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EXISTENCE AND MULTIPLICITY FOR QUASI-CRITICAL FOURTH ORDER QUASILINEAR PROBLEMS WITH GENERALIZED VANISHING POTENTIALS

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Abstract. We study the following fourth order quasilinear elliptic equation under Navier boundary conditions in the whole space $\mathbb{R}^N$

$$\begin{cases} -\Delta(-\Delta u)^{\frac{1}{p-1}} + \rho(x)u^{\frac{1}{p-1}} = \sigma(x)f(u) & \text{in } \mathbb{R}^N, \\ u, \Delta u \to 0 & \text{as } |x| \to +\infty, \end{cases}$$

with weights ρ and σ which are allowed to be vanishing at infinity and in presence of quasi-critical power nonlinearities f. Existence and multiplicity results are proved by means of Mountain Pass Theorem and its symmetric version thanks to some compact imbeddings in σ-weighted Lebesgue spaces. These results apply to an equivalent nonlinear elliptic system of Lane-Emden type and in particular to a biharmonic equation under Navier conditions in $\mathbb{R}^N$.

1. Introduction

In the last years we have treated the following class of fourth-order quasilinear elliptic problems

$$(1.1)\qquad \begin{cases} -\Delta(-\Delta u)^{\frac{1}{p-1}} + \rho(x)u^{\frac{1}{p-1}} = f(x,u) & \text{in } \Omega \text{ (resp. in } \mathbb{R}^N), \\ u = \Delta u = 0 \text{ (resp. } u, \Delta u \to 0) & \text{in } \partial\Omega \text{ (resp. as } |x| \to +\infty), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 3$, $1 < p \leq 2$, $\rho : \mathbb{R}^N \to \mathbb{R}$ is a Lebesgue measurable weight and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ (resp. $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$) satisfies suitable growth assumptions. For simplicity, we denote $s^\alpha = sgn(s)|s|^\alpha$ the odd extension of the power function. Related to the growth behavior of f we are interested in, we say that f has a $(\frac{p}{p-1} - 1)$-superlinear but subcritical growth if the growth of $f(x,u)$ is comparable with u^{q-1} with $q > 1$ and $1 - \frac{2}{N} < \frac{1}{p} + \frac{1}{q} < 1$, i.e. with $\frac{p}{p-1} < q < \frac{Np}{(N-2)p-N}$ while $f(x,u)$ has a $(\frac{p}{p-1} - 1)$-sublinear growth if its growth is comparable with u^{q-1} with $q > 1$ and $\frac{1}{p} + \frac{1}{q} > 1$, i.e. $q < \frac{p}{p-1}$; instead, $f(x,u)$ possesses a quasi-critical growth if it behaves like u^{q-1} with p and q belonging to the critical hyperbola

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$\frac{1}{p}+\frac{1}{q}=1-\frac{2}{N}$, that is $q=\frac{Np}{(N-2)p-N}$. The interest in this kind of problems arises mainly from the need to solve nonlinear Lane-Emden elliptic systems of the type

$$\left\{\begin{array}{ll} -\Delta u \ = \ v^{p-1} & \text{in } \Omega \text{ (resp. in } \mathbb{R}^N), \\ -\Delta v \ = \ -\rho(x)u^{\frac{1}{p-1}}+f(x,u) & \text{in } \Omega \text{ (resp. in } \mathbb{R}^N), \\ u=\Delta u=0 \text{ (resp. } u,\Delta u\to 0) & \text{in } \partial\Omega \text{ (resp. as } |x|\to+\infty), \end{array}\right. \tag{1.2}$$

which are equivalent to (1.1). Indeed, when $N\geq 3$ and $1<p\leq 2$ it is well known (see e.g. [18]) that the standard functional associated to systems (1.2) is strongly indefinite in interpolation spaces of infinite dimension. In order to overcome this problem, by arguing as in [16] (see also [17]) and by applying a decoupling technique i.e., by exploiting the "change of variable" $(-\Delta u)^{\frac{1}{p-1}}=v$ which works since $g(v)=v^{p-1}$ is an increasing function, it is possible to rewrite systems (1.2) as the equivalent fourth order quasilinear elliptic problems (1.1). Clearly, if u is a weak solution of (1.1), we get a weak solution of systems (1.2) which is the couple $(u,(-\Delta u)^{\frac{1}{p-1}})$. Existence, multiplicity and regularity results to (1.1) (resp. (1.2)) have been stated in bounded domains $\Omega\subset\mathbb{R}^N$ when $\rho(x)=0$ or in unbounded domains when $\rho(x)\neq 0$ both in the $(\frac{p}{p-1}-1)$-superlinear but subcritical case (see Barile and Salvatore [3, 4, 5], Candela and Salvatore [14], dos Santos [21] and references therein) and in the $(\frac{p}{p-1}-1)$-sublinear case (see Barile and Salvatore [6, 7, 8], Bonheure, dos Santos and Ramos [13] and Felmer and Martinez [22] and cited papers within). In the critical case, because of the lack of compactness of the problem, non-existence of solutions has been stated in [26] and [28] by using Pohozaev type arguments.

Besides the equivalence with classes of systems (1.2), problems like (1.1) are also interesting since, in the particular case $p=2$, they reduce to biharmonic equations with Navier boundary conditions of the form

$$\left\{\begin{array}{ll} \Delta^2 u=-\rho(x)u+f(x,u) & \text{in } \Omega \text{ (resp. in } \mathbb{R}^N), \\ u=\Delta u=0 \text{ (resp. } u,\Delta u\to 0) & \text{in } \partial\Omega \text{ (resp. as } |x|\to+\infty), \end{array}\right.$$

which have been studied in the last years and in different cases by several authors like Alves and do Ó [1], Bastos, Miyagaki and Vieira [9], Berchio and Gazzola [10], Bernís, Garcia Azorero and Peral [12], Chabrowski and do Ó [15], Demarque and Miyagaki [19], Deng and Shuai [20] and references within. Really, observe that when $1<p\leq 2$ the operator $-\Delta(-\Delta u)^{\frac{1}{p-1}}$ involved in (1.1) can be seen as an alternative generalization of the biharmonic operator $\Delta^2 u$ with respect to the classical definition of the polyharmonic operator $(-\Delta)^m u$ (see Gazzola, Grunau and Sweers [23]).

Specific aim of this paper is to study the following fourth order quasilinear elliptic problem under Navier conditions in $\mathbb{R}^N$

$$\left\{\begin{array}{ll} -\Delta(-\Delta u)^{\frac{1}{p-1}}+\rho(x)u^{\frac{1}{p-1}}=\sigma(x)f(u) & \text{in } \mathbb{R}^N, \\ u,\Delta u\to 0 & \text{as } |x|\to+\infty, \end{array}\right. \tag{1.3}$$

and therefore the equivalent nonlinear Lane-Emden elliptic problem

$$\left\{\begin{array}{ll} -\Delta u \ = \ v^{p-1} & \text{in } \mathbb{R}^N, \\ -\Delta v \ = \ -\rho(x)u^{\frac{1}{p-1}}+\sigma(x)f(u) & \text{in } \mathbb{R}^N, \\ u,v \ \to 0 & \text{as } |x|\to+\infty, \end{array}\right. \tag{1.4}$$

where $N\geq 5$ and $\frac{N}{N-2}<p\leq 2$, the continuous functions $\rho,\sigma:\mathbb{R}^N\to\mathbb{R}$ belong to a general class of weights such that decaying and vanishing ones are covered and $f:\mathbb{R}\to\mathbb{R}$ is a

suitable quasi-critical power nonlinear term with critical exponent

$$\left(\frac{p}{p-1}\right)^{**} = \frac{Np}{(N-2)p-N} \in \mathbb{R}$$

since $p > \frac{N}{N-2}$ (see Section 2). The strenght of this investigation is that, to the best of our knowledge, problem (1.3) (resp. (1.4)) has not been studied before under these general assumptions on the vanishing functions ρ and σ and on the nonlinearities f. These classes of potentials and nonlinearities allow us to recover the lack of the compactness of the problem. Moreover, as we will specify later, we extend in part or complement the results established in the case $p = 2$ for the biharmonic problem

$$\begin{cases} \Delta^2 u + \rho(x)u = \sigma(x)f(u) & \text{in } \mathbb{R}^N, \\ u, \Delta u \to 0 & \text{as } |x| \to +\infty, \end{cases} \tag{1.5}$$

in particular by Bastos, Miyagaki and Vieira [9] and references therein like Alves and do Ó [1] and Demarque and Miyagaki [19] and the ones stated by Deng and Shuai [20].
Specifically, we assume the following hypotheses on the potentials ρ and σ:

- (ρ_1) $\rho \in C(\mathbb{R}^N, \mathbb{R})$ and $\rho(x) > 0$ for every $x \in \mathbb{R}^N$;
- (σ_1) $\sigma \in C(\mathbb{R}^N, \mathbb{R})$, $\sigma(x) > 0$ for every $x \in \mathbb{R}^N$ and $\sigma \in L^\infty(\mathbb{R}^N)$;
- (σ_2) if $\{\Omega_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $\text{meas}(\Omega_n) \le R$ (where meas denotes the Lebesgue measure) for all $n \in \mathbb{N}$ and some $R > 0$, then

$$\lim_{r \to +\infty} \int_{\Omega_n \,\cap\, B_r^c(0)} \sigma(x) = 0 \quad \text{uniformly with respect to } n \in \mathbb{N}.$$

Furthermore, one of the below conditions relating ρ and σ occurs:

