) below).

Solimini's result was then rediscovered some years later by P. Gérard (see [12, Theorem 1.1]) and by S. Jaffard (see [13, Theorem 1]) in the framework of Sobolev spaces (of fractional order) with respect to the L^2 and L^p norm respectively, under a slightly weaker form (since the series of scaled profiles are replaced by larger and larger finite sums) and trough different methods.

Extensions of results of this type to other classes of functional spaces such as Besov, Triebel-Lizorkin and BMO spaces are presented in [2], [14] and [15] respectively.

It is worth to remark that, even if profile decomposition theorems allow to identify levels at which P.S. condition fails, its use goes far beyond the P.S. levels analysis. Indeed a substantially different use of profile decomposition theorems (see [8, Lemma 8]) allows to get the existence of infinitely many solutions to the equation $-\Delta u + a(x)u = u^p$ in $\mathbb{R}^2$ with $p > 1$, without asking on the continuous positive potential $a(x)$ any symmetry assumption as in [32] or any small oscillation assumption as in [5, 6, 33] (the strongest subsequent result in this direction is [19]). Even in [7] the use of a suitable profile decomposition (see [7, Theorem 8]) (jointly with a local Pohozaev identity) allows to prove the existence of infinitely many positive solutions to the Brezis-Nirenberg problem in a bounded domain of $\mathbb{R}^N$ when $N \geq 7$ by proving compactness for sequences of solutions of approximating problems which are something very similar to a P.S. sequence but not exactly the same.

3. Toward a functional analytic version

K. Tintarev and K.-H. Fiesler gave the first general functional analytic formulation of [24, Theorem 2] in the context of Hilbert spaces in [30]. Of course this setting just covers the case $p = 2$, as in Struwe Theorem, since the other values of p give only Banach Spaces. So, to set satisfactory functional-analytic grounds (which include at least the concrete functional spaces) profile decomposition has been formulated also in Banach spaces by S. Solimini and K. Tintarev in [25] and [26].

Without going into details of the above cited papers, one can state that the main difficulties in passing from Hilbert to Banach setting are connected with the need of recovering the two following bounds:

$$\|u_n - u\| \leq \|u_n\| + o(1) \quad \forall u_n \rightharpoonup u, \tag{NB}$$

which allows the "reduction" of the norm when subtracting from a sequence its weak limit, and

$$\sum_{i=1}^{+\infty} \|\varphi_i\|_{1,2}^2 \leq \liminf_{n\to+\infty} \|u_n\|_2^2, \tag{EB}$$

which gives an "energy" bound on the "profile bulk" of a sequence (and allows to quantify the above reduction, indeed, by combining (EB) with (NB), one gets the so called Kadec-Klee property, i.e. $\|u_n - u\| \to 0$ whenever $u_n \rightharpoonup u$ and $\|u_n\| \to \|u\|$).

Bound (EB) guarantees that the saturation process of a bounded sequence $(u_n)_{n\in\mathbb{N}}$ by its "deflated profiles" $(\rho_n^i(\varphi_i))_{n\in\mathbb{N}}$ (see Definition 2 below) must stop. Indeed, roughly speaking, when, scaling term by term the elements of $(u_n)_{n\in\mathbb{N}}$, one finds a nonzero weak limit (profile), by subtracting it from the scaled $(u_n)_{n\in\mathbb{N}}$ one gets a sequence with a better bound on the norm (see (NB)). Then, by using (EB), one reaches a sequence from which no other profile can be extracted.

Analogous estimates (see (5.3) and (7.13) below) will be employed to extend profile decomposition results in more general spaces. In some sense we can say that profile decomposition theorems or, more in general, what we shall call multiscale compactness theorems will be based on estimates like (EB).

Relation (NB) holds true in Hilbert Spaces by orthogonality

$$\|u_n\|^2 = \|u_n - u\|^2 + \|u\|^2 + o(1) \quad \forall u_n \rightharpoonup u \tag{3.1}$$

and in some concrete cases of Banach spaces, such as L^p spaces, by Brezis-Lieb Lemma (see [3]) if one replaces weak convergence by (the stronger) *a.e.* convergence

$$\|u_n\|_p^p = \|u_n - u\|_p^p + \|u\|_p^p + o(1) \quad \forall u_n \to u \ a.e. \tag{3.2}$$

In addition estimates of this kind have many and significant applications to PDEs, in particular, in a lot of problems with lack of compactness, they (jointly with the fact that profiles are solutions to the corresponding limit problem) allow to specify the "bad levels" for P.S. (i.e. levels at which the P.S. condition, for the corresponding functional, does not hold). In the last decades of the last century several variational methods have been developed in order to avoid such bad levels and get existence and multiplicity of solutions.

We remark that, in [24], which is concerned with the Banach space $H^{1,p}$, formula (3.1) is replaced by (3.2) and a bound analogous to (EB) is obtained by replacing $\|\cdot\|_{1,2}^2$ by $\|\cdot\|_{1,p}^p$ (i.e. by associating to any profile φ the energy $\|\varphi\|_{1,p}^p$ instead of $\|\varphi\|_{1,2}^2$). In both formulas the index p plays a fundamental role and a value of p is not expected in the generic case of Banach spaces.

