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ON THE REGULARIZING EFFECT OF SOME ABSORPTION AND SINGULAR LOWER ORDER TERMS IN CLASSICAL DIRICHLET PROBLEMS WITH L^1 DATA

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Abstract. We are interested in existence and regularity results concerning the solution to the following problem

$$\begin{cases} -\Delta u + u^s = \dfrac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$, $0 < \gamma \leq 1$, $s \geq 1$ and f is a nonnegative function that belongs to some Lebesgue space.

1. Introduction

In this paper we investigate the interaction between two *regularizing terms* in the following semilinear problem

$$(1) \qquad \begin{cases} -\Delta u + u^s = \dfrac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$, $N > 2$, $0 < \gamma \leq 1$, $s \geq 1$ and f is just a nonnegative $L^1(\Omega)$ function.
The motivations in the study of the above problem mainly arise by two papers, [3] and [4]. In [3] the authors investigate the regularizing effect of the term u^s on the solution to the following classical problem

$$(2) \qquad \begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when the term u^s is added in the left hand side of (2), namely when we consider the following

$$(3) \qquad \begin{cases} -\Delta u + u^s = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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We refer to the term u^s in (3) as the *absorption term.*
On the other side, in [4], the authors analyze the following singular problem

$$\begin{cases} -\Delta u = \dfrac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

discovering, among other things, a regularizing effect of the singular term $\dfrac{f(x)}{u^\gamma}$, to which we refer as the *singular sourcing term*, on the solution to the classical problem (2), when this term replaces the non-singular right hand side.

Here we want to focus on a possibly double regularization effect on the solution to (2) when we add both the absorption term on the left hand side and the singular sourcing term on the right hand side.

For the sake of completeness, we start recalling some literature regarding the above mentioned problems.
Firstly we consider the following

$$\begin{cases} -\Delta v = f(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $f \in L^r(\Omega)$, $r \geq 1$.
If $r = 1$, by Theorem 1 of [2] (see also Theorem 1 in [1]), we can ensure the existence of at least a distributional solution v to (5) such that $v \in W_0^{1,q}(\Omega$ for every $q < \dfrac{N}{N-1}$. Moreover, by the classical Calderon-Zygmund theory for infinite energy solution, if $1 < r < (2^*)'$ it is guaranteed the existence of at least a distributional solution $v \in W_0^{1,r^*}(\Omega)$ to (5) (see Theorem 3 of [2]). If instead $m \geq (2^*)'$, by the classical variational results, problem (5) has a finite energy solution.
Thus, the regularization effects mentioned above will be with respect to the Sobolev regularity of the solution v to the classical problem (5).
Now we turn our attention recalling some results contained in the papers [4] and [3].
In [4] the authors study the following problem

$$\begin{cases} -\Delta z = \dfrac{f(x)}{z^\gamma} & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

and they prove the existence of a solution z to (6) such that:

(i) if $\gamma = 1$ and $0 \leq f \in L^1(\Omega)$, then $z \in H_0^1(\Omega)$,

(ii) if $\gamma < 1$ and $0 \leq f \in L^{(\frac{2^*}{1-\gamma})'}(\Omega)$, then $z \in H_0^1(\Omega)$,

(iii) if $\gamma < 1$ and $0 \leq f \in L^r(\Omega)$ with $r < (\frac{2^*}{1-\gamma})'$, then $z \in W_0^{1,q}(\Omega)$ with $q = \frac{Nr(\gamma+1)}{N-r(1-\gamma)}$.

In all the above cases, the singular sourcing term has a regularizing effect on the Sobolev regularity of the solution z, compared to the one of the distributional solution to problem (5). Indeed, for example, it is immediate to see that, if $\gamma = 1$, $N > 2$ and $f \in L^1(\Omega)$, then (6) has a finite energy solution, while, if $\gamma = 0$, i.e. in the case of problem (5), the solution belongs only to $W_0^{1,q}(\Omega)$ for every $q < \dfrac{N}{N-1}$.
On the other hand, in [3], the authors perturb problem (5) with an absorption term. They consider the following

$$
\begin{cases} -\Delta w + w^s = f(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}
$$

and, in [3, Theorem 5], it is proved that, if $f \in L^1(\Omega)$ and $s > \dfrac{N}{N-2}$, then there exists w solution to (7) belonging to $W_0^{1,q}(\Omega)$ for all $q < \dfrac{2s}{(s+1)}$. Since

$$
\frac{N}{N-1} < \frac{2s}{s+1} \quad \Longleftrightarrow \quad s > \frac{N}{N-2},
$$

