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INFINITELY MANY PERIODIC SOLUTIONS FOR A FRACTIONAL PROBLEM UNDER PERTURBATION

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Abstract. We discuss the existence of infinitely many periodic weak solutions for a subcritical nonlinear problem involving the fractional operator $(-\Delta+I)^s$ on the torus $\mathbb{T}^N$. By using an abstract critical point result due to Clapp [14], we prove that, in spite of the presence of a perturbation $h \in L^2(\mathbb{T}^N)$ which breaks the symmetry of the problem under consideration, it is possible to find an unbounded sequence of periodic (weak) solutions.

1. Introduction

In the past years there has been a considerable amount of research related to the role of symmetry in obtaining multiple critical points of symmetric functionals associated to ordinary and partial differential equations. For instance, semilinear problems of the type

$$\begin{cases} Lu = f(x,u) + h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{1.1}$$

where L is uniformly elliptic, Ω is a smooth bounded domain in $\mathbb{R}^N$, $f(x,u)$ behaves like $|u|^{q-2}u$ with $q \in (2, \frac{2N}{N-2})$, and $h \in L^2(\Omega)$ is a perturbation, has been investigated by many authors by using topological and variational methods; see for instance [8, 9, 19, 25].
In this paper we focus our attention on the effect of a perturbation which destroys the symmetry of the following nonlinear fractional problem

$$(-\Delta + I)^s u = f(x,u) + h(x) \text{ on } \mathbb{T}^N, \tag{1.2}$$

where $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ is the N-dimensional torus, $N \geq 2$, $s \in (0,1)$, and $f : \mathbb{T}^N \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the following hypotheses:

- **$(f1)$:** f is a continuous function and $f(x,-t) = -f(x,t)$ for all $x \in \mathbb{T}^N$ and $t \in \mathbb{R}$;
- **$(f2)$:** there exist $p \in (1, 2^*_s - 1)$, where $2^*_s = \frac{2N}{N-2s}$, and $a_1, a_2 > 0$ such that for any $x \in \mathbb{T}^N$ and $t \in \mathbb{R}$
$$|f(x,t)| \leq a_1 + a_2|t|^p;$$
- **$(f3)$:** there exist $\mu > 2$ and $r_0 > 0$ such that
$$0 < \mu F(x,t) \leq t f(x,t)$$
for $x \in \mathbb{T}^N$, $|t| \geq r_0$, where $F(x,t) = \int_0^t f(x,\tau)d\tau$.

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Here we assume that the perturbation

$$h \in L^2(\mathbb{T}^N) \tag{1.3}$$

and that p satisfies the following condition

$$\frac{(N+2s)-p(N-2s)}{N(p-1)} > \frac{\mu}{\mu-1}. \tag{1.4}$$

We notice that $(f1)$ and $(f3)$ imply the existence of constants $a_3, a_4, a_5 > 0$ such that

$$\frac{1}{\mu}(tf(x,t)+a_3) \geq F(x,t)+a_4 \geq a_5|t|^{\mu} \tag{1.5}$$

for all $t \in \mathbb{R}$.
The operator $(-\Delta+I)^s$ on $\mathbb{T}^N$ is defined for any $u \in \mathcal{C}^{\infty}(\mathbb{T}^N)$ by setting

$$(-\Delta+I)^s u(x) = \sum_{k\in\mathbb{Z}^N} (|k|^2+1)^s c_k e^{\imath kx}$$

where $c_k = \int_{\mathbb{T}^N} u(x)e^{-\imath k\cdot x}dx$ are the Fourier coefficients of u. This operator can be extended by density on the Hilbert space

$$\mathbb{H}^s(\mathbb{T}^N) = \left\{ u = \sum_{k\in\mathbb{Z}^N} c_k e^{\imath k\cdot x} \in L^2(\mathbb{T}^N) \Big| [u]^2_{\mathbb{H}^s(\mathbb{T}^N)} := \sum_{k\in\mathbb{Z}^N} |k|^{2s}|c_k|^2 < \infty \right\}.$$

The study of fractional and non-local operators of elliptic type received immensely growing attention recently, because of their strong connection with real-world problems. These operators, arise in several contexts such as phase transition phenomena, population dynamics, game theory, mathematical finance, chemical reactions of liquids, geophysical fluid dynamics, quantum mechanics; see [16] and references therein for more details and applications.
In spite of the fact that there are many papers dealing with superlinear problems involving non-local operators [5, 10, 11, 17, 18, 22, 23], there are few results concerning the multiplicity of solutions for a non-local boundary problem under the effect of a perturbation. The only results which we know are due to Servadei [21], that proved the existence of infinitely many solutions to the problem

$$\begin{cases} (-\Delta)^s_{\mathbb{R}^N} u - \lambda u = f(x,u) + h & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases},$$

where $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ is a Lipschitz bounded open set, $(-\Delta)^s_{\mathbb{R}^N}$ is the fractional Laplacian, $\lambda \in \mathbb{R}$, f is a subcritical nonlinearity and $h \in L^2(\Omega)$ is a perturbation, and Colorado et al. [15] which studied existence and multiplicity of solutions for the following fractional critical problem involving the spectral Laplacian $(-\Delta)^s_{\Omega}$

$$\begin{cases} (-\Delta)^s_{\Omega} u = |u|^{\frac{4s}{N-2s}} u + h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

under appropriate conditions on the size of h. We point out that the non-local operators $(-\Delta)^s_{\mathbb{R}^N}$ and $(-\Delta)^s_{\Omega}$ appearing in the above problems are different; see [24].