4. How to recover (NB): the use of polar convergence

Unfortunately in more general Banach spaces weak convergence does not work properly, indeed we have not a "splitting formula" like (3.1) or (3.2) from which to get (NB). Indeed, if $u_n \rightharpoonup u$, we can only deduce that

$$\|u\| \leq \|u_n\| + o(1) \quad \text{and} \quad \|u_n - u\| \leq 2\|u_n\| + o(1)$$

which is clearly not enough.

So it is necessary to use a different mode of convergence of weak type. In the preliminary version of [25] a suitable notion, called polar convergence, was introduced, and substituted in the final version by the very similar notion of Δ-convergence introduced several years before by T.C. Lim in [16]. In a joint work with S. Solimini and K. Tintarev we have surveyed polar convergence with respect to other modes of convergence in [10] (where we also addressed the problem of the existence of a topology related to polar (or Δ) convergence) and below we rephrase [10, Definition 2.7].

Definition 1 (Polar limit)**.** Let $(u_n)_{n\in\mathbb{N}} \subset E$ be a sequence in a metric space (E, d). One says that $u \in E$ is the *polar limit* of $(u_n)_{n\in\mathbb{N}}$ and we shall write $u_n \rightharpoondown u$, if for every $v \neq u$ we have

$$d(u_n, u) < d(u_n, v) \text{ for all large } n. \tag{4.1}$$

Note that in the linear case, where d is the distance induced by the norm, (NB) is tautological if one replaces weak convergence by polar convergence (and takes $v = 0$ in (4.1)).

The notion of polar limit of a bounded sequence $(u_n)_{n\in\mathbb{N}}$ can be further clarified by means of the notions of *asymptotic centers* (denoted by $\mathrm{cen}_{as}((u_n)_{n\in\mathbb{N}})$) and *asymptotic radius* (denoted by $\mathrm{rad}_{as}((u_n)_{n\in\mathbb{N}})$). They are respectively the minimum points and the infimum value of the following functional (depending on $(u_n)_{n\in\mathbb{N}}$) defined on E by setting for all $v \in E$

$$I_{as}(v) = \limsup_n d(u_n, v). \tag{4.2}$$

So,

$$\mathrm{rad}_{as}((u_n)_{n\in\mathbb{N}}) := \inf_{v\in E} I_{as}(v) = \inf_{v\in E} \limsup_n d(u_n, v). \tag{4.3}$$

We emphasize that, while the asymptotic radius always exists and is uniquely determined, asymptotic centers may not exist or may be not uniquely determined. Therefore, the symbol $\mathrm{cen}_{as}((u_n)_{n\in\mathbb{N}})$ must be understood in the same sense as the limit symbol in a topological space which is not assumed to be Hausdorff. Note that if $u_n \rightharpoondown u$ then, see [10, Remark 2.4], u is one of the asymptotic centers of $(u_n)_{n\in\mathbb{N}}$, therefore the following equality holds true

$$\mathrm{rad}_{as}((u_n)_{n\in\mathbb{N}}) = \limsup_n d(u_n, u) \quad \forall u_n \rightharpoondown u. \tag{4.4}$$

When, as it will happen in the following section, the Banach space is assumed to be uniformly smooth and uniformly convex (see [21, Definition 1.e.1] or (5.1) below) polar convergence has many nice properties. Indeed, in such setting, polar convergence is of "dual" type. Indeed, suitably defining the dual element u' of $u \in E$ we have (see [10, Theorem 5.5]) that

$$u_n \rightharpoondown 0 \text{ in E} \qquad \text{iff} \qquad u_n' \rightharpoonup 0 \text{ in } E' \tag{4.5}$$

(this is the reason for which we have chosen to flip the symbol $\rightharpoonup$, used for weak convergence, getting $\rightharpoondown$ to denote polar convergence).

Moreover, since in Hilbert spaces each element coincide with its dual, it follows that, in such a setting, polar and weak convergence agree. It is worth to underline that when the two "weak" modes of convergence agree the space is not necessarily Hilbert. In such a case we say that the space is equipped with an Opial norm (see [22]). We should like to point out that the usual L^p norm is not an Opial norm for $p \neq 2$, (i.e. polar and weak convergence do not coincide in L^p spaces), but, on the other hand, as a particular case of a general result due to D. van Dulst (see [31]), there exists an equivalent Opial norm (of course, unlike weak convergence, polar convergence is not invariant on passing to equivalent norms).

Note that (4.5) allows to characterize polar convergence in L^p spaces by using its dual nature, i.e.

$$u_n \rightharpoondown 0 \text{ in } L^p \qquad \text{iff} \qquad |u_n|^{p-2}u_n \rightharpoonup 0 \text{ in } L^{p'}.$$

Since *a.e.* convergence always implies weak convergence and since, in the case $p = 2$, weak and polar convergence agree, one can think to recover (3.2) for a general p by replacing *a.e.* convergence with both weak and polar convergence. Indeed, S. Solimini and K. Tintarev in [25, Theorem 4.2] proved that, when $p \geq 3$ (or $p = 2$),

$$\|u_n\|_p^p \geq \|u\|_p^p + \|u_n - u\|_p^p + o(1) \quad \forall u_n \rightharpoonup u \text{ and } u_n \rightharpoondown u,$$

(from which (NB) follows) i.e. a partial extension to the space of p-summable functions (defined on any measurable space) of the well known Brezis-Lieb Lemma holds true.