also in this case there is a regularization effect of the absorption term with respect to the Sobolev regularity of v solution to (5).
It is worth recalling also the paper [5], where the author strengthens the assumption on the datum f and studies the regularity of a solution w to (7) depending simultaneously on the regularity of f and on the exponent s. To be more precise, in [5] it is proved that, if $s \geq 1$ and f is such that $|f|\log(1+|f|) \in L^1(\Omega)$, then w has the limiting regularity $w \in W_0^{1,\frac{2s}{(s+1)}}(\Omega)$ and that, if $f \in L^r(\Omega)$ with $r \in \big(1, (2^*)'\big)$, then $w \in W_0^{1,\frac{2sr}{(s+1)}}(\Omega)$ if $s \in \Big(\frac{1}{(2r-1)}, \frac{1}{(r-1)}\Big)$, while $w \in H_0^1(\Omega)$ if $s > \frac{1}{r-1}$.

As already said, we want to understand the mutual behavior of the two regularizing effects. In particular, we want to exploit if the combination of the two regularizations leads to an even more regular solution or, alternatively, if the regularization given by the singular sourcing term to the solution is too strong to expect some effects when we add also an absorption term. As we will see in the next sections, both the addition of a singular sourcing term to a problem with only an absorption term and the addition of an absorption term to a problem with only a singular sourcing term, will improve the regularity of the solution. Anyway we underline that, when we add a singular sourcing term to a problem with an absorption term, we improve the regularity of the solution anyhow we choose the singularity exponent $\gamma > 0$, while, when we add an absorption term to a problem with a singular sourcing term, in order to improve the regularity we have to consider an exponent $s \geq 1$ large enough, namely we are not able to improve the regularity of the solution for each $s \geq 1$. This tells us that a singular sourcing term gives rise to a more powerful regularization with respect to the one generated by an absorption term.

1.1. **Notations.** For a fixed $k > 0$, we define the truncation functions $T_k : \mathbb{R} \to \mathbb{R}$ and $G_k : \mathbb{R} \to \mathbb{R}$ as follows

$$T_k(s) := \max(-k, \min(s, k)),$$
$$G_k(s) := (|s| - k)^+ \operatorname{sign}(s).$$

We will denote with $\mathbb{R}^*$ the set $\mathbb{R}\backslash 0$, with $\mathbb{R}^+$ the set $\{t \in \mathbb{R} \text{ s.t. } t > 0\}$, with r^* the *Sobolev conjugate* of $1 \le r < \mathrm{N}$, given by $\dfrac{\mathrm{N}r}{\mathrm{N}-r}$, and with $r' = \dfrac{r}{r-1}$ the *Hölder conjugate* of $1 < r < \infty$ (if $r = 1$ we define $r' = \infty$, if $r = \infty$ we define $r' = 1$). Moreover, if no otherwise specified, we will denote by c several positive constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data (for instance c can depend on Ω, γ, s, N) but they will never depend on the indexes of the sequences we will introduce.

2. Existence of a regularized solution

We want to prove existence and regularity results for problem (1) in case Ω is an open bounded subset of $\mathbb{R}^N$ ($N > 2$), $0 < \gamma \le 1$, $s \ge 1$ and $0 \le f \in L^r(\Omega)$ with $r \ge 1$. We are looking for a distributional solution u that belongs to a smaller Sobolev space compared to the ones where v, z, w solutions to, respectively, (5), (6) and (7) belong.
For the sake of completeness we state what we mean for distributional solution (or simply solution) to (1).

Definition 1. A function $u \in W_0^{1,1}(\Omega)$ is a *distributional solution* to problem (1) in case $\gamma \le 1$, $s \ge 1$ and $f \in L^r(\Omega)$ with $r \ge 1$ if

$$\forall \omega \subset\subset \Omega \text{ exists } c_\omega > 0 \text{ s.t. } u \ge c_\omega \text{ a.e. in } \omega, \tag{8}$$

$$u^s \in L^1(\Omega),$$

and

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega u^s \varphi = \int_\Omega \frac{f\varphi}{u^\gamma} \quad \forall \varphi \in C_c^1(\Omega).$$

Our main result is the following.