$(\rho\sigma_1)$ $\dfrac{\sigma}{\rho} \in L^\infty(\mathbb{R}^N)$

or

$(\rho\sigma_2)$ there exists $m \in \left(\frac{p}{p-1}, \left(\frac{p}{p-1}\right)^{**}\right)$ such that

$$\lim_{|x| \to +\infty} \frac{\sigma(x)}{\rho(x)^\gamma} = 0 \quad \text{with } \gamma = \frac{\left(\frac{p}{p-1}\right)^{**} - m}{\left(\frac{p}{p-1}\right)^{**} - p} \in (0,1).$$

Regarding the nonlinear term f, we assume the following conditions in the origin and at infinity:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and if $(\rho\sigma_1)$ holds, then

$$\lim_{|s| \to 0^+} \frac{f(s)}{|s|^{\frac{1}{p-1}}} = 0$$

or

(f_2) $f \in C(\mathbb{R}, \mathbb{R})$ and if $(\rho\sigma_2)$ holds, then

$$\lim_{|s| \to 0^+} \frac{f(s)}{|s|^{m-1}} = 0$$

with $m \in \left(\frac{p}{p-1}, \left(\frac{p}{p-1}\right)^{**}\right)$ defined in $(\rho\sigma_2)$;

(f_3) f has a quasi-critical growth at infinity, namely,

$$\lim_{|s|\to+\infty}\frac{f(s)}{|s|^{\left(\frac{p}{p-1}\right)^{**}-1}}=0;$$

(f_4) there exists $\mu\in\left(\frac{p}{p-1},\left(\frac{p}{p-1}\right)^{**}\right)$ such that

$$0<\mu F(s)=\mu\int_0^s f(t)dt\le f(s)s\quad\text{for all } s\in\mathbb{R}\setminus\{0\};$$

(f_5) f is odd with respect to s, i.e. $f(-s)=-f(s)$ for every $s\in\mathbb{R}$.

Remark 1. By making some minor technical changes throughout the paper, we can replace the limit in assumptions (f_1), (f_2) and (f_3) with the limit superior and in addition in (f_3) we can require it is finite.

Remark 2. By assumptions (f_1) and (f_3), for any $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that

$$|f(s)|\le\varepsilon|s|^{\frac{1}{p-1}}+C_\varepsilon|s|^{\left(\frac{p}{p-1}\right)^{**}-1}\quad\text{for all } s\in\mathbb{R}\tag{1.6}$$

and by integration

$$|F(s)|\le\varepsilon\,\frac{(p-1)}{p}|s|^{\frac{p}{p-1}}+\frac{C_\varepsilon}{\left(\frac{p}{p-1}\right)^{**}}|s|^{\left(\frac{p}{p-1}\right)^{**}}\quad\text{for all } s\in\mathbb{R}.\tag{1.7}$$

Similarly, by hypotheses (f_2) and (f_3), for any $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that

$$|f(s)|\le\varepsilon|s|^{m-1}+C_\varepsilon|s|^{\left(\frac{p}{p-1}\right)^{**}-1}\quad\text{for all } s\in\mathbb{R}\tag{1.8}$$

and by integration

$$|F(s)|\le\varepsilon\,\frac{1}{m}|s|^{m}+\frac{C_\varepsilon}{\left(\frac{p}{p-1}\right)^{**}}|s|^{\left(\frac{p}{p-1}\right)^{**}}\quad\text{for all } s\in\mathbb{R}.\tag{1.9}$$

Remark 3. By hypothesis (f_4), fixed any $s_0>0$ we have that

$$F(s)\ge\frac{F(s_0)}{|s_0|^\mu}\,|s|^\mu\quad\text{for all } s\in\mathbb{R}\text{ such that } |s|\ge s_0.\tag{1.10}$$

Setting $C_{s_0,\mu}=\frac{F(s_0)}{|s_0|^\mu}>0$, we can find constants $C,C_1>0$ such that

$$F(s)\ge C\,|s|^\mu-C_1\quad\text{for all } s\in\mathbb{R}.\tag{1.11}$$

Therefore, since $\mu>\frac{p}{p-1}$ we deal with $\frac{p}{p-1}$-superquadratic functions F.

At this point we can state our results. For the definition of the working space $\mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ (resp. $\mathcal{D}^{2,2}(\mathbb{R}^N)$ in the case $p=2$) and the functional I associated to (1.3) we refer to Section 2.

Theorem 1. *Let $N \geq 5$ and $\frac{N}{N-2} < p \leq 2$. Suppose that (ρ_1), (σ_1), (σ_2), $(\rho\sigma_1)$ (resp. $(\rho\sigma_2)$) and (f_1) (resp. (f_2)), (f_3) and (f_4) hold. Then, I admits a non-trivial critical point $\overline{u} \in \mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ of Mountain Pass type therefore problem* (1.3) *admits a non-trivial weak solution $\overline{u} \in \mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N)$.*
Moreover, if f satisfies also (f_5), then I has an unbounded sequence $\{\overline{u}_n\}$ of non-trivial min-max type critical points $\overline{u}_n \in \mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ thus problem (1.3) *possesses an infinitely many non-trivial weak solutions $\overline{u}_n \in \mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N)$.*

Taking into account the equivalence of problem (1.3) with Lane-Emden type system (1.4), once we have found a weak solution $\overline{u} \in \mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ (resp. a sequence of weak solutions $\{\overline{u}_n\} \subset \mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N)$) of (1.3), we get a weak solution (resp. a sequence of weak solutions) of system (1.4) which is the couple $(\overline{u}, (-\Delta\overline{u})^{\frac{1}{p-1}})$ (resp. the couples sequence $\{(\overline{u}_n, (-\Delta\overline{u}_n)^{\frac{1}{p-1}})\}$). Therefore, as a direct consequence of Theorem 1 we can obtain the following result which is also new within the framework of Lane-Emden type systems.

Theorem 2. *Let $N \geq 5$ and $\frac{N}{N-2} < p \leq 2$. Suppose that (ρ_1), (σ_1), (σ_2), $(\rho\sigma_1)$ (resp. $(\rho\sigma_2)$) and (f_1) (resp. (f_2)), (f_3) and (f_4) hold. Then, system* (1.4) *admits a non-trivial weak solution $(\overline{u}, (-\Delta\overline{u})^{\frac{1}{p-1}})$ with $\overline{u} \in \mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N)$.*
If f satisfies also (f_5), problem (1.4) *has an unbounded sequence $\{(\overline{u}_n, (-\Delta\overline{u}_n)^{\frac{1}{p-1}})\}$ then infinitely many non-trivial weak solutions $(\overline{u}_n, (-\Delta\overline{u}_n)^{\frac{1}{p-1}})$ with $\overline{u}_n \in \mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N)$.*

Moreover, as observed above since in the case $p = 2$ problem (1.3) reduces to the biharmonic problem (1.5), we emphasize that Theorem 1 covers the next result which holds for $N \geq 5$ within the framework of biharmonic equations with quasi-critical growth and generalized vanishing potentials.

Corollary 1. *Let $N \geq 5$ and $p = 2$. Assume that (ρ_1), (σ_1), (σ_2), $(\rho\sigma_1)$ (resp. $(\rho\sigma_2)$) and (f_1) (resp. (f_2)), (f_3) and (f_4) hold (with $p = 2$). Then, problem* (1.5) *admits a non-trivial weak solution $\overline{u} \in \mathcal{D}^{2,2}(\mathbb{R}^N)$.*
If in addition f satisfies (f_5), problem (1.5) *has an unbounded sequence $\{\overline{u}_n\} \subset \mathcal{D}^{2,2}(\mathbb{R}^N)$ of non-trivial weak solutions.*

Note that we improve the existence of a solution in the paper by Bastos, Miyagaki and Vieira [9] since the authors establish it under a quasi-critical growth assumption on f in the origin which is stronger with respect to our (f_1) (resp. (f_2)). Since they are specifically interested in a ground state solution to (1.5), they work under a superquadraticity assumption on F at infinity which is weaker than (f_4) but under an additional monotonicity assumption on f. Moreover, we complete this paper with the multiplicity result. Consequently, we also complement the results in Demarque and Miyagaki [19] for nonradial weights ρ and σ, in Alves and do Ó [1] for more general weights and nonlinearities and some other references within [9]. Furthermore, we cover the existence result by Deng and Shuai [20] for $\mu = 1$ and $P(x) = 0$ and we establish in addition the multiplicity result. From these observations, we deduce therefore that Theorem 1 allows us to extend these results to the operator $-\Delta(-\Delta u)^{\frac{1}{p-1}}$ for exponents p such that $\frac{N}{N-2} < p \leq 2$ and dimensions N with $N \geq 5$.

The paper is organized as follows: in Section 2 we introduce the variational formulation of the problem and we prove some useful compactness results. In Section 3 we show the assumptions of Mountain Pass Theorem and its Symmetric version are satisfied thus concluding with the proof of Theorem 1.