The aim of this paper is to give a further result in this direction, considering a non-local problem with periodic boundary conditions, under the effect of a not small perturbation. Our main result can be stated as follows:

Theorem 1. *Let f satisfying $(f1)$-$(f3)$ and let $h \in L^2(\mathbb{T}^N)$. Assume that p satisfies the relation in (1.4). Then, (1.2) possesses an unbounded sequence of periodic solutions $(u_j)_{j\in\mathbb{N}} \subset \mathbb{H}^s(\mathbb{T}^N)$.*

Let us observe that when $h = 0$, the existence of infinitely many solutions to (1.2) can be obtained by using standard critical point theory for even functionals [1].
Our purpose is to investigate (1.2) in the case $h \neq 0$, that is when we have a lack of symmetry. In order to do this, we will use a variant of the Mountain Pass Theorem due to Clapp [14]:

Theorem 2. [14] *Let V be a G-Hilbert space with $V^G = \{0\}$, and let $V_1 \subset V_2 \subset \cdots \subset V_k \subset \dots$ be a sequence of finite dimensional G-invariant linear subspaces of V. Here $V^G := \{x \in V : gx = x \text{ for all } g \in G\}$ is the set of fixed points of V in G. Let $J : V \to \mathbb{R}$ be a $\mathcal{C}^1$-functional which satisfies the following conditions*

- **(i):** *J verifies the Palais-Smale condition $(PS)_a$ above a for some $a > 0$, that is any sequence (x_n) in V such that $J(x_n) \subset [a, b]$ for some $b \in \mathbb{R}$ and such that $J'(x_n) \to 0$ as $n \to \infty$ has a convergent subsequence;*
- **(ii):** *There are constants $\gamma > 0$ and $\mu > 1$ such that for all $x \in V$ and $g \in G$*
$$|J(x) - J(gx)| \leq \gamma(|J(x)|^{\frac{1}{\mu}} + 1);$$
- **(iii):** *There are constants $\beta > 0$, $\theta > \frac{\mu}{\mu-1}$, $j_0 \geq 1$ such that for all $j \geq j_0$*
$$\sup_{\rho\geq 0} \inf\{J(x) : x \in V_{j-1}^{\perp}, \|x\| = \rho\} \geq \beta j^{\theta};$$
- **(iv):** *For every $j \geq 1$ there exists $R_j > 0$ such that $\Phi(x) \leq 0$ for all $x \in V_j$: $\|x\| \geq R_j$;*
- **(v):** *There exists a fixed admissible representation W of G such that for all $j \geq j_0$, $V_j \cong \oplus_{i=1}^{j} W$.*

Then J has an unbounded sequence of critical values.

This result can be read as follows: if J is not too far away from being G-invariant and if the mountain range is steep enough, then J can still have an unbounded sequence of critical values.
In order to prove Theorem 1, we will introduce the following functionals defined on $\mathbb{H}^s(\mathbb{T}^N)$
$$I(u) = \frac{1}{2}\|u\|^2_{\mathbb{H}^s(\mathbb{T}^N)} - \int_{\mathbb{T}^N} F(x,u)dx - \int_{\mathbb{T}^N} hudx$$
and
$$J(u) = \frac{1}{2}\|u\|^2_{\mathbb{H}^s(\mathbb{T}^N)} - \int_{\mathbb{T}^N} F(x,u)dx - \int_{\mathbb{T}^N} \psi(u)hudx,$$
where ψ is a suitable functional such that $\psi(u) = 1$ if u is a critical point of I.
By considering the antipodal action of $G = \mathbb{Z}_2$ on $\mathbb{H}^s(\mathbb{T}^N)$, we will show that J satisfies the assumptions of Theorem 2 and that large critical values of the modified functional J are critical values of I.
We would like to note that in [2, 3, 4, 6, 7] the existence of periodic solutions to fractional problems of the type
$$(-\Delta + I)^s u = f(x,u) \text{ on } \mathbb{T}^N, \tag{1.6}$$
has been obtained by using variational methods after transforming (1.6) in a degenerate elliptic equation with nonlinear Neumann boundary conditions via a Caffarelli-Silvestre type

extension [12] in periodic setting. In this paper however, we prefer to analyze the problem directly in $\mathbb{H}^s(\mathbb{T}^N)$ so that we can adapt the techniques developed in [19].

The paper is organized as follows: in Section 2 we present some preliminary facts concerning the fractional Sobolev spaces on torus, and in Section 3 we give the proof of Theorem 1.

2. Preliminaries

2.1. **Fractional Sobolev spaces on torus.** In this section we collect some preliminary results concerning the fractional Sobolev spaces on torus.

Let $s \in (0,1)$ and $N \geq 2$. Let $u \in \mathcal{C}^\infty(\mathbb{T}^N)$. As usual, we identify $\mathbb{T}^N$ with $[0,2\pi]^N$, and the functions on $\mathbb{T}^N$ with functions on $\mathbb{R}^N$ that are periodic with period 2π in each coordinate $x_1, \dots, x_N$, that is $u(x + 2\pi e_i) = u(x)$ for all $x \in \mathbb{R}^N$ and $i = 1, \dots, N$.