Theorem 1. *Let $\gamma \le 1$, $s \ge 1$ and $0 \le f \in L^r(\Omega)$ with $r \ge 1$. Then there exists a distributional solution u to (1). Moreover u belongs to $H_0^1(\Omega) \cap L^{s+1}(\Omega)$ if*

(i) $\gamma = 1,\ f \in L^1(\Omega)$ *or*

(ii) $\gamma < 1,\ f \in L^r(\Omega)$ *for some* $r > 1$ *and* $s \ge \frac{1-r\gamma}{r-1}$ *or*

(iii) $\gamma < 1,\ f \in L^{\frac{s+1}{s+\gamma}}(\Omega)$

while if

(iv) $\gamma < 1$ *and* $f \in L^1(\Omega)$, *then* $u \in W_0^{1,\frac{2(s+\gamma)}{s+1}}(\Omega) \cap L^{s+\gamma}(\Omega)$.

2.1. **Approximating problems.** In order to prove Theorem 1, we will work by approximation, namely by introducing the following

$$\begin{cases} -\Delta u_{n,k} + T_k(|u_{n,k}|^{s-1}u_{n,k}) = \dfrac{f_n(x)}{\big(|u_{n,k}| + \frac{1}{n}\big)^{\gamma}} & \text{in } \Omega, \\ u_{n,k} = 0 & \text{on } \Omega, \end{cases} \tag{9}$$

where $n, k \in \mathbb{N}$, $0 \le f_n(x) := T_n(f(x)) \in L^\infty(\Omega)$, $\gamma \le 1$ and $s \ge 1$.
Thanks to [6, Théorème 2], we know that there exists $u_{n,k} \in H_0^1(\Omega)$ weak solution to (9) for each $n, k \in \mathbb{N}$ fixed. Moreover $u_{n,k} \in L^\infty(\Omega)$ for all $n, k \in \mathbb{N}$ since, if $m \ge 1$ is fixed, taking $G_m(u_{n,k}) \in H_0^1(\Omega)$ as test function in (9) and using that $G_m(u_{n,k})$ and $T_k(|u_{n,k}|^{s-1}u_{n,k})$ have the same sign of $u_{n,k}$, we immediately find that

$$\int_\Omega |\nabla G_m(u_{n,k})|^2 \le \int_\Omega f_n G_m(u_{n,k}),$$

and so we can proceed as in [7] to end up with $u_{n,k} \in L^\infty(\Omega)$. Moreover the previous L^∞ estimate is independent from $k \in \mathbb{N}$. Now taking $u_{n,k}$ as a test function in the weak formulation of (9), we find that $u_{n,k}$ is bounded in $H_0^1(\Omega)$ with respect to k for $n \in \mathbb{N}$ fixed. Since $u_{n,k}$ is bounded in $L^\infty(\Omega)$ independently on k, for each $n \in \mathbb{N}$ fixed we choose k_n large enough to obtain the following scheme of approximation

$$\begin{cases} -\Delta u_n + |u_n|^{s-1}u_n = \dfrac{f_n(x)}{\big(|u_n| + \frac{1}{n}\big)^{\gamma}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is given by u_{n,k_n}.
As concerns the sign of u_n, taking $u_n^- := \min(u_n, 0) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in (10), we find

$$\int_\Omega |\nabla u_n^-|^2 + \int_\Omega |u_n|^{s-1}(u_n^-)^2 = \int_\Omega \frac{f_n}{(|u_n| + \frac{1}{n})^{\gamma}} u_n^- \le 0,$$

and so that $u_n \ge 0$ almost everywhere in Ω.
Now we prove some local positivity property that will guarantee that the limit of the approximations (10) satisfies (8).

Proposition 1. *For each $n \in \mathbb{N}$ fixed, the nonnegative $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ weak solution to* (10) *is nondecreasing in $n \in \mathbb{N}$ and it results*

$$\forall\, \omega \subset\subset \Omega\ \exists\, c_\omega > 0 \text{ (independent of } n \in \mathbb{N}) \text{ s.t. } u_n \ge c_\omega \text{ in } \omega\ \forall\, n \in \mathbb{N}. \tag{11}$$

Proof. We can prove that the sequence u_n is nondecreasing in $n \in \mathbb{N}$ proceeding precisely as in [4, Lemma 2.2], namely taking $(u_n - u_{n+1})^+ := \max(u_n - u_{n+1}, 0) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in the difference between the problem solved by u_n and the one solved by u_{n+1}, so we will omit the details. To prove (11), we will instead use that

$$u_n \ge u_1 \quad \forall\, n \in \mathbb{N} \text{ a.e. in } \Omega \tag{12}$$

and we will apply the strong maximum principle to $u_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, that solves