2. Variational Tools

In order to prove that problem (1.3) has a variational structure, let us introduce the space

$$\mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N) = \{u \in L^{\left(\frac{p}{p-1}\right)^{**}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\Delta u|^{\frac{p}{p-1}}\, dx < \infty\} \hookrightarrow L^{\left(\frac{p}{p-1}\right)^{**}}(\mathbb{R}^N)$$

where the critical exponent is defined as follows

$$\left(\frac{p}{p-1}\right)^{**} = \frac{Np}{(N-2)p-N} \quad \text{since } p > \tfrac{N}{N-2}, \text{ i.e. } 2\tfrac{p}{p-1} < N.$$

This space is equipped with the norm

$$\|u\|_{\mathcal{D}^{2,\frac{p}{p-1}}} = \left(\int_{\mathbb{R}^N} |\Delta u|^{\frac{p}{p-1}}\, dx\right)^{\frac{p-1}{p}}.$$

In the following, we denote by $|\cdot|_t$ the usual norm on the Lebesgue space $L^t(\mathbb{R}^N)$ with $1 \le t \le +\infty$. As (ρ_1) holds, we can consider the space

$$\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N) = \{u \in \mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \rho(x)|u|^{\frac{p}{p-1}}\, dx < \infty\}$$

endowed with the norm

$$\|u\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} = \left(\int_{\mathbb{R}^N} |\Delta u|^{\frac{p}{p-1}}\, dx + \int_{\mathbb{R}^N} \rho(x)|u|^{\frac{p}{p-1}}\, dx\right)^{\frac{p-1}{p}}$$

and we denote by $\left(\mathcal{D}_\rho^{-2,\frac{p}{p-1}}(\mathbb{R}^N), \|\cdot\|_{\mathcal{D}_\rho^{-2,\frac{p}{p-1}}}\right)$ the normed dual space of $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$. From now on, let $1 \le t < \infty$ and

$$L_\rho^t(\mathbb{R}^N) = \{u \in L^t(\mathbb{R}^N) : \int_{\mathbb{R}^N} \rho(x)|u|^t\, dx < \infty\}$$

endowed with the norm

$$|u|_{t,\rho} = \left(\int_{\mathbb{R}^N} \rho(x)|u|^t\, dx\right)^{\frac{1}{t}}.$$

Clearly,

$$\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N) = \mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N) \cap L_\rho^{\frac{p}{p-1}}(\mathbb{R}^N)$$

therefore it easily follows

$$\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N) \hookrightarrow \mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N) \hookrightarrow L^{\left(\frac{p}{p-1}\right)^{**}}(\mathbb{R}^N) \quad \text{and} \quad \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N) \hookrightarrow L_\rho^{\frac{p}{p-1}}(\mathbb{R}^N). \tag{2.1}$$

Furthermore, for every $w \in \mathbb{R}$, $1 \le w < \infty$, let us define the Lebesgue space

$$L_\sigma^w(\mathbb{R}^N) = \left\{u \in L^w(\mathbb{R}^N) : \int_{\mathbb{R}^N} \sigma(x)|u|^w\, dx < +\infty\right\}$$

endowed with the norm

$$|u|_{w,\sigma} = \left(\int_{\mathbb{R}^N} \sigma(x)|u|^w\, dx\right)^{\frac{1}{w}}.$$

Recall that a weak solution of problem (1.3) is a function $u \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (-\Delta u)^{\frac{1}{p-1}}(-\Delta v)\,dx + \int_{\mathbb{R}^N} \rho(x)|u|^{\frac{1}{p-1}} v\,dx - \int_{\mathbb{R}^N} \sigma(x) f(u) v\,dx = 0$$

for every $v \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$.

From now on, c, c_i, C, C_i denote real positive constants changing line from line and

$$C_{\sigma,\rho} = \left|\frac{\sigma}{\rho}\right|_\infty = \operatorname{ess\,sup}\left\{\frac{\sigma(x)}{\rho(x)} : x \in \mathbb{R}^N\right\} \quad \text{and} \quad C_\sigma = |\sigma|_\infty = \operatorname{ess\,sup}\{\sigma(x) : x \in \mathbb{R}^N\}.$$

Now, let us recall the following classical result which will be useful in order to state the variational formulation of problem (1.3).

Lemma 1. *Let $\rho : \mathbb{R}^N \to \mathbb{R}$ be a function satisfying (ρ_1). Then, the Nemytskii operator $\widetilde{h}$ associated to $h(s) = s^{\frac{1}{p-1}}$ is continuous from $L_\rho^{\frac{p}{p-1}}(\mathbb{R}^N)$ to $L_\rho^p(\mathbb{R}^N)$.*

We can state now the following variational principle.

Proposition 1. *Assume that (ρ_1), (σ_1), $(\rho\sigma_1)$ (resp. $(\rho\sigma_2)$), (f_1) (resp. (f_2)) and (f_3) hold. Then, the weak solutions of problem (1.3) are the critical points of the energy functional defined on $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ by*

$$I(u) = \frac{p-1}{p}\int_{\mathbb{R}^N}\left(|\Delta u|^{\frac{p}{p-1}} + \rho(x)|u|^{\frac{p}{p-1}}\right)dx - \int_{\mathbb{R}^N}\sigma(x)F(u)\,dx, \quad u \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N).$$

More precisely, $I \in C^1(\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N))$ and its derivative $dI : \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N) \to \mathcal{D}_\rho^{-2,\frac{p}{p-1}}(\mathbb{R}^N)$ is defined as

$$dI(u)[v] = \int_{\mathbb{R}^N}\left[(-\Delta u)^{\frac{1}{p-1}}(-\Delta v) + \rho(x)|u|^{\frac{1}{p-1}}v - \sigma(x)f(u)v\right]dx \tag{2.2}$$

for all $u, v \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$.

Proof. First, we prove that the functional

$$I(u) = \frac{p-1}{p}\,\|u\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}}^{\frac{p}{p-1}} - \int_{\mathbb{R}^N}\sigma(x)F(u)\,dx \quad \text{for any } u \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$$

is well defined and its Fréchet derivative given in (2.2) is a continuous operator from $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ to $\mathcal{D}_\rho^{-2,\frac{p}{p-1}}(\mathbb{R}^N)$. We define and study separately the two maps

$$I_\rho(u) = \frac{p-1}{p}\,\|u\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}}^{\frac{p}{p-1}}, \qquad I_{\sigma,F}(u) = \int_{\mathbb{R}^N}\sigma(x)F(u)\,dx \qquad \text{for any } u \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N).$$

Clearly, $I_\rho \in C^1(\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N))$ since I_ρ is continuous from $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ to $\mathbb{R}$ and its Gâteaux derivative at u

$$dI_\rho(u)[v] \;=\; \int_{\mathbb{R}^N}(-\Delta u)^{\frac{1}{p-1}}(-\Delta v)\,dx + \int_{\mathbb{R}^N}\rho(x)u^{\frac{1}{p-1}}v\,dx$$

is a linear continuous map with respect to every $v \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$. Moreover, by adapting the arguments in [3, Proof of Proposition 2.7], we prove that $I_\rho \in C^1(\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N))$ since

$$\|dI_\rho(u_n) - dI_\rho(u)\|_{\mathcal{D}_\rho^{-2,\frac{p}{p-1}}} \to 0 \quad \text{if } u_n \to u \text{ in } \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N). \tag{2.3}$$

Indeed, by Hölder inequality and imbeddings in (2.1) we get for all $v \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$

$$\begin{aligned}
|(dI_\rho(u_n) - dI_\rho(u))[v]| &\leq \int_{\mathbb{R}^N} |(-\Delta u_n)^{\frac{1}{p-1}} - (-\Delta u)^{\frac{1}{p-1}}|\ |(-\Delta v)|\, dx \\
&\quad + \int_{\mathbb{R}^N} |\rho(x)(u_n^{\frac{1}{p-1}} - u^{\frac{1}{p-1}})|\ |v|\, dx \\
&\leq |(-\Delta u_n)^{\frac{1}{p-1}} - (-\Delta u)^{\frac{1}{p-1}}|_p\ |\Delta v|_{\frac{p}{p-1}} \\
&\quad + |(\rho(x))^{\frac{1}{p}}(u_n^{\frac{1}{p-1}} - u^{\frac{1}{p-1}})|_p\ |(\rho(x))^{\frac{p-1}{p}} v|_{\frac{p}{p-1}} \\
&\leq |(-\Delta u_n)^{\frac{1}{p-1}} - (-\Delta u)^{\frac{1}{p-1}}|_p\ \|v\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} \\
&\quad + |(\rho(x))^{\frac{1}{p}}(u_n^{\frac{1}{p-1}} - u^{\frac{1}{p-1}})|_p\ \|v\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}}.
\end{aligned}$$

Since $-\Delta u_n \to -\Delta u$ in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$, for the continuity of the Nemytskii operator associated to $h(s) = s^{\frac{1}{p-1}}$ from $L^{\frac{p}{p-1}}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$, we have that

$$|(-\Delta u_n)^{\frac{1}{p-1}} - (-\Delta u)^{\frac{1}{p-1}}|_p \to 0.$$

Since $u_n \to u$ in $L_\rho^{\frac{p}{p-1}}(\mathbb{R}^N)$, by Lemma 1 we have that

$$|u_n^{\frac{1}{p-1}} - u^{\frac{1}{p-1}}|_{p,\,\rho} = |(\rho(x))^{\frac{1}{p}}(u_n^{\frac{1}{p-1}} - u^{\frac{1}{p-1}})|_p \to 0.$$

Hence, passing to the supremum with respect to any $v \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ with $\|v\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} \leq 1$, we obtain (2.3) holds.