5. How to recover (EB): the use of uniform convexity

As recalled above, in a general Banach space we have no index p to use to get an energy estimate analogous to (EB) (where $p = 2$). To this aim we shall require uniform convexity and make use of the modulus of convexity (composed with the norm).

We recall that a normed vector space E is called uniformly convex if the following function $\delta : [0,2] \to \mathbb{R}$, called *modulus of convexity* of the space, defined by setting

$$\delta(\epsilon) = \inf_{\substack{u,v\in E,\ \|u\|=\|v\|=1,\\ \|u-v\|=\epsilon}} \left(1 - \left\|\frac{u+v}{2}\right\|\right) \tag{5.1}$$

is strictly positive on $(0, 2]$.

Recall that, by [21, Proposition 1.e.2], uniform smoothness corresponds to uniform convexity of the dual.

In the case of uniformly smooth and uniformly convex Banach spaces we have, see [25, Formula (1.4)], the following bound (for $\|u_n\|$, $\|u\| \le 1$)

$$\|u_n - u\| \le \|u_n\| - \delta(\|u\|) + o(1) \quad \forall u_n \rightharpoonup u \ne 0, \tag{5.2}$$

which quantifies the “reduction” of the norm caused by the subtraction of the polar limit. So, a suitable energy has been introduced in [25] which employes the modulus of convexity δ by associating to any profile φ the real number $\delta(\|\varphi\|)$. In this way, see [25, Formula (1.2)], the bound (EB) has been transformed (when $\liminf_{n\to+\infty} \|u_n\| \le 1$) in the following energy bound

$$\sum_{i=1}^{+\infty} \delta(\|\varphi_i\|) \le \liminf_{n\to+\infty} \|u_n\| \le 1. \tag{5.3}$$

Note that for every $1 < p < +\infty$ L^p spaces are uniformly convex (see Clarkson inequalities in [4]), moreover the asymptotic behavior of the modulus of convexity shows that, when $2 \le p < +\infty$, $\delta \circ \|\cdot\| \simeq p^{-1}2^{-p}\|\cdot\|_p^p$ and so (5.3) takes again the “shape” of the energy bound (EB).

6. Some recent terminology

The existence of the above mentioned distinct and slightly weaker formulations of [24, Theorem 2] due to P. Gérard and S. Jaffard (where the sum in the approximation is replaced by a finite sum whose number of addenda increases as $n \to +\infty$) has probably contributed to a misunderstanding concerning the two statements. Both of them are stated for “suitable” sequences of scalings but such sequences, as will be clarified later, are not the same in the two cases. The difference between the two statements has been underlined also by T. Tao in [29], in which the finite sum version (in another setting) has been defined *“more convenient”* while it clearly is a weaker result which can be stated under weaker assumptions (compare the statements of [9, Corollary 6.3] and [9, Corollary 6.4]). This circumstance does not facilitate the use of profile decomposition theorems in applications and partially justifies the still current specific arguments in giving the proof of the compactness of P.S. sequences related to distinct functionals, producing minor variants of Struwe Theorem which can all be easily deduced by profile decomposition, as we have mentioned for the original one.

We want to recall some terminology introduced in [9] and mainly used to underline the differences of the two statements specifying what the term “suitable” means in the two cases.

In [9] we introduced the notions of profile and profile system of a bounded sequence, and the notion of *s.t.s.* system (scale transition sequences system) or blowup system (related to a profile system $(\varphi_i)_{i\in I}$) and, among them, some specialized *s.t.s.* systems $((\rho_n^i)_{n\in\mathbb{N}})_{i\in I}$ which we call *routed s.t.s.* which (means that the sums $\sum_{i\in I}\rho_n^i(\varphi_i)$ are convergent unconditionally in i and uniformly with respect to n and which), roughly speaking, allows to treat the sum in (EB) as a finite sum.

In the remaining part of this section G will denote the group generated by the L^p invariant scalings, i.e. the group generated by the functions $g : L^p \to L^p$ such that, for all $u \in L^p$, $g(u) = \lambda^{\frac{N}{p}} u(\lambda(\cdot - x_0))$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^N$.

Definition 2 (Profiles and *s.t.s.* sequences or blowup sequences)**.** Let $(u_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ be a given bounded sequence. We say that $\varphi \in L^p(\mathbb{R}^N) \setminus \{0\}$ is a profile of the sequence $(u_n)_{n\in\mathbb{N}}$ if there exists $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} \subset G$ such that

$$\rho_n^{-1}(u_n) \rightharpoonup \varphi. \tag{6.1}$$

In such a case we shall say that $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ is a *scale transitions sequence* (*s.t.s.* for short) or a *blowup sequence* related to the profile φ.