Then we know that

$$u(x) = \sum_{k \in \mathbb{Z}^N} c_k e^{\imath k \cdot x},$$

where

$$c_k = \int_{\mathbb{T}^N} u(x) e^{-\imath k \cdot x} dx \quad (k \in \mathbb{Z}^N)$$

are the Fourier coefficients of u. We define the fractional Sobolev space $\mathbb{H}^s(\mathbb{T}^N)$ as the closure of $\mathcal{C}^\infty(\mathbb{T}^N)$ under the norm

$$\|u\|^2_{\mathbb{H}^s(\mathbb{T}^N)} := \sum_{k \in \mathbb{Z}^N} (|k|^2 + 1)^s \, |c_k|^2.$$

Let us observe that $\mathbb{H}^s(\mathbb{T}^N)$ is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_{\mathbb{H}^s(\mathbb{T}^N)} = \sum_{k \in \mathbb{Z}^N} (|k|^2 + 1)^s c_k \bar{d}_k$$

for any $u = \sum_{k \in \mathbb{Z}^N} c_k e^{\imath k \cdot x}$ and $v = \sum_{k \in \mathbb{Z}^N} d_k e^{\imath k \cdot x}$ belonging to $\mathbb{H}^s(\mathbb{T}^N)$. Finally we use the notation

$$[u]^2_{\mathbb{H}^s(\mathbb{T}^N)} = \sum_{k \in \mathbb{Z}^N} |k|^{2s} \, |c_k|^2$$

to indicate the semi-norm of u.

Now, we recall the following embeddings

Theorem 3. *(Fractional Sobolev embeddings on torus) The inclusion of $\mathbb{H}^s(\mathbb{T}^N)$ in $L^q(\mathbb{T}^N)$ is continuous for any $q \in [1, 2^*_s]$ and compact for any $q \in [1, 2^*_s)$.*

Proof. We give a simple proof of this result. Let $u = \sum_{k \in \mathbb{Z}^N} c_k e^{\imath k \cdot x}$ be a smooth function on $\mathbb{T}^N$ such that $\int_{\mathbb{T}^N} u \, dx = 0$, and let $v = \sum_{k \in \mathbb{Z}^N} d_k e^{\imath k \cdot x} \in L^{\frac{2N}{N+2s}}(\mathbb{T}^N)$.

By applying the Cauchy-Schwartz inequality, we can see that

$$\begin{aligned} |(u,v)_{L^2(\mathbb{T}^N)}| &= \Big| \sum_{|k| \geq 1} c_k \bar{d}_k \Big| = \Big| \sum_{|k| \geq 1} |k|^s |k|^{-s} c_k \bar{d}_k \Big| \\ &\leq \Big(\sum_{|k| \geq 1} |k|^{2s} |c_k|^2 \Big)^{\frac{1}{2}} \Big(\sum_{|k| \geq 1} |k|^{-2s} |d_k|^2 \Big)^{\frac{1}{2}} \\ &= [u]_{\mathbb{H}^s(\mathbb{T}^N)} \|(-\Delta)^{-\frac{s}{2}} v\|_{L^2(\mathbb{T}^N)}. \qquad (2.1) \end{aligned}$$

Now, by the Hardy-Littlewood-Sobolev inequality we know that

$$\|(-\Delta)^{-\frac{s}{2}}v\|_{L^2(\mathbb{T}^N)} \leq C\|v\|_{L^{\frac{2N}{N+2s}}(\mathbb{T}^N)} \tag{2.2}$$

for some constant $C > 0$.
Combining (2.1) and (2.2) we get

$$|(u,v)_{L^2(\mathbb{T}^N)}| \leq C[u]_{\mathbb{H}^s(\mathbb{T}^N)}\|v\|_{L^{\frac{2N}{N+2s}}(\mathbb{T}^N)}. \tag{2.3}$$

Taking $v = |u|^{\frac{N+2s}{N-2s}-1}u \in L^{\frac{2N}{N+2s}}(\mathbb{T}^N)$, we have

$$|(u,v)_{L^2(\mathbb{T}^N)}| = \|u\|^{\frac{2N}{N-2s}}_{L^{\frac{2N}{N-2s}}(\mathbb{T}^N)}$$

and

$$\|v\|_{L^{\frac{2N}{N+2s}}(\mathbb{T}^N)} = \|u\|^{\frac{N+2s}{N-2s}}_{L^{\frac{2N}{N-2s}}(\mathbb{T}^N)},$$

so (2.3) becomes

$$\|u\|_{L^{\frac{2N}{N-2s}}(\mathbb{T}^N)} \leq C[u]_{\mathbb{H}^s(\mathbb{T}^N)}. \tag{2.4}$$