Now, we have to prove that also $I_{\sigma,F} \in C^1(\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N))$ with

$$dI_{\sigma,F}(u)[v] = \int_{\mathbb{R}^N} \sigma(x) f(u) v\, dx \qquad \text{for all } u, v \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N). \tag{2.4}$$

First suppose that $(\rho\sigma_1)$ and (f_1) hold. Then, by (σ_1) and inequality (1.7) in Remark 2 it is

$$\begin{aligned}
|I_{\sigma,F}(u)| &\leq C_{\sigma,\rho}\, \varepsilon\, \frac{(p-1)}{p} \int_{\mathbb{R}^N} \rho(x)|u|^{\frac{p}{p-1}}\, dx + C_\sigma\, C_\varepsilon\, \frac{1}{\left(\frac{p}{p-1}\right)^{**}} \int_{\mathbb{R}^N} |u|^{\left(\frac{p}{p-1}\right)^{**}}\, dx \\
&= C_{\sigma,\rho}\, \varepsilon\, \frac{(p-1)}{p} |u|^{\frac{p}{(p-1)}}_{\frac{p}{(p-1)},\,\rho} + C_\sigma\, C_\varepsilon\, \frac{1}{\left(\frac{p}{p-1}\right)^{**}} |u|^{\left(\frac{p}{p-1}\right)^{**}}_{\left(\frac{p}{p-1}\right)^{**}}
\end{aligned}$$

so by (2.1) we get $I_{\sigma,F}(u) \in \mathbb{R}$ for every $u \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$. Similarly, by (σ_1) and (1.6) in Remark 2 we get

$$|dI_{\sigma,F}(u)[v]| \leq C_{\sigma,\rho}\ \varepsilon \int_{\mathbb{R}^N} \rho(x)|u|^{\frac{1}{p-1}}|v|\, dx + C_\varepsilon\, C_\sigma \int_{\mathbb{R}^N} |u|^{\left(\frac{p}{p-1}\right)^{**}-1}|v|\, dx.$$

As concerns as the first integral, let us observe that by Hölder inequality it is

$$\begin{aligned}
\int_{\mathbb{R}^N} \rho(x)|u|^{\frac{1}{p-1}}|v|\, dx &= \int_{\mathbb{R}^N} \left(\rho(x)^{\frac{1}{p}}|u|^{\frac{1}{p-1}}\right)\left(\rho(x)^{\frac{p-1}{p}}|v|\right) dx \\
&\leq |u|^{\frac{1}{p-1}}_{\frac{p}{p-1},\,\rho}\ |v|_{\frac{p}{p-1},\,\rho}
\end{aligned} \tag{2.5}$$

while for the second integral we get

$$(2.6)\qquad \begin{aligned}&\int_{\mathbb{R}^N} |u|^{\left(\frac{p}{p-1}\right)^{**}-1}|v|\,dx\\ &\le \left(\int_{\mathbb{R}^N} |u|^{\left(\frac{p}{p-1}\right)^{**}}dx\right)^{\frac{\left(\frac{p}{p-1}\right)^{**}-1}{\left(\frac{p}{p-1}\right)^{**}}} \left(\int_{\mathbb{R}^N} |v|^{\left(\frac{p}{p-1}\right)^{**}}dx\right)^{\frac{1}{\left(\frac{p}{p-1}\right)^{**}}}\\ &= |u|^{\left(\frac{p}{p-1}\right)^{**}-1}_{\left(\frac{p}{p-1}\right)^{**}}\ |v|_{\left(\frac{p}{p-1}\right)^{**}}.\end{aligned}$$

Consequently,

$$|dI_{\sigma,F}(u)[v]| \le C_{\sigma,\rho}\ \varepsilon\,|u|^{\frac{1}{p-1}}_{\frac{p}{p-1},\rho}\ |v|_{\frac{p}{p-1},\rho} + C_\varepsilon\, C_\sigma\, |u|^{\left(\frac{p}{p-1}\right)^{**}-1}_{\left(\frac{p}{p-1}\right)^{**}}\ |v|_{\left(\frac{p}{p-1}\right)^{**}}$$

and by (2.1) it follows that $dI_{\sigma,F}(u)[v] \in \mathbb{R}$ for all $u, v \in \mathcal{D}^{2,\frac{p}{p-1}}_\rho(\mathbb{R}^N)$.
Now, suppose that $(\rho\sigma_2)$ and (f_2) hold. Then, by (σ_1) and inequality (1.9) in Remark 2, choosen a radius $r > 0$ we get

$$(2.7)\qquad \begin{aligned}|I_{\sigma,F}(u)| &\le \frac{\varepsilon}{m}\int_{\mathbb{R}^N}\sigma(x)|u|^m\,dx + \frac{C_\varepsilon}{\left(\frac{p}{p-1}\right)^{**}}\int_{\mathbb{R}^N}\sigma(x)|u|^{\left(\frac{p}{p-1}\right)^{**}}dx\\ &\le \frac{\varepsilon}{m}\left(C_\sigma\int_{B_r(0)}|u|^m\,dx + \int_{B_r^c(0)}\sigma(x)|u|^m\,dx\right) + \frac{C_\varepsilon}{\left(\frac{p}{p-1}\right)^{**}}C_\sigma|u|^{\left(\frac{p}{p-1}\right)^{**}}_{\left(\frac{p}{p-1}\right)^{**}}.\end{aligned}$$

Since $m \in \left(\frac{p}{p-1}, \left(\frac{p}{p-1}\right)^{**}\right)$ it is $L^{\left(\frac{p}{p-1}\right)^{**}}(B_r(0)) \hookrightarrow L^m(B_r(0))$ from which it follows

$$(2.8)\qquad \begin{aligned}\int_{B_r(0)}|u|^m\,dx &\le c\left(\int_{B_r(0)}|u|^{\left(\frac{p}{p-1}\right)^{**}}dx\right)^{\frac{m}{\left(\frac{p}{p-1}\right)^{**}}}\\ &\le c\left(\int_{\mathbb{R}^N}|u|^{\left(\frac{p}{p-1}\right)^{**}}dx\right)^{\frac{m}{\left(\frac{p}{p-1}\right)^{**}}} = c\,|u|^m_{\left(\frac{p}{p-1}\right)^{**}}.\end{aligned}$$

Now, in order to estimate the integral on $B_r^c(0)$ in (2.7) let us define for any fixed $x \in \mathbb{R}^N$ the function

$$g(s) = \rho(x)s^{\frac{p}{p-1}-m} + s^{\left(\frac{p}{p-1}\right)^{**}-m} \quad \text{for every } s > 0$$

whose minimum value is $C_m\rho(x)^\gamma$ with γ defined in $(\rho\sigma_2)$ and

$$C_m = \left(\frac{\left(\frac{p}{p-1}\right)^{**} - \frac{p}{p-1}}{\left(\frac{p}{p-1}\right)^{**} - m}\right)\left(\frac{m - \frac{p}{p-1}}{\left(\frac{p}{p-1}\right)^{**} - \frac{p}{p-1}}\right)^{\frac{\frac{p}{p-1}-m}{\left(\frac{p}{p-1}\right)^{**}-\frac{p}{p-1}}}.$$

Hence,

$$C_m\,\rho(x)^\gamma \le \rho(x)\,s^{\frac{p}{p-1}-m} + s^{\left(\frac{p}{p-1}\right)^{**}-m} \quad \text{for every } x \in \mathbb{R}^N \text{ and } s > 0.$$

Then, in combination with $(\rho\sigma_2)$, in correspondence of any $\varepsilon > 0$ we can find a positive radius $r > 0$ sufficiently large such that

$$(2.9)\qquad \sigma(x)|s|^m \le \varepsilon\,\rho(x)^\gamma|s|^m \le \varepsilon C_m^{-1}\left(\rho(x)|s|^{\frac{p}{p-1}} + |s|^{\left(\frac{p}{p-1}\right)^{**}}\right)$$
$$\text{for every } s \in \mathbb{R} \text{ and } |x| \ge r.$$

In particular, for every $u \in \mathcal{D}^{2,\frac{p}{p-1}}_{\rho}(\mathbb{R}^N)$ we get

$$\int_{B_r^c(0)} \sigma(x)|u|^m\, dx \le \varepsilon\, C_m^{-1} \left(|u|^{\frac{p}{p-1}}_{\frac{p}{p-1},\rho} + |u|^{\left(\frac{p}{p-1}\right)^{**}}_{\left(\frac{p}{p-1}\right)^{**}} \right). \tag{2.10}$$

Therefore, by substituting (2.8) and (2.10) in (2.7) we get

$$\begin{aligned} |I_{\sigma,F}(u)| \quad \le \quad & \frac{\varepsilon}{m} \left(C_\sigma\, c\, |u|^m_{\left(\frac{p}{p-1}\right)^{**}} + \varepsilon\, C_m^{-1} \left(|u|^{\frac{p}{p-1}}_{\frac{p}{p-1},\rho} + |u|^{\left(\frac{p}{p-1}\right)^{**}}_{\left(\frac{p}{p-1}\right)^{**}} \right) \right) \\ & + \frac{C_\varepsilon\, C_\sigma}{\left(\frac{p}{p-1}\right)^{**}} |u|^{\left(\frac{p}{p-1}\right)^{**}}_{\left(\frac{p}{p-1}\right)^{**}}. \end{aligned} \tag{2.11}$$

and by (2.1) we obtain $I_{\sigma,F}(u)$ is well defined for every $u \in \mathcal{D}^{2,\frac{p}{p-1}}_{\rho}(\mathbb{R}^N)$.
Reasoning in a similar way, we are going also to prove that $dI_{\sigma,F}(u)[v]$ is well posed for every $u, v \in \mathcal{D}^{2,\frac{p}{p-1}}_{\rho}(\mathbb{R}^N)$. Indeed, by (1.8) in Remark 2, (2.6) and chosen a radius $r > 0$ we get

$$\begin{aligned} |dI_{\sigma,F}(u)[v]| &\le \varepsilon \int_{\mathbb{R}^N} \sigma(x)|u|^{m-1}|v|\, dx + C_\varepsilon\, C_\sigma \int_{\mathbb{R}^N} |u|^{\left(\frac{p}{p-1}\right)^{**}-1}|v|\, dx \\ &\le \varepsilon \left(C_\sigma \int_{B_r(0)} |u|^{m-1}|v|\, dx + \int_{B_r^c(0)} \sigma(x)|u|^{m-1}|v|\, dx \right) \\ &\qquad + C_\varepsilon\, C_\sigma\, |u|^{\left(\frac{p}{p-1}\right)^{**}-1}_{\left(\frac{p}{p-1}\right)^{**}} |v|_{\left(\frac{p}{p-1}\right)^{**}}. \end{aligned} \tag{2.12}$$