Note that if φ is a profile of the sequence $(u_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ is a *s.t.s.* related to φ, then any $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n^{-1} \circ \rho_n \to i_d$ (i_d denotes the identity map on E) is still a *s.t.s.* related to φ, while for all $g \in G$, $g(\varphi)$ is still a profile of the sequence $(u_n)_{n\in\mathbb{N}}$ and $(\rho_n \circ g^{-1})_{n\in\mathbb{N}}$ is a *s.t.s.* related to the profile $g(\varphi)$.

Therefore we shall say that two profiles φ and ψ of a sequence $(u_n)_{n\in\mathbb{N}}$ are *distinct* if $\psi \neq g(\varphi)$ for all $g \in G$ while they are *copies* if exists $g \in G$ such that $\psi = g(\varphi)$. So any profile can be thought as a whole orbit of copies $G(\varphi) := (g(\varphi))_{g\in G}$. Finally, by taking into account [9, Remark 2.1], we deduce that if $(\rho_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ are *s.t.s.* related to distinct profiles they must be mutually diverging or quasi orthogonal (i.e. $(\sigma_n^{-1} \circ \rho_n)_{n\in\mathbb{N}}$ is diverging, which, in turn, means that $\sigma_n^{-1} \circ \rho_n \to 0$).

Definition 3 (Profile system)**.** Let $(u_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ be a bounded sequence. A family $(\varphi_i)_{i\in I}$ of profiles of the sequence $(u_n)_{n\in\mathbb{N}}$ is said to be a *profile system* (in $L^p(\mathbb{R}^N)$) of the sequence $(u_n)_{n\in\mathbb{N}}$ if, for any profile φ, all elements φ_i which are copies of φ are equal and their number is (finite and) less or equal to the multiplicity $m(\varphi)$ of the profile. ($m(\varphi)$ is the supremum of the cardinality of the sets of mutually diverging sequences related to φ).

It is worth to remark that any profile system is also a profile system of every subsequence.

Definition 4 (*s.t.s.* system (or blowup system), concentration system)**.** Let $\Phi = (\varphi_i)_{i\in I}$ be a profile system of a (bounded) sequence $(u_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$. A *s.t.s. system* or a *blowup system* related to the profile system Φ is any family $\boldsymbol{P} = (\boldsymbol{\rho}_i)_{i\in I}$ such that

i) for all $i \in I$, $\boldsymbol{\rho}_i = (\rho_n^i)_{n\in\mathbb{N}} \subset G$ is a *s.t.s.* sequence related to the profile φ_i;
ii) for all $i, j \in I$, $i \neq j$, $\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}_j$ are mutually diverging.

In such a case the pair $(\Phi, \boldsymbol{P})$ will be called a *concentration system* of $(u_n)_{n\in\mathbb{N}}$.

Definition 5 (Complete profile system, profile convergent sequence)**.** We shall say that a (possibly empty) profile system $(\varphi_i)_{i\in I}$ of a bounded sequence $(u_n)_{n\in\mathbb{N}}$ is *complete* if no subsequence $(u_{k_n})_{n\in\mathbb{N}}$ has a "richer profile system" (i.e. a profile system which has a new profile or a profile with a bigger multiplicity). If a sequence admits a complete profile system we shall say that it is *profile convergent.* Therefore a given bounded sequence $(u_n)_{n\in\mathbb{N}}$ is profile convergent if it does not admit any subsequence with a bigger number of profiles, or with profiles with higher multiplicity.

The existence of a complete profile system for a bounded sequence in L^p is guaranteed by [9, Theorem 3.1] which is a "multiscale" version of Banach Alaouglu Theorem and is reported below for the reader's convenience.

Theorem 1 (Multiscale weak compactness)**.** *Any bounded sequence in $L^p(\mathbb{R}^N)$ admits a profile convergent subsequence.*

Note that the multiscale weak compactness theorem, combined with the "theorem of alternative" [24, Theorem 1] allows to immediately deduce profile decomposition in Sobolev spaces, namely [24, Theorem 2] from the "cocompact embedding" of $H^{1,p}$ in L^{p^*}. Such cocompactness result, thanks to the Sobolev embedding in Lorentz spaces, admits a simple proof which is carried out on the Marcinkiewicz space of index p^*. This is not just a technical device because this result is false in the case of the optimal embedding in the wider category of Lorentz spaces, as analogously happens with Rellich Theorem in the category of Lebesgue spaces.

7. Passing to metric spaces

Now, we have concrete examples of functional spaces and a theory sufficiently general to include them. It may make sense to think to the more general context of metric spaces. Note that in the metric settlement the main cited theorems have no meaning since they are based on the notion of sum and therefore they heavily exploit the (now lacking) algebraic structure of the space. We shall show that, thanks to the reinterpretation of all relevant notions (such as profile, scale transitions sequence or blowup sequence, profile system, etc.), given in [9] and recalled in the previous section, we can approach the problem by taking as scalings the elements of a suitable group G of isometries which acts on the metric space (E, d) and which will surrogate the group of the invariant scalings in L^p.

The lack of a zero (in the algebraic sense) can be then compensated by the request that all elements in G admit a unique common fixed point z which leads to the request of the following axiom.

Axiom G1. *There exists a unique ("zero") $z \in E$ such that $g(z) = z$ for all $g \in G$.*

This request, by following the concrete example given by the L^p invariant scalings, can be reduced to the group H of translations which is a commutative invariant subgroup of G as shown in the following result.