This allows us to deduce that the embedding of $\mathbb{H}^s(\mathbb{T}^N)$ into $L^q(\mathbb{T}^N)$ is continuous for any $q \in [1, 2^*_s]$.
Finally, we show that $\mathbb{H}^s(\mathbb{T}^N)$ is compactly embedded in $L^q(\mathbb{T}^N)$ for any $q \in [1, 2^*_s)$. By using the interpolation inequality and (2.4), we know that for every $q \in (2, 2^*_s)$

$$\|u\|_{L^q(\mathbb{T}^N)} \leq \|u\|^{\theta}_{L^2(\mathbb{T}^N)}\|u\|^{1-\theta}_{L^{2^*_s}(\mathbb{T}^N)} \leq C\|u\|^{\theta}_{L^2(\mathbb{T}^N)}\|u\|^{1-\theta}_{\mathbb{H}^s(\mathbb{T}^N)}$$

for some $\theta \in (0,1)$. Therefore, it suffices to verify that $\mathbb{H}^s(\mathbb{T}^N)$ is compactly embedded in $L^2(\mathbb{T}^N)$ to obtain the desired result.
Let $u^j \rightharpoonup 0$ in $\mathbb{H}^s(\mathbb{T}^N)$ as $j \to \infty$. Then

$$\lim_{j\to\infty} |c^j_k|^2(|k|^2+1)^s = 0 \quad \forall k \in \mathbb{Z}^N \tag{2.5}$$

and

$$\sum_{k\in\mathbb{Z}^N} |c^j_k|^2(|k|^2+1)^s \leq C \quad \forall j \in \mathbb{N}. \tag{2.6}$$

Fix $\varepsilon > 0$. Then there exists $\nu > 0$ such that $(|k|^2+1)^{-s} < \varepsilon$ for $|k| > \nu$. By (2.6) we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^N} |c^j_k|^2 &= \sum_{|k|\leq\nu} |c^j_k|^2 + \sum_{|k|>\nu} |c^j_k|^2 \\ &= \sum_{|k|\leq\nu} |c^j_k|^2 + \sum_{|k|>\nu} |c^j_k|^2(|k|^2+1)^s(|k|^2+1)^{-s} \\ &\leq \sum_{|k|\leq\nu} |c^j_k|^2 + C\varepsilon.\end{aligned}$$

By (2.5) we deduce that $\sum_{|k|\leq\nu} |c^j_k|^2 < \varepsilon$ for j large. So $u^j \to 0$ in $L^2(\mathbb{T}^N)$ as $j \to \infty$.

□

It is well known (see [20, 26]) that the powers of a non-negative and self-adjoint operator in a bounded domain are defined through the spectral decomposition using the powers of the eigenvalues of the original operator. Since $(-\Delta+I)^{-s}$ is a positive compact self-adjoint operator in $L^2(\mathbb{T}^N)$, it is easy to show that the following result holds:

Theorem 4 (Spectral Theorem)**.**

(i): *The operator $(-\Delta+I)^s$ has a countable family of eigenvalues $\{\lambda_h\}_{h\in\mathbb{N}}$ which can be written as an increasing sequence of positive numbers*

$$0<\lambda_1<\lambda_2\leq\cdots\leq\lambda_h\leq\lambda_{h+1}\leq\dots.$$

Each eigenvalue is repeated a number of times equal to its multiplicity (which is finite).

(ii): $\lambda_h=\mu_h^s$ *for all $h\in\mathbb{N}$, where $\{\mu_h\}_{h\in\mathbb{N}}$ is the increasing sequence of eigenvalues of $-\Delta+I$.*

(iii): $\lambda_1=1$ *is simple, $\lambda_h=\mu_h^s\to+\infty$ as $h\to+\infty$,*

(iv): *The sequence $\{u_h\}_{h\in\mathbb{N}}$ of eigenfunctions corresponding to λ_h is an orthonormal basis of $L^2(\mathbb{T}^N)$ and an orthogonal basis of $\mathbb{H}^s(\mathbb{T}^N)$.*
Let us note that $\{u_h,\mu_h\}_{h\in\mathbb{N}}$ are the eigenfunctions and eigenvalues of $-\Delta+I$.

(v): *For any $h\in\mathbb{N}$, λ_h has finite multiplicity, and there holds*

$$\lambda_h=\min_{u\in V_h^{\perp}\setminus\{0\}}\frac{\|u\|^2_{\mathbb{H}^s(\mathbb{T}^N)}}{\|u\|^2_{L^2(\mathbb{T}^N)}}\quad\textit{(Rayleigh's principle)}$$

where

$$V_h=span\{u_1,\cdots,u_h\}$$

and

$$V_h^{\perp}=\{u\in\mathbb{H}^s(\mathbb{T}^N):\langle u,u_j\rangle_{\mathbb{H}^s(\mathbb{T}^N)}=0,\ for\ j=1,\dots,h-1\}.$$

(vi): *For any $h\in\mathbb{N}$, the h-eigenvalue can be characterized as follows:*

$$\lambda_h=\max_{u\in V_h\setminus\{0\}}\frac{\|u\|^2_{\mathbb{H}^s(\mathbb{T}^N)}}{\|u\|^2_{L^2(\mathbb{T}^N)}}.$$

3. Proof of Theorem 1

This last section is devoted to the proof of our main result.
Let us introduce the following functional

$$I(u)=\frac{1}{2}\|u\|^2_{\mathbb{H}^s(\mathbb{T}^N)}-\int_{\mathbb{T}^N}F(x,u)\,dx-\int_{\mathbb{T}^N}hu\,dx \tag{3.1}$$

defined for $u\in\mathbb{H}^s(\mathbb{T}^N)$. Clearly, $I\in\mathcal{C}^1(\mathbb{H}^s(\mathbb{T}^N),\mathbb{R})$ in view of the assumptions on f.
We begin proving the following