Now, by Hölder inequality and since $m < \left(\frac{p}{p-1}\right)^{**}$ implies

$$L^{\left(\frac{p}{p-1}\right)^{**}}(B_r(0)) \hookrightarrow L^{\frac{(m-1)\left(\frac{p}{p-1}\right)^{**}}{\left(\frac{p}{p-1}\right)^{**}-1}}(B_r(0)),$$

we get

$$\begin{aligned} &\int_{B_r(0)} |u|^{m-1}|v|\, dx \\ &\quad \le \left(\int_{B_r(0)} |u|^{\frac{(m-1)\left(\frac{p}{p-1}\right)^{**}}{\left(\frac{p}{p-1}\right)^{**}-1}}\, dx \right)^{\frac{\left(\frac{p}{p-1}\right)^{**}-1}{\left(\frac{p}{p-1}\right)^{**}}} \left(\int_{B_r(0)} |v|^{\left(\frac{p}{p-1}\right)^{**}}\, dx \right)^{\frac{1}{\left(\frac{p}{p-1}\right)^{**}}} \\ &\quad \le c \left(\int_{B_r(0)} |u|^{\left(\frac{p}{p-1}\right)^{**}}\, dx \right)^{\frac{m-1}{\left(\frac{p}{p-1}\right)^{**}}} |v|_{\left(\frac{p}{p-1}\right)^{**}} \\ &\quad \le c\, |u|^{m-1}_{\left(\frac{p}{p-1}\right)^{**}} |v|_{\left(\frac{p}{p-1}\right)^{**}}. \end{aligned} \tag{2.13}$$

On the other hand, by (2.9) we have

$$\sigma(x) \le \varepsilon \left(\rho(x)|s|^{\frac{p}{p-1}-m} + |s|^{\left(\frac{p}{p-1}\right)^{**}-m} \right) \quad \text{for every } s \in \mathbb{R}\setminus\{0\} \text{ and } |x| \ge r \tag{2.14}$$

so that

$$\sigma(x)|s|^{m-1} \le \varepsilon \left(\rho(x)|s|^{\frac{1}{p-1}} + |s|^{\left(\frac{p}{p-1}\right)^{**}-1} \right) \quad \text{for every } s \in \mathbb{R} \text{ and } |x| \ge r.$$

Therefore, by exploiting (2.5) and (2.6) we have

$$
\begin{aligned}
&\int_{B_r^c(0)} \sigma(x)|u|^{m-1}|v|\,dx \\
&\quad \le \varepsilon \left(\int_{B_r^c(0)} \rho(x)|u|^{\frac{1}{p-1}}|v|\,dx + \int_{B_r^c(0)} |u|^{\left(\frac{p}{p-1}\right)^{**}-1}|v|\,dx \right) \\
&\quad \le \varepsilon \left(|u|^{\frac{1}{p-1}}_{\frac{p}{p-1},\rho}\, |v|_{\frac{p}{p-1},\rho} + |u|^{\left(\frac{p}{p-1}\right)^{**}-1}_{\left(\frac{p}{p-1}\right)^{**}}\, |v|_{\left(\frac{p}{p-1}\right)^{**}} \right).
\end{aligned} \tag{2.15}
$$

Consequently, by substituting (2.13) and (2.15) in (2.12) we conclude

$$
\begin{aligned}
|dI_{\sigma,F}(u)[v]| \le \varepsilon \Bigg(& C_\sigma\, c\, |u|^{m-1}_{\left(\frac{p}{p-1}\right)^{**}}\, |v|_{\left(\frac{p}{p-1}\right)^{**}} \\
& + \varepsilon \left(|u|^{\frac{1}{p-1}}_{\frac{p}{p-1},\rho}\, |v|_{\frac{p}{p-1},\rho} + |u|^{\left(\frac{p}{p-1}\right)^{**}-1}_{\left(\frac{p}{p-1}\right)^{**}}\, |v|_{\left(\frac{p}{p-1}\right)^{**}} \right)\Bigg) \\
& + C_\varepsilon\, C_\sigma\, |u|^{\left(\frac{p}{p-1}\right)^{**}-1}_{\left(\frac{p}{p-1}\right)^{**}}\, |v|_{\left(\frac{p}{p-1}\right)^{**}}
\end{aligned}
$$

and by (2.1) we get $dI_{\sigma,F}(u)[v]$ is well defined for every $u, v \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$.
Moreover, standard arguments imply that the Gâteaux derivative of $I_{\sigma,F}$ at u is as in (2.4) and it is linear and continuous from $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ to $\mathbb{R}$.
As concerns as the continuity of $dI_{\sigma,F}$ from $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ to $\mathcal{D}_\rho^{-2,\frac{p}{p-1}}(\mathbb{R}^N)$, i.e.

$$
\|d\varphi_1(u_n) - d\varphi_1(u)\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} \to 0 \quad \text{if } u_n \to u \text{ in } \mathcal{D}_\rho^{2,\frac{p}{p-1}},
$$

we can refer to and exploit Proposition 3 where the compactness of $d\varphi_1 = dI_{\sigma,F}$ from $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ to $\mathcal{D}_\rho^{-2,\frac{p}{p-1}}(\mathbb{R}^N)$ is stated. □

As ensured by the following result, the presence of the interacting weights $\rho(x)$ and $\sigma(x)$ allows us to overcome the lack of compact imbeddings of the space $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ into the Lebesgue spaces $L^t(\mathbb{R}^N)$ with $t \in \left(\frac{p}{p-1}, \left(\frac{p}{p-1}\right)^{**}\right)$. Here below we provide all the details of the proof since we extend to $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ the arguments employed in Bastos, Miyagaki and Vieira [9] for the space $\mathcal{D}_\rho^{2,2}(\mathbb{R}^N)$. These arguments follow the line of the compactness results proved in Opic and Kufner [27] for $\mathcal{D}_\rho^{1,2}(\mathbb{R}^N)$ in presence of vanishing potentials $\rho(x)$ and $\sigma(x)$ which are included in our setting assumptions.

Proposition 2. *Suppose that* (ρ_1), (σ_1) *and* (σ_2) *are satisfied.*

(*i*) *If* $(\rho\sigma_1)$ *holds, then*

$$
\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N) \text{ is compactly embedded in } L^w_\sigma(\mathbb{R}^N) \text{ for any } w \in \left(\tfrac{p}{p-1}, \left(\tfrac{p}{p-1}\right)^{**}\right);
$$

(*ii*) *if* $(\rho\sigma_2)$ *holds, then*

$$
\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N) \text{ is compactly embedded in } L^m_\sigma(\mathbb{R}^N)
$$

with $m \in \left(\frac{p}{p-1}, \left(\frac{p}{p-1}\right)^{**}\right)$ *defined in* $(\rho\sigma_2)$.

Proof. First, assume that $(\rho\sigma_1)$ holds. Fixed $\varepsilon > 0$, since $w > \frac{p}{p-1}$ there exists $s_0 > 0$ such that

$$|s|^w \leq \varepsilon\, |s|^{\frac{p}{p-1}} \quad \text{for all } s \in \mathbb{R},\ |s| \leq s_0.$$

Then, by $(\rho\sigma_1)$ it is

$$\sigma(x)|s|^w \leq C_{\sigma,\rho}\, \rho(x)\, |s|^w \leq \varepsilon\, C_{\sigma,\rho}\, |s|^{\frac{p}{p-1}} \quad \text{for all } s \in \mathbb{R},\ |s| \leq s_0 \text{ and } x \in \mathbb{R}^N.$$

In correspondence of $\varepsilon > 0$, from $\sigma \in L^\infty(\mathbb{R}^N)$ and $r < \left(\frac{p}{p-1}\right)^{**}$ we can find $s_1 > 0$ such that

$$\sigma(x)|s|^w \leq \varepsilon\, C_\sigma\, |s|^{\left(\frac{p}{p-1}\right)^{**}} \quad \text{for all } s \in \mathbb{R},\ |s| \geq s_1 \text{ and } x \in \mathbb{R}^N.$$

Since of continuity of the functions involved, there exists a constant $c > 0$ such that

$$\sigma(x)|s|^w \leq c\, \sigma(x)\, \chi_{[s_0,s_1]}(|s|)\, |s|^{\left(\frac{p}{p-1}\right)^{**}} \quad \text{for all } s \in \mathbb{R},\ s_0 \leq |s| \leq s_1 \text{ and } x \in \mathbb{R}^N$$

where $c = \max_{s_0 \leq |s| \leq s_1} \frac{|s|^w}{|s|^{\left(\frac{p}{p-1}\right)^{**}}}$ and $\chi_{[s_0,s_1]}$ denotes the characteristic function in the interval $[s_0, s_1]$.