Theorem 2. *Let G be a group of isometries on a metric space E. Let $H \subset G$ be an invariant commutative subgroup of G which satisfies*

i) *$\forall f \in H, f \neq i_d, f$ has at most two fixed points,*

ii) *$\exists h \in H$ which admits a unique fixed point $z \in E$.*

Then G admits a unique fixed point z.

Proof. We start by claiming that z in ii) is the unique fixed point of every $f \in H$, $f \neq i_d$. Indeed, given $f \in H$, we have, since z is fixed by h and H is commutative, that $f(z) = f(h(z)) = h(f(z))$ and deduce that also $f(z)$ is fixed by h, then by assumption ii), we deduce $f(z) = z$, i.e. z is fixed by f. Moreover, if $f(x) = x$ for some $x \in E$, and $f \in H$, being H

commutative, we have $f(h(x)) = h(f(x)) = h(x)$ i.e. $h(x)$ is fixed by f. If $x \neq z$ we have $h(x) \neq h(z) = z$ and, by ii), since x cannot be fixed by h, that $h(x) \neq x$. So x, $h(x)$ and z are three distinct fixed points of f, in contradiction to i) and this proves the claim.

We prove now that z is fixed by every map $g \in G$. Indeed, given $g \in G$ and $f \in H \setminus \{i_d\}$, we have, since H is invariant, that the map $\varphi = g \circ f \circ g^{-1} \in H$ and $\varphi(g(z)) = g(f(z)) = g(z)$, so $g(z)$ is fixed by $\varphi \in H$ and so $g(z) = z$. The uniqueness of z is a trivial consequence of ii). □

On the other hand, to simulate the behavior of L^p scalings we shall ask also the following compactness axiom on G of dichotomy (alternative) type which furthermore guarantees that the group G is closed with respect to the (strong pointwise) convergence (see [11]).

Axiom G2. *For any sequence $(g_n)_{n\in\mathbb{N}} \subset G$, if $(g_n)_{n\in\mathbb{N}}$ is not diverging (i.e. if for all $u \in E$ $(g_n(u))_{n\in\mathbb{N}}$ does not polarly converge to the "zero" $z \in E$), then there exists a subsequence $(g_{k_n})_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ which is converging.*

Since L^p spaces are, for $1 < p < +\infty$, uniformly smooth and uniformly convex Banach spaces (see [4]), we shall consider the corresponding metric counterpart known as Staple Rotundity (see [10, Definition 3.1 and Remark 3.2]).

Definition 6. A metric space (E, d) is a (uniform) SR ("Staples rotund") metric space (or satisfies (uniformly) property SR) if there exists a continuous function $\delta : (\mathbb{R}_+)^2 \to \mathbb{R}_+$ (called *modulus of rotundity* of the space) such that for any r, $\overline{d} > 0$, for any $u, v \in E$ with $d(u, v) \geq \overline{d}$:

$$\text{(SR)} \qquad \operatorname{rad}(B_r(u) \cap B_r(v)) \leq r - \delta(r, \overline{d}),$$

where, for any subset $X \subset E$, $\operatorname{rad}(X) := \inf_{u\in E} \sup_{v\in X} d(u, v)$ is the Chebyshev radius of the set X.

We take the opportunity to recall that the Chebyshev center of a set $X \subset E$, denoted by $\operatorname{cen}(X)$, is one of the points (if they exist) in E such that $\sup_{v\in X} d(\operatorname{cen}(X), v) = \operatorname{rad}(X)$ (i.e. are the centers (if they exist) of balls with minimal radius (the Chebyshev radius) containing the set X).

Note that, given $r > 0$ and $\overline{d} > \overline{d}' > 0$, since if $d(u, v) \geq \overline{d}$ then also $d(u, v) > \overline{d}'$ we get, by (SR), that $\operatorname{rad}(B_r(u) \cap B_r(v))$ not only is bounded by $r - \delta(r, \overline{d})$ but also by $r - \delta(r, \overline{d}')$. Therefore we deduce that $\delta(r, \overline{d}')$ cannot be bigger than $\delta(r, \overline{d})$ and so we shall always assume that the modulus of rotundity in monotone increasing with respect to $\overline{d}$.

It is worth to recall that, in complete metric spaces, property SR guarantees, among other things, existence and uniqueness of the asymptotic center of a bounded sequence (see [27, Theorem 2.5 and Theorem 3.3]). Moreover, see [10, Section 3-Statement a) and Remark 2.4], if the sequence is polar convergent, then the polar limit coincides with the asymptotic center of the sequence and therefore (4.4) holds true. Finally the space is (sequentially) compact with respect to polar convergence as stated in the following result proved in [10].

Theorem 3. *Let (E, d) be a complete SR metric space. Then every bounded sequence in E has a polar convergent subsequence.*

At this point the notions of profile, profile system, *s.t.s.* systems, etc. recalled in Section 6 can be extended in the metric setting (by replacing L^p and 0 by a complete SR metric space and by z respectively) and by replacing weak convergence ($\rightharpoonup$ sign) by polar convergence ($\rightharpoondown$ sign).