Lemma 1. *Let u be a critical point of I. Then there is a constant a_6 depending on $\|h\|_{L^2(\mathbb{T}^N)}$ such that*

$$\int_{\mathbb{T}^N}[F(x,u)+a_4]\,dx\leq\frac{1}{\mu}\int_{\mathbb{T}^N}[uf(x,u)+a_3]\,dx\leq a_6(I(u)^2+1)^{1/2} \tag{3.2}$$

Proof. By using (1.5) and the fact that u is a critical point of I, we can see that

$$\begin{aligned} I(u) &= I(u) - \frac{1}{2} I'(u)u \\ &\geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big) \int_{\mathbb{T}^N} [uf(x,u) + a_3]\,dx - \frac{1}{2}\|h\|_{L^2(\mathbb{T}^N)} \|u\|_{L^2(\mathbb{T}^N)} - a_7. \end{aligned} \tag{3.3}$$

Since $\mu > 2$ and by applying the Hölder and Young inequalities we deduce that for any $\varepsilon > 0$

$$I(u) \geq a_8 \int_{\mathbb{T}^N} (uf(x,u) + a_3)\,dx - a_9 - C_\varepsilon \|h\|^\nu_{L^2(\mathbb{T}^N)} - \varepsilon \|u\|^\mu_{L^\mu(\mathbb{T}^N)}, \tag{3.4}$$

where $\frac{1}{\mu} + \frac{1}{\nu} = 1$. Choosing ε such that $2\varepsilon = \mu a_5 a_8$, and by using (3.4), (1.5) and the Schwartz inequality, we obtain the claim.

□

Now, we modify the functional I as follows. Let $\chi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(t) = 1$ for $t \leq 1$, $\chi(t) = 0$ for $t > 2$ and $-2 < \chi' < 0$ for $t \in (1, 2)$.
For $u \in \mathbb{H}^s(\mathbb{T}^N)$, we set

$$Q(u) = 2a_6(I(u)^2 + 1)^{\frac{1}{2}}$$

and we define the following functionals on $\mathbb{H}^s(\mathbb{T}^N)$

$$\psi(u) = \chi\left(Q(u)^{-1} \int_{\mathbb{T}^N} [F(x,u) + a_4]\,dx\right)$$

and

$$J(u) = \frac{1}{2}\|u\|^2_{\mathbb{H}^s(\mathbb{T}^N)} - \int_{\mathbb{T}^N} F(x,u)\,dx - \int_{\mathbb{T}^N} \psi(u) hu\,dx.$$

We notice that (3.2) implies that $\psi(u) = 1$ if u is a critical point of I, and in particular $J(u) = I(u)$.
Let us consider the antipodal action of $G = \mathbb{Z}_2$ on $W = \mathbb{R}$, which is admissible by the Borsuk-Ulam Theorem.
In order to show that J verifies the condition (*ii*) of Theorem 2, we give the following preliminary result

Lemma 2. *If $u \in \operatorname{supp}\psi$, then*

$$\Big|\int_{\mathbb{T}^N} hu\,dx\Big| \leq \alpha_1(|I(u)|^{\frac{1}{\mu}} + 1) \tag{3.5}$$

where α_1 depends on $\|h\|_{L^2(\mathbb{T}^N)}$.

Proof. By using the Schwartz and Hölder inequalities and (1.5), we obtain that for any $u \in \mathbb{H}^s(\mathbb{T}^N)$

$$\begin{aligned} \Big|\int_{\mathbb{T}^N} hu\,dx\Big| &\leq \|h\|_{L^2(\mathbb{T}^N)} \|u\|_{L^2(\mathbb{T}^N)} \leq \alpha_2 \|u\|_{L^\mu(\mathbb{T}^N)} \\ &\leq \alpha_3 \Big(\int_{\mathbb{T}^N} (F(x,u) + a_4)\,dx\Big)^{\frac{1}{\mu}}. \end{aligned} \tag{3.6}$$

Let us note that, if $u \in \operatorname{supp}\psi$, we get

$$\int_{\mathbb{T}^N} (F(x,u) + a_4)\,dx \leq 4a_6(I(u)^2 + 1)^{1/2} \leq \alpha_4(|I(u)| + 1). \tag{3.7}$$

Then, taking into account (3.6) and (3.7), we have the thesis. □

At this point we can prove that J satisfies the following property:

Lemma 3. *There is a constant β_1, depending on $\|h\|_{L^2(\mathbb{T}^N)}$, such that for all $u \in \mathbb{H}^s(\mathbb{T}^N)$,*

$$|J(u) - J(-u)| \leq \beta_1(|J(u)|^{\frac{1}{\mu}} + 1). \tag{3.8}$$

Proof. By using the definition of J and the assumption $(f1)$, we can see that

$$|J(u) - J(-u)| = (\psi(u) + \psi(-u))\Big|\int_{\mathbb{T}^N} hu\,dx\Big|. \tag{3.9}$$

Then, by Lemma 2, we deduce that

$$\psi(-u)\Big|\int_{\mathbb{T}^N} hu\,dx\Big| \leq \alpha_1\psi(-u)(|I(u)|^{\frac{1}{\mu}} + 1). \tag{3.10}$$