$$\begin{cases} -\Delta u_1 + u_1^s = \dfrac{f_1(x)}{(u_1 + 1)^{\gamma}} \ge \dfrac{f_1(x)}{\big(\|u_1\|_{L^\infty(\Omega)} + 1\big)^{\gamma}} \ge 0 & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Indeed, since $u_1, \Delta u_1 \in L^1_{loc}(\Omega)$, $u_1 \geq 0$ almost everywhere in Ω, $\Delta u_1 \leq u_1^s$ and

$$\int_0 \left(t^{s+1}\right)^{-\frac{1}{2}} = \infty \qquad \Longleftrightarrow \qquad s \geq 1,$$

we can apply [8, Theorem 1] and deduce that

$$\forall\, \omega \subset\subset \Omega\ \exists\, c_\omega > 0 \text{ s.t. } u_1 \geq c_\omega \text{ in } \omega.$$

Then (11) follows from (12). $\square$

2.2. **A priori estimates.** Now we need some compactness results on the sequence of approximating solutions u_n, at least up to subsequences.

Proposition 2. *Let $n \in \mathbb{N}$ and $u_n \in H^1_0(\Omega) \cap L^\infty(\Omega)$ be a solution to (10) where $s \geq 1$.*
(i) If one of the following holds

$$\begin{cases} \gamma = 1,\ f \in L^1(\Omega)\ , \\ \gamma < 1,\ f \in L^r(\Omega) \textit{ for some } r > 1 \textit{ and } s \geq \frac{1-r\gamma}{r-1}, \\ \gamma < 1,\ f \in L^{\frac{s+1}{s+\gamma}}(\Omega)\ , \end{cases}$$

then u_n is bounded in $H^1_0(\Omega) \cap L^{s+1}(\Omega)$.
(ii) If instead $\gamma < 1$ and $f \in L^1(\Omega)$, then u_n is bounded in $W_0^{1,\frac{2(s+\gamma)}{s+1}}(\Omega) \cap L^{s+\gamma}(\Omega)$.

Proof. The case (i). Let us take $u_n \in H^1_0(\Omega) \cap L^\infty(\Omega)$ as test function in (10). We obtain

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega u_n^{s+1} \leq \int_\Omega f_n u_n^{1-\gamma}. \tag{13}$$

If $\gamma = 1$, we immediately find that u_n is bounded in $H^1_0(\Omega)$ and in $L^{s+1}(\Omega)$.
If $\gamma < 1$, we apply Young's inequality with weights $(\epsilon, c(\epsilon))$ and exponents (r, r') on the right hand side of the previous, obtaining

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega u_n^{s+1} \leq \frac{1}{c(\epsilon)} \int_\Omega f_n^r + \epsilon \int_\Omega u_n^{(1-\gamma)r'} \leq \frac{1}{c(\epsilon)} \int_\Omega f^r + \epsilon c \int_\Omega u_n^{s+1}.$$

If ϵ is small enough, we deduce the following estimate

$$\int_\Omega |\nabla u_n|^2 + c(\Omega, \epsilon) \int_\Omega u_n^{s+1} \leq \frac{1}{c(\epsilon)} \int_\Omega f^r \leq c.$$

If $\gamma < 1$ and $f \in L^{\frac{s+1}{s+\gamma}}(\Omega)$, we apply Young's inequality with weights $(\epsilon, c(\epsilon))$ and exponents

$$\left(\frac{s+1}{s+\gamma}, \frac{s+1}{1-\gamma}\right)$$

on the right hand side of (13). Proceeding as before, we can easily prove the last assertion.
The case (ii). Let us take $(u_n + \epsilon)^\gamma - \epsilon^\gamma \in H^1_0(\Omega) \cap L^\infty(\Omega)$ as test function in (10), where $0 < \epsilon < \frac{1}{n}$. We obtain

$$\gamma \int_\Omega |\nabla u_n|^2 (u_n + \epsilon)^{\gamma - 1} + \int_\Omega u_n^s((u_n + \epsilon)^\gamma - \epsilon^\gamma) \leq \int_\Omega f_n$$

and so, in particular, we deduce that

$$\int_\Omega u_n^s((u_n + \epsilon)^\gamma - \epsilon^\gamma) \leq \int_\Omega f,$$

that, letting $\epsilon \to 0$, implies

$$\int_\Omega u_n^{s+\gamma} \le \int_\Omega f.$$

Moreover we deduce

$$\int_\Omega \frac{|\nabla u_n|^2}{(u_n+\epsilon)^{1-\gamma}} \le c.$$

Now, if $q<2$, applying Hölder inequality with exponents $\dfrac{2}{q}$ and $\dfrac{2}{2-q}$, we find

$$\int_\Omega |\nabla u_n|^q = \int_\Omega \frac{|\nabla u_n|^q}{(u_n+\epsilon)^{(1-\gamma)\frac{q}{2}}}(u_n+\epsilon)^{(1-\gamma)\frac{q}{2}} \le c\left(\int_\Omega (u_n+\epsilon)^{\frac{(1-\gamma)q}{2-q}}\right)^{1-\frac{q}{2}}.$$