Next step is to recover the polar version of Theorem 1.

Let us briefly recall what we have get about profile decomposition in Banach spaces:

- a notion of polar convergence which perfectly fits in the metric context;
- a notion of uniform convexity which has in the Staple rotundity its metric counterpart;
- a energy function $\delta \circ \| \cdot \|$ which is based on the modulus of convexity (and on the norm) and which can be adapted to metric spaces by means of the modulus of Staple rotundity (and of the distance $d(\cdot, z)$ from the point z given by Axiom G1);
- a notion of profile system (and related *s.t.s.* system) and a weak multiscale compactness (see Theorem 1) which do not require any algebraic structure.

So, it has been quite natural to ask for the existence of profile decomposition theorems in the more general context of metric spaces and in [11], with S. Solimini and K. Tintarev, we have positively answered the question acting on suitable (and more general) isometries group.

The main difficulty to overcome has been the inability to "subtract" a profile from a profile system in order to get a (less "rich") new profile system (with a lacking profile). Indeed, in the linear setting, by subtracting the weak limit u (which is a particular profile) of a sequence $(u_n)_{n\in\mathbb{N}}$ one gets a sequence $(u_n - u)_{n\in\mathbb{N}}$ which admits a profile system Φ which, due to (5.3) has an energy which, by (NB), is less than the original one $\Phi \cup \{u\}$ (see Remark 1 below). This property, of course, corresponds to some "additivity property" of the distance function which, in general, does not hold since there is no additivity structure on the space.

So we have to reinterpret both (NB) and (5.3). In particular since our norm will be replaced by the distance from z, i.e. $\| \cdot \| \simeq d(\cdot, z)$, in order to evaluate the right hand side of (NB) we introduce the following *"asymptotic norm" or "z-radius"*, denoted by $\mathrm{rad}_z((u_n)_{n\in\mathbb{N}})$, of a (bounded) sequence $(u_n)_{n\in\mathbb{N}}$ by setting

$$\mathrm{rad}_z((u_n)_{n\in\mathbb{N}}) := \limsup_n d(u_n, z). \tag{7.1}$$

Moreover, since by (6.1) profiles of a bounded sequence $(u_n)_{n\in\mathbb{N}}$ are obtained as polar limit of sequences $(\rho_n^{-1}(u_n))_{n\in\mathbb{N}}$ with $(\rho_n)_{n\in\mathbb{N}} \subset G$, we introduce also the *multiscale asymptotic radius*, denoted by $\mathrm{rad}_G((u_n)_{n\in\mathbb{N}})$, of a (bounded) sequence $(u_n)_{n\in\mathbb{N}}$ by setting

$$\mathrm{rad}_G((u_n)_{n\in\mathbb{N}}) := \inf_{(\rho_n)_{n\in\mathbb{N}}\in\mathcal{G}} \mathrm{rad}_{as}((\rho_n(u_n))_{n\in\mathbb{N}}). \tag{7.2}$$

Note that, for any sequence $(u_n)_{n\in\mathbb{N}} \subset E$, we have

$$\mathrm{rad}_G((u_n)_{n\in\mathbb{N}}) \le \mathrm{rad}_{as}((u_n)_{n\in\mathbb{N}}) \le \mathrm{rad}_z((u_n)_{n\in\mathbb{N}}), \tag{7.3}$$

where $\mathrm{rad}_{as}((u_n)_{n\in\mathbb{N}})$ is defined by (4.3).

It is worth to remark that, in the linear case (where $z = 0$), since it is possible to subtract u from each element of the sequence, we have, by (4.4),

$$\mathrm{rad}_{as}((u_n)_{n\in\mathbb{N}}) = \limsup_n d(u_n - u, 0) = \mathrm{rad}_0((u_n - u)_{n\in\mathbb{N}}) = \mathrm{rad}_z((u_n - u)_{n\in\mathbb{N}}).$$

So in the linear case, by subtracting the polar limit u of a bounded sequence $(u_n)_{n\in\mathbb{N}}$ one gets a new sequence $(u_n - u)_{n\in\mathbb{N}}$ which has a less rich profile system (it just misses the polar limit we have subtracted) and whose z-radius rad_z is equal to the asymptotic radius rad_{as} of the given sequence. If, instead of polar limit we want to speak more generally about profiles, we shall consider rad_G instead of rad_{as} so, every time we subtract a profile we get a new sequence for which the rad_z is equal to the rad_G of the previous sequence.