Consequently, for any $w \in \left(\frac{p}{p-1}, \left(\frac{p}{p-1}\right)^{**}\right)$ and $\varepsilon > 0$, there exist $0 < s_0 < s_1$ and a suitable constant $C > 0$ such that

$$\sigma(x)\, |s|^w \leq \varepsilon\, C\, (\rho(x)|s|^{\frac{p}{p-1}} + |s|^{\left(\frac{p}{p-1}\right)^{**}}) + C\sigma(x)\, \chi_{[s_0,s_1]}(|s|)\, |s|^{\left(\frac{p}{p-1}\right)^{**}}$$
$$\text{for all } s \in \mathbb{R} \text{ and for every } x \in \mathbb{R}^N.$$

For every $u \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ denote by

$$Q(u) = C\, \left(\int_{\mathbb{R}^N} \rho(x)|u|^{\frac{p}{p-1}}\, dx + \int_{\mathbb{R}^N} |u|^{\left(\frac{p}{p-1}\right)^{**}}\, dx\right)$$

and $\Omega = \left\{x \in \mathbb{R}^N : s_0 \leq |u(x)| \leq s_1\right\}$. Then, for any given radius $r_1 > 0$ we get

$$\begin{aligned} \int_{B^c_{r_1}(0)} \sigma(x)|u|^w\, dx &\leq \varepsilon\, Q(u) + C \int_{B^c_{r_1}(0)} \sigma(x)\, \chi_{[s_0,s_1]}(|u|)\, |u|^{\left(\frac{p}{p-1}\right)^{**}}\, dx \\ &= \varepsilon\, Q(u) + C \int_{B^c_{r_1}(0)\,\cap\,\Omega} \sigma(x)\, |u|^{\left(\frac{p}{p-1}\right)^{**}}\, dx \\ &\leq \varepsilon\, Q(u) + C\, s_1^{\left(\frac{p}{p-1}\right)^{**}} \int_{B^c_{r_1}(0)\,\cap\,\Omega} \sigma(x)\, dx. \end{aligned} \tag{2.16}$$

Now, if $\{u_n\}$ is a sequence such that $u_n \rightharpoonup u$ in $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$, by (2.1) there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} \rho(x)|u_n|^{\frac{p}{p-1}}\, dx \leq C_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |u_n|^{\left(\frac{p}{p-1}\right)^{**}}\, dx \leq C_1 \quad \text{for all } n \in \mathbb{N}. \tag{2.17}$$

Therefore, $\{Q(u_n)\}$ is bounded from above in $\mathbb{R}$ by $2C_1$, i.e.

$$Q(u_n) \leq 2C_1 \quad \text{for all } n \in \mathbb{N} \tag{2.18}$$

and, denoted by $\Omega_n = \{x \in \mathbb{R}^N : s_0 \leq |u_n(x)| \leq s_1\}$, it follows that

$$s_0^{\left(\frac{p}{p-1}\right)^{**}}\, \text{meas}(\Omega_n) \leq \int_{\Omega_n} |u_n|^{\left(\frac{p}{p-1}\right)^{**}}\, dx \leq C_1 \quad \text{for all } n \in \mathbb{N}$$

from which we get $\sup_{n\in\mathbb{N}} |\,\mathrm{meas}(\Omega_n)| < +\infty$. Consequently, from (σ_2) in correspondence of $\frac{\varepsilon}{C\, s_1^{\left(\frac{p}{p-1}\right)^{**}}} > 0$ there exists a radius $r_2 = r_2(\varepsilon) > 0$ such that

$$\int_{\Omega_n \cap B^c_{r_2}(0)} \sigma(x)\, dx < \frac{\varepsilon}{s_1^{\left(\frac{p}{p-1}\right)^{**}}} \quad \text{for all } n \in \mathbb{N}. \tag{2.19}$$

By exploiting both the boundedness of $\{Q(u_n)\}$ stated in (2.18) and (2.19) in (2.16) with $u = u_n$, choosen a suitable radius $r > 0$ (e.g. $r = \max\{r_1, r_2\}$) it follows

$$\int_{B^c_r(0)} \sigma(x)|u_n|^w\, dx \le \varepsilon(2C_1 + 1) \quad \text{for all } n \in \mathbb{N} \tag{2.20}$$

and by Fatou' Lemma we get also

$$\int_{B^c_r(0)} \sigma(x)|u|^w\, dx \le \varepsilon(2C_1 + 1) \quad \text{for all } n \in \mathbb{N}. \tag{2.21}$$

Now, since by (2.1) it is $u \in \mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ and $\mathcal{D}^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ can be also endowed with the norm

$$|D^2u|_{\frac{p}{p-1}} = \left(\int_{\mathbb{R}^N} |D^2u|^{\frac{p}{p-1}}\, dx\right)^{\frac{p-1}{p}}$$

(see [24, page164]), we easily get $D^2u \in L^{\frac{p}{p-1}}(B_r(0))$. Then, by [25, Corollary, page 3] it is $D^\alpha u \in L^{\frac{p}{p-1}}(B_r(0))$ for $|\alpha| = 0, 1$ then $u \in W^{2,\frac{p}{p-1}}(B_r(0))$. By compact embeddings in bounded domains it is in particular

$$W^{2,\frac{p}{p-1}}(B_r(0)) \hookrightarrow\hookrightarrow L^w(B_r(0)) \quad \text{for any } w \in \left(\tfrac{p}{p-1}, \left(\tfrac{p}{p-1}\right)^{**}\right).$$

Therefore, by (σ_1) it follows

$$\lim_{n\to+\infty} \int_{B_r(0)} \sigma(x)|u_n|^w\, dx = \int_{B_r(0)} \sigma(x)|u|^w\, dx \tag{2.22}$$

for any $w \in \left(\frac{p}{p-1}, \left(\frac{p}{p-1}\right)^{**}\right)$. Then, from (2.20) and (2.21) for $\varepsilon > 0$ small enough and (2.22) we deduce

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \sigma(x)|u_n|^w\, dx = \int_{\mathbb{R}^N} \sigma(x)|u|^w\, dx$$

from which we conclude that

$$u_n \to u \quad \text{in } L^w_\sigma(\mathbb{R}^N), \text{ for every } w \in \left(\tfrac{p}{p-1}, \left(\tfrac{p}{p-1}\right)^{**}\right).$$

If $(\rho\sigma_2)$ holds, by (2.9) we get for any $\varepsilon > 0$ the existence of a radius $r > 0$ sufficiently large such that

$$\int_{B^c_r(0)} \sigma(x)|u|^m\, dx \le \varepsilon\, C_m^{-1} \left(\int_{B^c_r(0)} \rho(x)|u|^{\frac{p}{p-1}}\, dx + \int_{B^c_r(0)} |u|^{\left(\frac{p}{p-1}\right)^{**}}\, dx\right)$$

for all $u \in \mathcal{D}^{2,\frac{p}{p-1}}_\rho(\mathbb{R}^N)$. If $\{u_n\}$ is a sequence such that $u_n \rightharpoonup u$ in $\mathcal{D}^{2,\frac{p}{p-1}}_\rho(\mathbb{R}^N)$ there exists $C_1 > 0$ such that inequalities in (2.17) hold therefore

$$\int_{B^c_r(0)} \sigma(x)|u_n|^m\, dx \le 2\,\varepsilon\, C_1\, C_m^{-1} \quad \text{for all } n \in \mathbb{N} \tag{2.23}$$

and by Fatou' Lemma it is

$$\int_{B_r^c(0)} \sigma(x)|u_n|^m\,dx \le 2\,\varepsilon\, C_1\, C_m^{-1} \quad \text{for all } n \in \mathbb{N}. \tag{2.24}$$

Reasoning similarly as above in order to get (2.22), since $m \in \left(\frac{p}{p-1}, \left(\frac{p}{p-1}\right)^{**}\right)$ we obtain

$$\lim_{n\to+\infty} \int_{B_r(0)} \sigma(x)|u_n|^m\,dx = \int_{B_r(0)} \sigma(x)|u|^m\,dx. \tag{2.25}$$

Then, from (2.23) and (2.24) for $\varepsilon > 0$ small enough and (2.25) it follows

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \sigma(x)|u_n|^m\,dx = \int_{\mathbb{R}^N} \sigma(x)|u|^m\,dx$$

and we obtain

$$u_n \to u \quad \text{in } L^m_\sigma(\mathbb{R}^N).$$

□

Thanks to Proposition 2, we can prove the following compactness result related to $dI_{\sigma,F}$.

Proposition 3. *Suppose that* (ρ_1), (σ_1), (σ_2), $(\rho\sigma_1)$ *(resp.* $(\rho\sigma_2)$*) and* (f_1) *(resp.* (f_2)*) and* (f_3) *hold. Then, if* $\{u_n\}$ *is a sequence such that* $u_n \rightharpoonup u$ *in* $\mathcal{D}^{2,\frac{p}{p-1}}_\rho(\mathbb{R}^N)$*, we get*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \sigma(x)f(u_n)v\,dx = \int_{\mathbb{R}^N} \sigma(x)f(u)v\,dx \quad \textit{for every } v \in \mathcal{D}^{2,\frac{p}{p-1}}_\rho(\mathbb{R}^N);$$

consequently,

$$dI_{\sigma,F} \textit{ is compact from } \mathcal{D}^{2,\frac{p}{p-1}}_\rho(\mathbb{R}^N) \textit{ in } \mathcal{D}^{-2,\frac{p}{p-1}}_\rho(\mathbb{R}^N).$$

Proof. Let $\{u_n\}$ be a sequence such that $u_n \rightharpoonup u$ in $\mathcal{D}^{2,\frac{p}{p-1}}_\rho(\mathbb{R}^N)$. Fixed an arbitrary radius $r > 0$ we aim to prove that for any $v \in \mathcal{D}^{2,\frac{p}{p-1}}_\rho(\mathbb{R}^N)$

$$\lim_{n\to+\infty} \int_{B_r(0)} \sigma(x)f(u_n)v\,dx = \int_{B_r(0)} \sigma(x)f(u)v\,dx \tag{2.26}$$

and

$$\lim_{n\to+\infty} \int_{B_r^c(0)} \sigma(x)f(u_n)v\,dx = \int_{B_r^c(0)} \sigma(x)f(u)v\,dx. \tag{2.27}$$

Let's start by showing (2.26). Denoted by $\Theta_n = \{x \in \mathbb{R}^N : |u_n(x)| \le 1\}$ (resp. $\Theta = \{x \in \mathbb{R}^N : |u(x)| \le 1\}$) and χ_{Θ_n} (resp. χ_Θ) its characteristic function, we first prove that

$$\lim_{n\to+\infty} \int_{B_r(0)} \sigma(x)f(u_n)\chi_{\Theta_n}v\,dx = \int_{B_r(0)} \sigma(x)f(u)\chi_\Theta v\,dx. \tag{2.28}$$