Finally we choose q such that

$$\frac{(1-\gamma)q}{2-q} = s+\gamma.$$

It is easy to verify that

$$q = \frac{2(s+\gamma)}{(s+1)} < 2,$$

and this gives us the result. □

2.3. **Proof of Theorem 1.**

Proof of Theorem 1. We have to pass to the limit the approximation (10).
Thanks to the a priori estimates of Proposition 2, by weak convergence we can pass to the limit in the left hand side of the distributional formulations of the approximating problems with $C^1_c(\Omega)$ test functions. For what concerns the right hand side, using (11) we find

$$\left|\frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}\right| \le \left|\frac{f\varphi}{c^\gamma_{\text{supp}\varphi}}\right| \qquad \forall \varphi \in C^1_c(\Omega).$$

Then, thanks to Lebesgue Theorem, we can pass to the limit also in the right hand side of the distributional formulation of (10). This concludes the proof. □

2.4. **Some comments on the regularizing effect.** First of all, it is easy to verify that, if

$$\gamma < 1 \quad \text{and} \quad s > \frac{N+2}{N-2} \tag{14}$$

then

$$\frac{s+1}{s+\gamma} < \left(\frac{2^*}{1-\gamma}\right)'.$$

Since $f \in L^{\left(\frac{2^*}{1-\gamma}\right)'}(\Omega)$ is the weaker assumption on the datum in order to find a priori estimates in $H^1_0(\Omega)$ for the sequence of approximating solutions to (4) (see [4, Theorem 5.1]), it follows that, if we add the term u^s, with s satisfying (14), in the left hand side of (4), we find a priori estimates in $H^1_0(\Omega)$ for the sequence of approximating solutions also for less regular data.

Furthermore, if $f \in L^1(\Omega)$ and $\gamma < 1$, the Sobolev space in which the sequence of approximating solutions to (4) is bounded is given by $W_0^{1,\frac{N(\gamma+1)}{N-(1-\gamma)}}(\Omega)$ (see [4, Theorem 5.6]). It is easy to verify that, if

$$\gamma < 1, \quad \text{and} \quad s > \frac{N+2\gamma}{N-2} \tag{15}$$

then

$$\frac{N(\gamma+1)}{N-(1-\gamma)} < \frac{2(s+\gamma)}{s+1}.$$

So we have another regularizing effect of the lower order term u^s, with s such that (15) holds, on the a priori estimates for the approximating solutions.
Finally we recall that, if $f \in L^1(\Omega)$ and $s > \frac{N}{N-2}$, then the sequence of approximating solutions to (3) is bounded in $W_0^{1,q}(\Omega)$ for all $q \in \left[1, \frac{2s}{(s+1)}\right)$ (see [3, Theorem 5]). Since

$$\frac{2s}{(s+1)} < \frac{2(s+\gamma)}{(s+1)} \quad \Longleftrightarrow \quad \gamma > 0,$$

we immediately obtain the, if we perturb the right hand side of (3) through the singular term $\frac{1}{u^\gamma}$ with $\gamma > 0$, we find more regular a priori estimates on the sequence of approximating solutions.

3. More general singular and absorption terms: combined effect on the existence and regularity of the solution

Here we intend to analyze a more general case, namely the following

$$\begin{cases} -\Delta u + g(u) = h(u)f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

The absorption term $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing and continuous function on $\mathbb{R}$ satisfying the assumptions listed below:

$$g(t)t \geq 0 \quad \text{for all } t \in \mathbb{R}, \tag{g1}$$

$$\int_0 (g(t)t)^{-\frac{1}{2}} = \infty, \tag{g2}$$

$$g(t) \geq t^s \quad \text{for all } t \geq 0 \text{ and for some } s \geq 1. \tag{g3}$$

The singular sourcing term $h : \mathbb{R}^* \to \mathbb{R}^+$ is a nonincreasing function on $\mathbb{R}^*$, continuous and bounded on $\mathbb{R}^*$ such that

$$\lim_{t\to 0} h(t) = +\infty, \tag{h1}$$

and such that

(h2) there exist $c > 0,\ \delta' > \delta > 0$ and $\gamma, \theta \in (0,1]$ s.t. $\begin{cases} h(t) \le \dfrac{c}{t^\theta} & \text{if} \quad t < \delta \\ h(t) \le \dfrac{c}{t^\gamma} & \text{if} \quad t > \delta'. \end{cases}$

The example we have in mind for the function g is

$$g(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t^s & \text{if } 0 \le t \le 1 \\ t^{s+1} & \text{if } t \ge 1, \end{cases}$$

while the function h is a general singular nonincreasing function with two different behaviors near zero and near infinite.
This time the definition of distributional solution is as follows.