Let us observe that by the definitions of $I(u)$ and $J(u)$ we know that

$$|I(u)| \leq |J(u)| + 2\Big|\int_{\mathbb{T}^N} hu\,dx\Big|, \tag{3.11}$$

so, by using (3.10) we deduce

$$\psi(-u)\Big|\int_{\mathbb{T}^N} hu\,dx\Big| \leq \alpha_2\psi(-u)\Big(|J(u)|^{\frac{1}{\mu}} + \Big|\int_{\mathbb{T}^N} hu\,dx\Big|^{\frac{1}{\mu}} + 1\Big). \tag{3.12}$$

Thus, by using the Young's inequality, we can see that the term $\int_{\mathbb{T}^N} hu\,dx$ on the right-hand side of (3.12) can be absorbed by the left-hand side. Similarly, we can deduce a corresponding estimate for the $\psi(-u)$ term in (3.9), so we can infer that (3.8) holds. □

Now, we show that large critical values of J are critical values of I. Firstly we prove the following preliminary result:

Lemma 4. *There are constants $M_0, \alpha_0 > 0$, depending on $\|h\|_{L^2(\mathbb{T}^N)}$, such that if $M \geq M_0$, $J(u) \geq M$ and $u \in \operatorname{supp}\psi$, then $I(u) \geq \alpha_0 M$.*

Proof. Clearly, if $u \in \operatorname{supp}\psi$, then

$$I(u) \geq J(u) - \Big|\int_{\mathbb{T}^N} hu\,dx\Big|. \tag{3.13}$$

Hence (3.5) and (3.13) imply

$$I(u) + \alpha_1|I(u)|^{1/\mu} \geq J(u) - \alpha_1 \geq M/2 \tag{3.14}$$

for M_0 large enough. If $I(u) \leq 0$,

$$\frac{\alpha_1^\nu}{\nu} + \frac{1}{\mu}|I(u)| \geq \alpha_1|I(u)|^{1/\mu} \geq M/2 + |I(u)| \tag{3.15}$$

which gives a contradiction if $M_0 > 2\alpha_1^\nu\nu^{-1}$. As a consequence, $I(u) > 0$ and

$$I(u) > M/4 \text{ or } I(u) \geq \Big(\frac{M}{4\alpha_1}\Big)^\mu$$

which implies the Lemma since $\mu > 2$. □

Lemma 5. *There is a constant $M_1 > 0$ such that if $J(u) \geq M_1$ and $J'(u) = 0$, then $J(u) = I(u)$ and $I'(u) = 0$.*

Proof. We are going to prove that $\psi(u) = 1$ and $\psi'(u) = 0$. Taking into account the definition of ψ, this happens if

$$Q(u)^{-1} \int_{\mathbb{T}^N} (F(x,u) + a_4)\, dx \leq 1. \tag{3.16}$$

Now, we show that (3.16) is satisfied.
Let us note that

$$J'(u)u = \|u\|^2_{\mathbb{H}^s(\mathbb{T}^N)} - \int_{\mathbb{T}^N} uf(x,u) - (\psi(u) + \psi'(u)u)hu\, dx, \tag{3.17}$$

where

$$\psi'(u)u = \chi'\Big(Q(u)^{-1} \int_{\mathbb{T}^N} (F(x,u) + a_4)\, dx\Big)$$
$$Q(u)^{-2}\Big[Q(u) \int_{\mathbb{T}^N} uf(x,u)\, dx - (2a_6)^2 \Big(\int_{\mathbb{T}^N} (F(x,u) + a_4)\, dx\Big) Q(u)^{-1} I(u) I'(u)u\Big].$$

Then, we can regroup the terms as
(3.18)

$$J'(u)u = (1 + T_1(u))\|u\|^2_{\mathbb{H}^s(\mathbb{T}^N)} - (1 + T_2(u)) \int_{\mathbb{T}^N} uf(x,u)\, dx - (\psi(u) + T_1(u)) \int_{\mathbb{T}^N} hu\, dx,$$

where

$$T_1(u) = \chi'\Big(Q(u)^{-1} \int_{\mathbb{T}^N} (F(x,u) + a_4)\, dx\Big)(2a_6)^2 Q(u)^{-3} I(u) \int_{\mathbb{T}^N} (F(x,u) + a_4)\, dx \int_{\mathbb{T}^N} hu\, dx$$

and

$$T_2(u) = \chi'\Big(Q(u)^{-1} \int_{\mathbb{T}^N} (F(x,u) + a_4)\, dx\Big)\Big[Q(u)^{-1} \int_{\mathbb{T}^N} hu\, dx\Big] + T_1(u).$$

Now, we consider

$$J(u) - \frac{1}{2(1 + T_1(u))} J'(u)u. \tag{3.19}$$

If $T_1(u) = T_2(u) = 0$ and $\psi(u) = 1$, then (3.19) reduces to the left-hand side of (3.3), so (3.16) follows from (3.2). Since $0 \leq \psi(u) \leq 1$, if $T_1(u)$ and $T_2(u)$ are both small enough, the calculation made in (3.3) when carried out for (3.19) leads to (3.2) with a_6 replaced by a larger constant which is smaller than $2a_6$. But this gives (3.16). So, in order to conclude the proof of Lemma, it is enough to prove that $T_1(u), T_2(u) \to 0$ as $M_1 \to \infty$.
Firstly, we can note that

$$|T_1(u)| \leq |\chi'(\cdots)| 4a_6 Q(u)^{-1} \Big| \int_{\mathbb{T}^N} hu\, dx \Big|.$$

If $u \notin \operatorname{supp} \psi$, $T_1(u) = 0 = T_2(u)$. Otherwise, by using Lemma 2 and Lemma 4 we get

$$|T_1(u)| \leq \alpha_2 Q(u)^{\frac{1}{\mu} - 1} \leq (M_1 + 1)^{\frac{1}{\mu} - 1} \to 0 \text{ as } M_1 \to \infty.$$

By the structure of T_2, we also deduce $T_2(u) \to 0$ as $M_1 \to \infty$.