Suppose $(\rho\sigma_1)$ holds and f satisfies (f_1). Since $\sigma(x)^{\frac{p-1}{p}} \le C_{\sigma,\rho}^{\frac{p-1}{p}}\,\rho(x)^{\frac{p-1}{p}}$, we get easily $\sigma(x)^{\frac{p-1}{p}}v \in L^{\frac{p}{p-1}}(B_r(0))$ and by the following relation

$$\left|\sigma(x)^{\frac{1}{p}} f(u_n(x))\chi_{\Theta_n}\right|^p \le C_{\sigma,\rho}\, C\, \rho(x)|u_n(x)|^{\frac{p}{p-1}}$$

we deduce the sequence $\{\sigma(x)^{\frac{1}{p}} f(u_n(x))\chi_{\Theta_n}\}$ is bounded in $L^p(B_r(0))$. Then, since

$$\int_{B_r(0)} \sigma(x) f(u_n)\chi_{\Theta_n} v\, dx = \int_{B_r(0)} \left(\sigma(x)^{\frac{1}{p}} f(u_n)\chi_{\Theta_n}\right)\left(\sigma(x)^{\frac{p-1}{p}} v\right) dx,$$

by pointwise convergence we get (2.28). Now, assume $(\rho\sigma_2)$ holds and f satisfies (f_2). By Proposition 2 (ii) we have $\sigma(x)^{\frac{1}{m}} v \in L^m(B_r(0))$ for any $v \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(B_r(0))$ and the sequence $\{\sigma(x)^{\frac{m-1}{m}} f(u_n(x))\chi_{\Theta_n}\}$ is bounded in $L^{\frac{m}{m-1}}(B_r(0))$ since

$$\left|\sigma(x)^{\frac{m-1}{m}} f(u_n(x))\chi_{\Theta_n}\right|^{\frac{m}{m-1}} \le C\,\sigma(x)\,|u_n(x)|^m.$$

Then, pointwise convergence together with

$$\int_{B_r(0)} \sigma(x) f(u_n)\chi_{\Theta_n} v\, dx = \int_{B_r(0)} \left(\sigma(x)^{\frac{m-1}{m}} f(u_n)\chi_{\Theta_n}\right)\left(\sigma(x)^{\frac{1}{m}} v\right) dx$$

allow us to show (2.28) also in this second case. Finally, it remains to prove that

$$\lim_{n\to+\infty} \int_{B_r(0)} \sigma(x) f(u_n)\chi_{\Theta_n^c} v\, dx = \int_{B_r(0)} \sigma(x) f(u)\chi_{\Theta^c} v\, dx \tag{2.29}$$

where Θ_n^c (resp. Θ^c) denotes the complement of Θ_n (resp. Θ) and $\chi_{\Theta_n^c}$ (resp. $\chi_{\Theta_n^c}$) its characteristic function. In both cases, by (f_3) the sequence $\{\sigma(x) f(u_n(x))\chi_{\Theta_n^c}\}$ is bounded in $L^{\frac{\left(\frac{p}{p-1}\right)^{**}}{\left(\frac{p}{p-1}\right)^{**}-1}}(B_r(0))$ since

$$\left|\sigma(x) f(u_n(x))\chi_{\Theta_n^c}\right|^{\frac{\left(\frac{p}{p-1}\right)^{**}}{\left(\frac{p}{p-1}\right)^{**}-1}} \le C_\sigma^{\frac{\left(\frac{p}{p-1}\right)^{**}}{\left(\frac{p}{p-1}\right)^{**}-1}} C_1 |u_n(x)|^{\left(\frac{p}{p-1}\right)^{**}},$$

therefore (2.29) follows by pointwise convergence and since $v \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(B_r(0))$. By (2.28) and (2.29) we conclude (2.26) is satisfied.

At this point we prove (2.27) by distinguishing here also two cases. In the first case, assume that $(\rho\sigma_1)$ holds. From (f_1) and (f_3), fixing $w \in \left(\frac{p}{p-1}, \left(\frac{p}{p-1}\right)^{**}\right)$ and taking $\varepsilon > 0$, there exists $C > 0$ such that

$$\sigma(x)|f(s)| \le \varepsilon\, C\left(\rho(x)|s|^{\frac{1}{p-1}} + |s|^{\left(\frac{p}{p-1}\right)^{**}-1}\right) + C\,\sigma(x)|s|^{w-1} \tag{2.30}$$

$$\text{for all } s \in \mathbb{R} \text{ and } x \in \mathbb{R}^N.$$

Indeed, given $\varepsilon > 0$, from (f_1) and $(\rho\sigma_1)$ there exists $s_0 > 0$ such that

$$\sigma(x)|f(s)| \le \varepsilon\, C_{\sigma,\rho}\, \rho(x)|s|^{\frac{1}{p-1}} \quad \text{for all } s \in \mathbb{R},\ |s| \le s_0 \text{ and } x \in \mathbb{R}^N.$$

Moreover, from (σ_1) and by (f_3) we can find $s_1 > 0$ such that

$$\sigma(x)|f(s)| \le \varepsilon\, C_\sigma\, |s|^{\left(\frac{p}{p-1}\right)^{**}-1} \quad \text{for all } s \in \mathbb{R},\ |s| \ge s_1 \text{ and } x \in \mathbb{R}^N.$$

Since the functions involved are continuous we get the existence of a constant $c > 0$ such that

$$\sigma(x)|f(s)| \le C\,\sigma(x)\,|s|^{w-1} \quad \text{for all } s \in \mathbb{R},\ s_0 \le |s| \le s_1 \text{ and } x \in \mathbb{R}^N.$$

Therefore, for any fixed $w \in \left(\frac{p}{p-1}, \left(\frac{p}{p-1}\right)^{**}\right)$, in correspondence of $\varepsilon > 0$ there exists a constant $C > 0$ such that (2.30) holds.

Now, since $\{u_n\}$ is a sequence such that $u_n \rightharpoonup u$ in $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$, for any radius $r > 0$ we get by Hölder inequality, continuous embeddings in (2.1), Proposition 2 (i), (2.5) and (2.6) the following

$$\begin{aligned}\int_{B_r^c(0)} \sigma(x)|f(u_n)|\,|v|\,dx &\leq \varepsilon\, C \left(\int_{B_r^c(0)} \rho(x)|u_n|^{\frac{1}{p-1}}|v|\,dx + \int_{B_r^c(0)} |u_n|^{\left(\frac{p}{p-1}\right)^{**}-1}|v|\,dx\right) \\ &\quad + C\int_{B_r^c(0)} \sigma(x)|u_n|^{w-1}|v|\,dx \\ &\leq \varepsilon\, C \left(|u_n|^{\frac{1}{p-1}}_{\frac{p}{p-1},\rho}\, |v|_{\frac{p}{p-1},\rho} + |u_n|^{\left(\frac{p}{p-1}\right)^{**}-1}_{\left(\frac{p}{p-1}\right)^{**}}\, |v|_{\left(\frac{p}{p-1}\right)^{**}}\right) \\ &\quad + C\int_{B_r^c(0)} (\sigma(x))^{\frac{w-1}{w}}|u_n|^{w-1}\,(\sigma(x))^{\frac{1}{w}}|v|\,dx \\ &\leq \varepsilon\, C_1 \left(|u_n|^{\frac{1}{p-1}}_{\frac{p}{p-1},\rho} + |u|^{\left(\frac{p}{p-1}\right)^{**}-1}_{\left(\frac{p}{p-1}\right)^{**}}\right) \|v\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} \\ &\quad + C_1 \left(\int_{B_r^c(0)} \sigma(x)|u_n|^w\,dx\right)^{\frac{w-1}{w}} \|v\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}}\end{aligned}$$

for any $v \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$. Denoted by

$$R(u_n) = C_1 \left(|u_n|^{\frac{1}{p-1}}_{\frac{p}{p-1},\rho} + |u|^{\left(\frac{p}{p-1}\right)^{**}-1}_{\left(\frac{p}{p-1}\right)^{**}}\right),$$

it is not difficult to observe that by (2.1) we get $R(u_n)$ is bounded from above by a positive constant C_2 for all $n \in \mathbb{N}$ while since Proposition 2 (i) implies

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \sigma(x)|u_n|^w\,dx = \int_{\mathbb{R}^N} \sigma(x)|u|^w\,dx,$$

there exists a radius $r_1 > 0$ such that

$$\left(\int_{B_{r_1}^c(0)} \sigma(x)|u_n|^w\,dx\right)^{\frac{w-1}{w}} \leq \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Then,

$$\int_{B_{r_1}^c(0)} \sigma(x)|f(u_n)|\,|v|\,dx \leq \varepsilon\, C_1\,(C_2+1)\,\|v\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} \quad \text{for all } n \in \mathbb{N}$$

and choosen a radius $r_2 \geq r_1 >$ by Fatou's Lemma it is also

$$\int_{B_{r_2}^c(0)} \sigma(x)|f(u)|\,|v|\,dx \leq \varepsilon\, C_1\,(C_2+1)\,\|v\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} \quad \text{for all } n \in \mathbb{N}.$$

Consequently, for a suitable choise of the radius $r > 0$ we conclude (2.27) holds.
Now we prove the second case; indeed, we assume $(\rho\sigma_2)$ holds. From (2.14), given $\varepsilon > 0$ there exists $r > 0$ such that

$$\sigma(x)|f(s)| \leq \varepsilon\, C\left(\rho(x)|f(s)||s|^{\frac{p}{p-1}-m} + |f(s)||s|^{\left(\frac{p}{p-1}\right)^{**}-m}\right) \quad \text{for all } s \in \mathbb{R} \text{ and } |x| \geq r.$$