Definition 2. Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and continuous function satisfying (g1), (g2) and (g3) and let $h : \mathbb{R}^* \to \mathbb{R}^+$ be a nonincreasing function on $\mathbb{R}^*$, continuous and bounded on $\mathbb{R}^*$, satisfying (h1) and (h2). Under these assumptions a function $u \in W_0^{1,1}(\Omega)$ is a *distributional solution* to problem (16), with $f \in L^r(\Omega)$ and $r \ge 1$, if

$$\forall \omega \subset\subset \Omega \text{ exists } c_\omega > 0 \text{ s.t. } u \ge c_\omega \text{ a.e. in } \omega, \tag{17}$$

$$g(u) \in L^1(\Omega),$$

and

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega g(u)\varphi = \int_\Omega h(u) f \varphi \quad \forall \varphi \in C_c^1(\Omega). \tag{18}$$

Remark 1. We underline that property (17) implies in particular that $h(u)f \in L^1_{loc}(\Omega)$, as it was in the case with monotone singularity given by $h(u) = \dfrac{1}{u^\gamma}$ $(\gamma > 0)$. Consequently, the right hand side of (18) is well defined.

We intend to prove the following result.

Theorem 2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and continuous function satisfying* (g1), (g2) *and* (g3) *and let $h : \mathbb{R}^* \to \mathbb{R}^+$ be a nonincreasing function on $\mathbb{R}^*$, continuous and bounded on $\mathbb{R}^*$, satisfying* (h1) *and* (h2). *If $f \in L^r(\Omega)$ with $r \ge 1$, then there exists a distributional solution u to* (16). *Moreover u belongs to $H_0^1(\Omega) \cap L^{s+1}(\Omega)$ if*

(i) $\theta, \gamma = 1,\ f \in L^1(\Omega)$ *or*

(ii) $\theta, \gamma < 1,\ f \in L^r(\Omega)$ *for some* $r > 1$ *and* $s \ge \frac{1-r\gamma}{r-1}$ *or*

(iii) $\theta, \gamma < 1,\ f \in L^{\frac{s+1}{s+\gamma}}(\Omega)$,

while if

(iv) $\theta < \gamma < 1$ *and* $f \in L^1(\Omega)$, *then* $u \in W_0^{1,\frac{2(s+\gamma)}{s+1}}(\Omega) \cap L^{s+\gamma}(\Omega)$.

Since Theorem 2 can be seen as an extension of Theorem 1, for shortness we will omit some details in the proof, referring to the corresponding ones in the proof of Theorem 1.

Proof of Theorem 2. As before, we introduce the following scheme of approximation for problem (16)

$$\begin{cases} -\Delta u_{n,k} + T_k(g(u_{n,k})) = h_n(u_{n,k}) f_n(x) & \text{in } \Omega, \\ u_{n,k} = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

where $n, k \in \mathbb{N}$, $0 \le f_n(x) := T_n(f(x)) \in L^\infty(\Omega)$ and $h_n(x) := T_n(h(x))$.
Thanks to [6, Théorème 2], we know that there exists $u_{n,k} \in H_0^1(\Omega)$ weak solution to (19) for each $n, k \in \mathbb{N}$ fixed. As in Theorem 1, we can show that $u_{n,k} \in L^\infty(\Omega)$ for all $n, k \in \mathbb{N}$ and that this L^∞-estimate is independent on k and, taking $u_{n,k}$ as a test function in the weak formulation of the previous, we find once again that $u_{n,k}$ is bounded in $H_0^1(\Omega)$ with respect to k for $n \in \mathbb{N}$ fixed. Since $u_{n,k}$ is bounded in $L^\infty(\Omega)$ independently on k, as already done for problem (9) we choose k large enough to obtain the following scheme of approximation

$$\begin{cases} -\Delta u_n + g(u_n) = h_n(u_n) f_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