□

Taking into account the previous lemma, in order to verify (i) of Theorem 2, we need to prove the following result

Lemma 6. *$J \in \mathcal{C}^1(\mathbb{H}^s(\mathbb{T}^N), \mathbb{R})$ and there is a constant $M_2 > 0$ such that J satisfies (PS) on $\hat{A}_{M_2} \equiv \{u \in \mathbb{H}^s(\mathbb{T}^N) : J(u) \geq M_2\}$.*

Proof. By using $(f1)$ and $(f2)$, it is clear that $I \in \mathcal{C}^1(\mathbb{H}^s(\mathbb{T}^N), \mathbb{R})$. Since $\chi \in \mathcal{C}^\infty$, and f verifies $(f1)$ and $(f2)$, we can see that ψ and therefore $J \in \mathcal{C}^1(\mathbb{H}^s(\mathbb{T}^N), \mathbb{R})$.
Now, let $(u_m) \subset \mathbb{H}^s(\mathbb{T}^N)$ such that $M_2 \leq J(u_m) \leq K$ and $J'(u_m) \to 0$.
Then for all large m,

$$\begin{aligned} \rho\|u_m\|_{\mathbb{H}^s(\mathbb{T}^N)} + K &\geq J(u_m) - \rho J'(u_m)u_m \\ &= \Big(\frac{1}{2} - \rho(1 + T_1(u_m))\Big)\|u_m\|^2_{\mathbb{H}^s(\mathbb{T}^N)} \\ &\quad + \rho(1 + T_2(u_m)) \int_{\mathbb{T}^N} u_m f(x, u_m)\, dx - \int_{\mathbb{T}^N} F(x, u_m)\, dx \\ &\quad + [\rho(\psi(u_m) + T_1(u_m)) - \psi(u_m)] \int_{\mathbb{T}^N} h u_m\, dx \end{aligned} \tag{3.20}$$

where ρ is free for the moment.
For M_2 sufficiently large, and therefore T_1, T_2 small, by $(f3)$ we can choose $\rho \in (\frac{1}{\mu}, \frac{1}{2})$ and $\varepsilon > 0$ such that

$$\frac{1}{2(1 + T_1(u_m))} > \rho + \varepsilon > \rho - \varepsilon > \frac{1}{\mu(1 + T_2(u_m))} \tag{3.21}$$

uniformly in m.
Putting together (3.20), (3.21) and (1.5), and by using the Hölder and Young inequalities as in (3.4), we obtain

$$\rho\|u_m\|_{\mathbb{H}^s(\mathbb{T}^N)} + K \geq \varepsilon\|u_m\|^2_{\mathbb{H}^s(\mathbb{T}^N)} + c_1\|u_m\|^\mu_{L^\mu(\mathbb{T}^N)} - c_2\|u_m\|_{\mathbb{H}^s(\mathbb{T}^N)} - c_3 \tag{3.22}$$

which yields $\{u_m\}$ is bounded in $\mathbb{H}^s(\mathbb{T}^N)$.
Now, it is easy to see that

$$J'(u_m) = (1 + T_1(u_m))u_m - \mathcal{P}(u_m) \tag{3.23}$$

where $\mathcal{P}$ is a compact operator. Taking M_2 so large such that $|T_1(u_m)| \leq \frac{1}{2}$ and by using the facts (u_m) is bounded and $J'(u_m) \to 0$, we can infer that $(1 + T_1(u_m))^{-1}\mathcal{P}(u_m)$ converges along a subsequence. In virtue of (3.23), also (u_m) converges along a subsequence, and we can conclude that J fulfills (PS) on $\hat{A}_{M_2}$.

□

Therefore, in order to prove Theorem 1, it is enough to show that J has an unbounded sequence of critical values. For this reason, we are going to check (iii) and (iv) of Theorem 2. Regarding the condition (iv), it is easy to see that for every $u \in \mathcal{W}$, with $\mathcal{W} \subset \mathbb{H}^s(\mathbb{T}^N)$ finite dimensional, there exist positive constants c_1, c_2, c_3 and c_4 (depending on $\mathcal{W}$) such that

$$J(u) \leq c_1\|u\|^2_{\mathbb{H}^s(\mathbb{T}^N)} - c_2\|u\|^\mu_{\mathbb{H}^s(\mathbb{T}^N)} + c_3\|u\|_{\mathbb{H}^s(\mathbb{T}^N)} + c_4 \to -\infty \text{ as } \|u\|_{\mathbb{H}^s(\mathbb{T}^N)} \to \infty \tag{3.24}$$

since $\mu > 2$. We note that in (3.24), we used (1.5), $|\psi(u)| < 1$ and Hölder inequality to estimate the term $\int_{\mathbb{T}^N} hu\,dx$.