Such result can be postulated, without any additive structure, by Axiom G3 below which deals with the extension of the above notions to the set of all sequences which admit a given profile system (and a related *s.t.s.* system). More precisely, given $I \subset \mathbb{N}$, $\Phi = (\varphi_i)_{i\in I}$ a family in $E \setminus \{z\}$, and $\boldsymbol{P} = (\boldsymbol{\rho_i})_{i\in I} = ((\rho^i_n)_{n\in\mathbb{N}})_{i\in I}$ a family of sequences of G, we define

$$U(\Phi) := \{(u_n)_{n\in\mathbb{N}} \mid \Phi \text{ is a profile system of } (u_n)_{n\in\mathbb{N}}\} \tag{7.4}$$

and

$$U(\Phi, \boldsymbol{P}) := \{(u_n)_{n\in\mathbb{N}} \mid (\Phi, \boldsymbol{P}) \text{ is a concentration system of } (u_n)_{n\in\mathbb{N}}\}. \tag{7.5}$$

When $U(\Phi) \neq \emptyset$, i.e. when there exists a sequence of which Φ is a profile system, we shall abuse Definition 3 by saying that $\Phi = (\varphi_i)_{i\in I} \subset E \setminus \{z\}$ is a "profile system". Analogously, we shall abuse Definition 4 by saying that $\boldsymbol{P}$ is a "*s.t.s.* system" or a "blowup system" related to Φ if $U(\Phi, \boldsymbol{P}) \neq \emptyset$. Moreover, in such a case, we shall also say that $\boldsymbol{P}$ is *compatible* with Φ or that $(\Phi, \boldsymbol{P})$ is a "concentration system". So, when $(\Phi, \boldsymbol{P})$ is a "concentration system" the following properties hold true:

i) if $\exists g \in G$ and $\exists i, j \in I$ such that $\varphi_j = g(\varphi_i)$ then g is the identity map i_d on E,
ii) $\forall i, j \in I$, $i \neq j$, $\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}_j$ are quasiorthogonal.

With a clear abuse of terminology that does not give rise to possible misunderstandings, we shall say that a system Φ' (resp. a concentration system $(\Phi', \boldsymbol{P}')$) *is included in* - or *is a subsystem of* - Φ (resp. $(\Phi, \boldsymbol{P})$) and we shall write $\Phi' \subset \Phi$ (resp. $(\Phi', \boldsymbol{P}') \subset (\Phi, \boldsymbol{P})$) if there exists $J \subset I$ such that $\Phi' = (\varphi_i)_{i\in J}$ (and, in the respective case, $\boldsymbol{P}' = (\boldsymbol{\rho}_i)_{i\in J}$). Finally, we shall say that Φ' (resp. $(\Phi', \boldsymbol{P}')$) is a *maximal subsystem* of Φ (resp. $(\Phi, \boldsymbol{P})$)) if the set $I \setminus J$ reduces to a single point.

In the remaining part of the paper we shall reserve the notation $U(\Phi)$ (resp. $U(\Phi, \boldsymbol{P})$) to profile systems Φ (resp. concentration systems $(\Phi, \boldsymbol{P})$).

Remark 1. When $(u_n)_{n\in\mathbb{N}} \in U(\Phi)$ and $\overline{\varphi}$ is a profile of $(u_n)_{n\in\mathbb{N}}$ and $\overline{\boldsymbol{\rho}} = (\overline{\rho}_n)_{n\in\mathbb{N}}$ is a related *s.t.s.* we shall write that $(u_n)_{n\in\mathbb{N}} \in U(\Phi \cup \{\overline{\varphi}\})$ and we shall say that $\Phi \cup \{\overline{\varphi}\}$ is the profile system of $(u_n)_{n\in\mathbb{N}}$ obtained by "adding" $\overline{\varphi}$ to Φ if one of the following alternatives hold true:

a) $\overline{\varphi} \neq \rho(\varphi_i)$ for all $i \in I$, $\rho \in G$ (i.e. when we are really adding a new profile);
b) if there exists $i \in I$ and $\rho \in G$ such that $\overline{\varphi} = \rho(\varphi_i)$ then $\overline{\boldsymbol{\rho}}$ is quasiorthogonal to every $\boldsymbol{\rho}_j$ such that $\varphi_j = \varphi_i$ (i.e. when we are increasing the multiplicity of $\overline{\varphi}$ since it already is one of the profiles in Φ).

Denoting by $\mathcal{U}$ one of the sets $U(\Phi)$ or $U(\Phi, \boldsymbol{P})$, we define the following *asymptotic radius notions* of the set $\mathcal{U}$.

$$\mathrm{rad}_{as}(\mathcal{U}) := \inf_{(u_n)_{n\in\mathbb{N}}\in\mathcal{U}} \mathrm{rad}_{as}((u_n)_{n\in\mathbb{N}}), \tag{7.6}$$

$$\mathrm{rad}_{z}(\mathcal{U}) := \inf_{(u_n)_{n\in\mathbb{N}}\in\mathcal{U}} \mathrm{rad}_{z}((u_n)_{n\in\mathbb{N}}), \tag{7.7}$$

and

$$\mathrm{rad}_G(\mathcal{U}) := \inf_{(u_n)_{n\in\mathbb{N}}\in\mathcal{U}} \mathrm{rad}_G((u_n)_{n\in\mathbb{N}}). \tag{7.8}$$

Note that

$$\mathrm{rad}_{as}(U(\Phi)) = \inf_{(u_n)_{n\in\mathbb{N}}\in U(\Phi)} \mathrm{rad}_G((u_n)_{n\in\mathbb{N}}) = \mathrm{rad}_G(U(\Phi)), \tag{7.9}$$

and that (at the light of the surrogate $d(\cdot, z)$ for the "norm") $\mathrm{rad}_z(U(\Phi))$ will be the good candidate to replace the right hand side of (5.3) (and therefore of (EB)).