where $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is given by a certain $u_{n,k}$ for a k large enough.
Moreover, analogously to what done in Proposition 1, we are able to prove that the sequence u_n is nonnegative and nondecreasing in $n \in \mathbb{N}$.
In order to prove the local positivity of u_n, we will use, once again, the strong maximum principle applied to $u_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, that solves

$$\begin{cases} -\Delta u_1 + g(u_1) = h_1(u_1) f_1 \ge h_1(||u_1||_{L^\infty(\Omega)}) f_1 \ge 0 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, thanks to (g2), we can apply [8, Theorem 1] and deduce that

$$\forall\, \omega \subset\subset \Omega \; \exists\, c_\omega > 0 \text{ s.t. } u_1 \ge c_\omega \text{ in } \omega.$$

From now on we divide the proof in the different cases with respect to the parameters.

Proof of (i). In this case, it is sufficient to take u_n as a test function in the weak formulation of (20) obtaining

$$\begin{aligned} \int_\Omega |\nabla u_n|^2 + \int_\Omega u_n^{s+1} &\le \int_\Omega |\nabla u_n|^2 + \int_\Omega g(u_n) u_n \\ &\le c \int_{\{u_n \le \delta\}} f_n + \int_{\{\delta < u_n < \delta'\}} h_n(u_n) f_n u_n + c \int_{\{u_n \ge \delta'\}} f_n \\ &\le 2c \int_\Omega f + \delta' ||h||_{L^\infty(\delta, \delta')} \int_\Omega f. \end{aligned}$$

Thus, u_n is bounded in $H_0^1(\Omega) \cap L^{s+1}(\Omega)$ and we have no problems in passing to the limit in the approximation (20) with $C_c^1(\Omega)$ test functions, using Lebesgue Theorem for the singular sourcing term and monotone convergence Theorem for the absorption term.

Proof of (ii). If $\theta, \gamma < 1$, we take u_n as test function and we apply Young's inequality with weights $(\epsilon, c(\epsilon))$ and exponents (r, r') on the last term of the right hand side of the weak formulation, obtaining

$$\begin{aligned}\int_\Omega |\nabla u_n|^2 + \int_\Omega u_n^{s+1} &\le \int_\Omega |\nabla u_n|^2 + \int_\Omega g(u_n)u_n \\ &\le c\int_{\{u_n\le\delta\}} f_n u_n^{1-\theta} + \delta' \int_{\{\delta<u_n<\delta'\}} h_n(u_n)f_n + c\int_{\{u_n\ge\delta'\}} f_n u_n^{1-\gamma} \\ &\le c\delta^{1-\theta}\int_\Omega f + \delta' ||h||_{L^\infty((\delta,\delta'))}\int_\Omega f + \frac{1}{c(\epsilon)}\int_\Omega f_n^r + c\epsilon\int_\Omega u_n^{(1-\gamma)r'} \\ &\le c\delta^{1-\theta}\int_\Omega f + \delta' ||h||_{L^\infty((\delta,\delta'))}\int_\Omega f + \frac{1}{c(\epsilon)}\int_\Omega f^r + c\epsilon\int_\Omega u_n^{s+1}.\end{aligned}$$

Thus, if ϵ is small enough, we deduce

$$\int_\Omega |\nabla u_n|^2 + c\int_\Omega u_n^{s+1} \le c. \tag{21}$$

As already done in case (i), we can pass to the limit the approximation scheme (20) in order to prove the existence of a solution $u \in H_0^1(\Omega) \cap L^{s+1}(\Omega)$ to problem (16).

Proof of (iii). For this case we can proceed in a similarly to what done for (ii), obtaining

$$\begin{aligned}\int_\Omega |\nabla u_n|^2 + \int_\Omega u_n^{s+1} &\le \int_\Omega |\nabla u_n|^2 + \int_\Omega g(u_n)u_n \\ &\le c\int_{\{u_n\le\delta\}} f_n u_n^{1-\theta} + \delta' \int_{\{\delta<u_n<\delta'\}} h_n(u_n)f_n + c\int_{\{u_n\ge\delta'\}} f_n u_n^{1-\gamma} \\ &\le c\delta^{1-\theta}\int_\Omega f + \delta' ||h||_{L^\infty((\delta,\delta'))}\int_\Omega f + \frac{1}{c(\epsilon)}\int_\Omega f_n^{\frac{s+1}{s+\gamma}} + c\epsilon\int_\Omega u_n^{s+1}.\end{aligned}$$

This allows us to deduce (21) once again and to obtain a solution $u \in H_0^1(\Omega) \cap L^{s+1}(\Omega)$ to (16).