Finally, we prove the following result:

Lemma 7. *There are constants $\beta_2 > 0$ and $j_0 \in \mathbb{N}$ depending on $\|h\|_{L^2(\mathbb{T}^N)}$ such that for all $j \geq j_0$,*

$$\sup_{\rho \geq 0} \inf\{J(u) : u \in V_{j-1}^{\perp}, \|u\|_{\mathbb{H}^s(\mathbb{T}^N)} = \rho\} \geq \beta_2 j^{\frac{(N+2s)-(N-2s)p}{N(p-1)}}. \tag{3.25}$$

Proof. Let $u \in \partial B_\rho \cap V_{j-1}^{\perp}$. Then by $(f2)$, we can deduce that

$$J(u) \geq \frac{1}{2}\rho^2 - \alpha_2 \|u\|^{p+1}_{L^{p+1}(\mathbb{T}^N)} - \alpha_3 - \|h\|_{L^2(\mathbb{T}^N)} \|u\|_{L^2(\mathbb{T}^N)}. \tag{3.26}$$

By using the interpolation inequality and Theorem 3, we get for all $u \in \mathbb{H}^s(\mathbb{T}^N)$

$$\|u\|_{L^{p+1}(\mathbb{T}^N)} \leq a_7 \|u\|^a_{\mathbb{H}^s(\mathbb{T}^N)} \|u\|^{1-a}_{L^2(\mathbb{T}^N)}, \tag{3.27}$$

where $2a = \frac{N(p-1)}{s(p+1)}$.

From Theorem 4, we also have

$$\|u\|_{L^2(\mathbb{T}^N)} \leq \lambda_j^{-\frac{1}{2}} \|u\|_{\mathbb{H}^s(\mathbb{T}^N)} \tag{3.28}$$

for all $u \in V_{j-1}^{\perp}$.

Putting together (3.26), (3.27) and (3.28), we can see that

$$J(u) \geq \frac{1}{2}\rho^2 - \alpha_4 \lambda_j^{-\frac{(1-a)(p+1)}{2}} \rho^{p+1} - \alpha_3 - \|h\|_{L^2(\mathbb{T}^N)} \lambda_j^{-\frac{1}{2}} \rho. \tag{3.29}$$

Taking

$$\rho = \rho_j = \frac{1}{(4\alpha_4)^{\frac{1}{p-1}}} \lambda_j^{\frac{(1-a)}{2}(\frac{p+1}{p-1})},$$

we deduce that

$$J(u) \geq \frac{1}{4}\rho_j^2 - \|h\|_{L^2(\mathbb{T}^N)} \lambda_j^{-\frac{1}{2}} \rho_j - \alpha_3. \tag{3.30}$$

Recalling [13] that for the compact manifold $\mathcal{M} = \mathbb{T}^N$ the following Weyl's formula for the asymptotic distribution of the eigenvalues $\mu_j(\mathcal{M})$ of $-\Delta$ on $\mathcal{M}$ holds

$$\mu_j^{\frac{N}{2}}(\mathcal{M}) \sim \frac{(2\pi)^N}{\omega_N} \frac{j}{Vol(\mathcal{M})} \text{ as } j \to \infty,$$

and by using (ii) of Theorem 4, we can see that there exist $j_0 \in \mathbb{N}$ and α_5 independent of j such that

$$\lambda_j \geq \alpha_5 j^{\frac{2s}{N}} \text{ for } j \geq j_0.$$

This together with (3.30) completes the proof of lemma. □

Proof of Theorem 1. We consider the antipodal action of $G = \mathbb{Z}_2$ on $\mathbb{H}^s(\mathbb{T}^N)$, and we take $W = \mathbb{R}$. Let us observe that for all $j \in \mathbb{N}$, V_j is a G-invariant linear subspace of $\mathbb{H}^s(\mathbb{T}^N)$, $\dim V_j = j$ and that $V^G = \{0\}$. Putting together Lemma 1-Lemma 7, and (3.24), we can see that the assumptions of Theorem 2 are satisfied. Then, there exist a sequence of critical values $(d_j) \subset \mathbb{R}$ and $(u_j) \subset \mathbb{H}^s(\mathbb{T}^N)$ such that $I(u_j) = d_j \to \infty$ and $I'(u_j) = 0$. In particular, being $I'(u_j)u_j = 0$, we have

$$
\begin{aligned}
\|u_j\|^2_{\mathbb{H}^s(\mathbb{T}^N)} &= \int_{\mathbb{T}^N} f(x,u_j)u_j dx + \int_{\mathbb{T}^N} hu_j dx \\
&= 2d_j + 2\int_{\mathbb{T}^N} F(x,u_j)dx + \int_{\mathbb{T}^N} hu_j dx. \quad (3.31)
\end{aligned}
$$

Then, by using (3.31), $(f1)$, $(f3)$ and $h \in L^2(\mathbb{T}^N)$, it is easy to show that there exist $\alpha, \beta > 0$ independent of $j \in \mathbb{N}$ such that $\|u_j\|^2_{\mathbb{H}^s(\mathbb{T}^N)} \geq \alpha d_j - \beta \to \infty$ as $j \to \infty$.

□

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