Finally, remark that the above defined radii are monotone increasing functions with respect to inclusion. More precisely, if rad denotes one of the radii defined by (7.6), (7.7) and (7.8), then

$$\mathrm{rad}(U(\Phi', \boldsymbol{P}')) \le \mathrm{rad}(U(\Phi, \boldsymbol{P})) \quad \forall(\Phi', \boldsymbol{P}') \subset (\Phi, \boldsymbol{P}). \tag{7.10}$$

When strict inequality holds in (7.10) we shall say that rad is *strictly increasing* with respect to inclusion.

Now we are able to state the last axiom required to the group G.

Axiom G3. *The function* rad_z *is strictly increasing with respect to inclusion. Moreover, any finite profile system* Φ *admits a maximal profile subsystem* Φ' *such that*

$$\mathrm{rad}_z(U(\Phi')) \le \mathrm{rad}_{as}(U(\Phi)). \tag{7.11}$$

By making use of the modulus of rotundity δ of the space, for any $R > 0$ we can define the function $\boldsymbol{\delta}_R :]0, 2R] \to \mathbb{R}_+$, by setting for all $0 < \overline{d} \le 2R$

$$\boldsymbol{\delta}_R(\overline{d}) := \min_{\frac{\overline{d}}{2}\le r\le R} \delta(r, \overline{d}). \tag{7.12}$$

Note that, fixed $R > 0$, since the modulus of rotundity δ is monotone increasing with respect to the variable $\overline{d}$ and since the interval $[2^{-1}\,\overline{d}, R]$ reduces when $\overline{d}$ increases, we deduce that the function $\boldsymbol{\delta}_R$ is monotone increasing (in the variable $\overline{d}$). On the contrary, fixed $\overline{d} > 0$, the value $\boldsymbol{\delta}_R(\overline{d})$ decreases if $R \ge 2^{-1}\,\overline{d}$ increases. Finally, since $\boldsymbol{\delta}_R(\overline{d}) \le \delta(2^{-1}\,\overline{d}, \overline{d}) \le 2^{-1}\,\overline{d}$, we can extend $\boldsymbol{\delta}_R$ to 0 by setting $\boldsymbol{\delta}_R(0) = 0$.

This monotone cost function, depending on R (and composed with the "norm" $d(\cdot, z)$) allows to associate an energy $V_R(\varphi) = \boldsymbol{\delta}_R(d(\varphi, z))$ to any profile φ of a given (bounded) sequence $(u_n)_{n\in\mathbb{N}}$ such that $\mathrm{rad}_z((u_n)_{n\in\mathbb{N}}) < R$ and to provide an energy estimate (of the sum of the energies of the profiles of the sequence), similar to formula (5.3) (and therefore to (EB)) stated in the following lemma which is proved (in a more general approach in [11]).

Lemma 1 (Energy estimate)**.** *Let* $R > 0$ *be given. Then, for any profile system* $\Phi = (\varphi_i)_{i\in I}$ *such that* $\mathrm{rad}_z(U(\Phi)) < R$*, we have*

$$V_R(\Phi) := \sum_{i\in I} V_R(\varphi_i) = \sum_{i\in I} \boldsymbol{\delta}_R(d(\varphi_i, z)) \le \mathrm{rad}_z(U(\Phi)) < R. \tag{7.13}$$

By taking into account that, as remarked above, the function $\boldsymbol{\delta}_R$ is decreasing with respect to R, we get that the energy bound (7.13) still holds true if one replaces R by any larger real number, indeed the left hand side decreases while the right hand side increases (the central term is independent on R). Therefore smaller is R more significant is (7.13).

Thanks to the above estimate it is possible to recover without any relevant effort the maximality argument used in [9] to prove the multiscale compactness theorem in L^p spaces.

Theorem 4 (Multiscale polar compactness). *Let E be a complete SR metric space, with an admissible group G of scalings satisfying axioms G1, G2 and G3. Then any bounded sequence in E admits a (polar) profile convergent subsequence.*

8. What about profile decomposition in metric spaces

At this point the metric structure does not allow to speak about profile reconstruction as the sum of "deflated" profiles used in L^p (see [9, Definition 4.3]) and which can of course be written, without big troubles (the sum becomes a series), in the linear setting. Since in general, in a metric space, we have not any algebraic structure, we shall use instead a suitable counterpart of the characterizing formula, given for L^p spaces, in [9, Formula (4.18)]. Therefore, as a first attempt, one can give as a definition of profile reconstruction (of a given concentration system $(\Phi, \boldsymbol{P})$) the sequence in $U(\Phi, \boldsymbol{P})$ which is closer to z, i.e. which minimizes $\mathrm{rad}_z(U(\Phi, \boldsymbol{P}))$.

By introducing some extra axioms on the space, it is possible to prove that given any concentration system $(\Phi, \boldsymbol{P})$ the related profile reconstruction is unique (modulo infinitesimal terms and modulo subsequences). Moreover the profile reconstruction approximates, modulo an infinitesimal term, a given profile convergent sequence which admits $(\Phi, \boldsymbol{P})$ as a complete concentration system.

In this way we get a metric version of profile decomposition theorems. The reader can refer to [11] for more details.

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