Proof of (iv). This time we take $(u_n + \epsilon)^\gamma - \epsilon^\gamma \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in (20), where $0 < \epsilon < \frac{1}{n}$. We obtain

$$\begin{aligned}&\gamma\int_\Omega |\nabla u_n|^2(u_n+\epsilon)^{\gamma-1} + \int_\Omega u_n^s((u_n+\epsilon)^\gamma - \epsilon^\gamma) \\ &\le \gamma\int_\Omega |\nabla u_n|^2(u_n+\epsilon)^{\gamma-1} + \int_\Omega g(u_n)((u_n+\epsilon)^\gamma - \epsilon^\gamma) \\ &\le c\int_{\{u_n\le\delta\}} \frac{f_n(u_n+\epsilon)^\gamma}{u_n^\theta} + \int_{\{\delta\le u_n\le\delta'\}} f_n h_n(u_n)(u_n+\epsilon)^\gamma + c\int_{\{u_n\ge\delta'\}} \frac{f_n(u_n+\epsilon)^\gamma}{u_n^\gamma},\end{aligned}$$

and so we have in particular

$$\begin{aligned}\int_\Omega u_n^s((u_n+\epsilon)^\gamma - \epsilon^\gamma) \le c\int_{\{u_n\le\delta\}} \frac{f_n(u_n+\epsilon)^\gamma}{u_n^\theta} &+ \int_{\{\delta\le u_n\le\delta'\}} f_n h_n(u_n)(u_n+\epsilon)^\gamma \\ &+ c\int_{\{u_n\ge\delta'\}} \frac{f_n(u_n+\epsilon)^\gamma}{u_n^\gamma},\end{aligned}$$

that, letting $\epsilon \to 0$, implies
$$\int_\Omega u_n^{s+\gamma} \le c\delta^{\gamma-\theta} + c.$$
Moreover, we can reason as in proof case (ii) of Proposition 2 obtaining that u_n is bounded in $W_0^{1,\frac{2(s+\gamma)}{s+1}}(\Omega)$ with respect to n. From here we conclude as in case (i) of the current Theorem. This concludes the proof. □

3.1. **A final remark.** Here we want to explain the motivations that brought us to investigate only the case with singularity indexes $\theta, \gamma \le 1$.
First of all, we had in mind the idea of understanding the regularization effect of a singular sourcing term joined to an absorption term and, the regularization effect of a singular sourcing term with an exponent of singularity far from the origin greater than one, is itself very strong, also without the addition of an absorption term.
Indeed, we consider the following

$$\begin{cases} -\Delta u = h(u)f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}$$

where the singularity $h : \mathbb{R}^* \to \mathbb{R}^+$ is assumed to be of class $C_b(\mathbb{R}^*)$ along with the following growth conditions

$$\begin{cases} \exists\, c, \delta > 0, \theta \le 1 \quad \text{s.t.} \quad h(s) \le \frac{c}{s^\theta} \quad \text{if} \quad s < \delta \\ \exists\, c > 0, \delta' > \delta, \gamma \ge 1 \quad \text{s.t.} \quad h(s) \le \frac{c}{s^\gamma} \quad \text{if} \quad s > \delta'. \end{cases}$$

Formally, if we take u itself as a test function in (22) we obtain

$$\int_\Omega |\nabla u|^2 \le c\int_{\{u\le\delta\}} fu^{1-\theta} + \int_{\{\delta<u<\delta'\}} h(u)fu + c\int_{\{u\ge\delta'\}} fu^{1-\gamma}$$
$$\le c\delta^{1-\theta}\int_{\{u\le\delta\}} f + \delta'||h||_{L^\infty((\delta,\delta'))}\int_{\{\delta<u<\delta'\}} f + c\delta'^{1-\gamma}\int_{\{u\ge\delta'\}} f,$$

that gives an H^1-estimate for the solution u. Clearly, the above argument can be tightened passing to the limit in a suitable approximation scheme. For the motivation explained above, it was not interesting for us the case $\gamma > 1$.
For what concerns the case $\theta > 1$, the regularizing effect of the singular sourcing term is only local for problem (22), as well explained for the model case in [4] where they find solutions $u \in H^1_{loc}(\Omega)$, and we have chosen to omit the study of any local regularization effect obtained by adding an absorption term.
Therefore, for exponents greater than one, it seemed to be not of interest the combination of the two regularization effects, the one of absorption term with the one of the singular sourcing term.

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