By (f_2), for any $\varepsilon > 0$ there exists $s_0 > 0$ such that

$$|f(s)| \leq \varepsilon |s|^{m-1} \quad \text{for all } s \in \mathbb{R},\ |s| \leq s_0$$

then,

$$\sigma(x)|f(s)| \leq \varepsilon\, C\left(\rho(x)|s|^{\frac{1}{p-1}} + |s|^{\left(\frac{p}{p-1}\right)^{**}-1}\right) \text{ for all } s \in \mathbb{R},\ |s| \leq s_0 \text{ and } |x| \geq r.$$

Moreover, from (f_3) and (σ_1) for any given $\varepsilon > 0$ there exists $s_1 > s_0 > 0$ such that

$$\frac{\sigma(x)|f(s)|}{\rho(x)|s|^{\frac{1}{p-1}} + |s|^{\left(\frac{p}{p-1}\right)^{**}-1}} \leq C_\sigma\, \frac{|f(s)|}{|s|^{\left(\frac{p}{p-1}\right)^{**}-1}} \leq \varepsilon\, C_\sigma \quad \text{for all } s \in \mathbb{R},\ |s| \geq s_1$$

which implies

$$\sigma(x)|f(s)| \leq \varepsilon\, C\left(\rho(x)|s|^{\frac{1}{p-1}} + |s|^{\left(\frac{p}{p-1}\right)^{**}-1}\right) \text{ for all } s \in \mathbb{R},\ |s| \geq s_1 \text{ and } x \in \mathbb{R}^N.$$

Therefore, there exist $C > 0$ and $s_1 > s_0 > 0$ satisfying

$$\sigma(x)|f(s)| \leq \varepsilon\, C\left(\rho(x)|s|^{\frac{1}{p-1}} + |s|^{\left(\frac{p}{p-1}\right)^{**}-1}\right) \quad \text{for every } s \in \Omega^c \text{ and } |x| \geq r \tag{2.31}$$

where $\Omega^c = \{s \in \mathbb{R} : |s| < s_0 \quad \text{or} \quad |s| > s_1\}$.
In correspondence of the radius $r > 0$ by (2.31) and (1.8) in Remark 2 we can write

$$\begin{aligned}
\int_{B_r^c(0)} \sigma(x)|f(u_n)||v|\,dx &= \int_{B_r^c(0)\,\cap\,\Omega_n^c} \sigma(x)|f(u_n)||v|\,dx + \int_{B_r^c(0)\,\cap\,\Omega_n} \sigma(x)|f(u_n)||v|\,dx \\
&\leq \varepsilon\, C \int_{B_r^c(0)\,\cap\,\Omega_n^c} \left(\rho(x)|u_n|^{\frac{1}{p-1}} + |u_n|^{\left(\frac{p}{p-1}\right)^{**}-1}\right)|v|\,dx \\
&\quad + \varepsilon \int_{B_r^c(0)\,\cap\,\Omega_n} \sigma(x)|u_n|^{m-1}|v|\,dx + C_\varepsilon \int_{B_r^c(0)\,\cap\,\Omega_n} \sigma(x)|u_n|^{\left(\frac{p}{p-1}\right)^{**}-1}|v|\,dx
\end{aligned}$$

where we denote by $\Omega_n = \left\{x \in \mathbb{R}^N : s_0 \leq |u_n(x)| < s_1\right\}$ and its complement by $\Omega_n^c = \left\{x \in \mathbb{R}^N : |u_n(x)| < s_0 \quad \text{or} \quad |u_n(x)| > s_1\right\}$. So, reasoning similarly as in the previous step, by Hölder inequality and the definition of $R(u_n)$ we obtain

$$\begin{aligned}
\int_{B_r^c(0)} \sigma(x)|f(u_n)||v|\,dx &\leq \varepsilon\, C\, R(u_n)\, \|v\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} \\
&\quad + \varepsilon\, s_1^{m-1}\, c\left(\int_{B_r^c(0)\,\cap\,\Omega_n} \sigma(x)\,dx\right)^{\frac{m-1}{m}} \|v\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} + C_\varepsilon\, C_\sigma\, |u_n|_{\left(\frac{p}{p-1}\right)^{**}}^{\left(\frac{p}{p-1}\right)^{**}-1} |v|_{\left(\frac{p}{p-1}\right)^{**}}
\end{aligned}$$

and in particular

$$\begin{aligned}
\int_{B_r^c(0)} \sigma(x)|f(u_n)||v|\,dx &\leq (\varepsilon\, C + C_\varepsilon\, C_\sigma\, c)\, R(u_n)\, \|v\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} \\
&\qquad + \varepsilon\, s_1^{m-1}\, c\left(\int_{B_r^c(0)\,\cap\,\Omega_n} \sigma(x)\,dx\right)^{\frac{m-1}{m}} \|v\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}}.
\end{aligned}$$

By the boundedness of $R(u_n)$ for all $n \in \mathbb{N}$ and the fact that $\sup_{n\in\mathbb{N}} |\operatorname{meas}(\Omega_n)| < +\infty$ which allows us to apply (σ_2), we can exploit some arguments as in the proof of Proposition 2 thus obtaining (2.27). □

3. Proof of Theorem 1

At this point we prove that I satisfies the Mountain Pass geometry (see [2]).

Lemma 2. *Under assumptions* (ρ_1), (σ_1), $(\rho\sigma_1)$ *(resp.* $(\rho\sigma_2)$*),* (f_1) *(resp.* (f_2)*),* (f_3) *and* (f_4)*, the functional* I *has a mountain pass geometry, that is*

- (I_0) $I(0) = 0$;
- (I_{δ_0}) *there exist* $\rho_0, \delta_0 > 0$ *such that* $I(u) \geq \delta_0$ *for all* $u \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ *with* $\|u\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} = \rho_0$;
- (I_{u_0}) *there exists* $u_0 \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ *such that* $\|u_0\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} > \rho_0$ *and* $I(u_0) \leq 0$.

Proof. (I_0) By (1.7) (resp. (1.9)) in Remark 2 we get easily $I(0) = 0$.
(I_{δ_0}) First suppose that $(\rho\sigma_1)$ holds. Then, by (σ_1), (1.7) in Remark 2 and (2.1) in correspondence of $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} I(u) &= \frac{p-1}{p}\,\|u\|^{\frac{p}{p-1}}_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} - \int_{\mathbb{R}^N} \sigma(x) F(u)\,dx \\ &\geq \frac{p-1}{p}\,\|u\|^{\frac{p}{p-1}}_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} - C_{\sigma,\rho}\,\varepsilon\,\frac{(p-1)}{p}|u|^{\frac{p}{(p-1)}}_{\frac{p}{(p-1)},\rho} - C_\sigma\, C_\varepsilon\,\frac{1}{\left(\frac{p}{p-1}\right)^{**}}|u|^{\left(\frac{p}{p-1}\right)^{**}}_{\left(\frac{p}{p-1}\right)^{**}} \\ &\geq \frac{p-1}{p}\,(1-\varepsilon\, C_{\sigma,\rho}\, c)\,\|u\|^{\frac{p}{p-1}}_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} - C_\sigma\, C_\varepsilon\,\frac{1}{\left(\frac{p}{p-1}\right)^{**}}\, c_1\,\|u\|^{\left(\frac{p}{p-1}\right)^{**}}_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} \end{aligned}$$

for every $u \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$. Since $\frac{p}{p-1} < \left(\frac{p}{p-1}\right)^{**}$, taken $\|u\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} = \rho$ with ρ and $\varepsilon > 0$ small enough, we get (I_{δ_0}) holds for a suitable δ_0.
Now, suppose that $(\rho\sigma_2)$ holds. Then, by (σ_1), (1.9) in Remark 2 and choosen a radius $r > 0$, in correspondence of $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that (2.7) holds and by substituting (2.8) and (2.10) in (2.7) we get (2.11) then

$$\begin{aligned} I(u) &\geq \frac{p-1}{p}\,\|u\|^{\frac{p}{p-1}}_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} - \frac{\varepsilon}{m}\left(C_\sigma\, c\,|u|^m_{\left(\frac{p}{p-1}\right)^{**}} + \varepsilon\, C_m^{-1}\left(|u|^{\frac{p}{p-1}}_{\frac{p}{p-1},\rho} + |u|^{\left(\frac{p}{p-1}\right)^{**}}_{\left(\frac{p}{p-1}\right)^{**}}\right)\right) \\ &\qquad - \frac{C_\varepsilon\, C_\sigma}{\left(\frac{p}{p-1}\right)^{**}}|u|^{\left(\frac{p}{p-1}\right)^{**}}_{\left(\frac{p}{p-1}\right)^{**}} \\ &\geq \left(\frac{p-1}{p} - \frac{\varepsilon^2}{m}\, C_m^{-1}\, C_1\right)\|u\|^{\frac{p}{p-1}}_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} - \frac{\varepsilon}{m}\, C_\sigma\, C_2\,\|u\|^m_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} \\ &\qquad - \left(\frac{\varepsilon^2}{m}\, C_m^{-1}\, C_3 - \frac{C_\varepsilon\, C_\sigma}{\left(\frac{p}{p-1}\right)^{**}}\, C_4\right)\|u\|^{\left(\frac{p}{p-1}\right)^{**}}_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} \end{aligned}$$

where in the last inequality we have exploited (2.1). Since $\frac{p}{p-1} < m < \left(\frac{p}{p-1}\right)^{**}$, taken $\|u\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} = \rho$ with ρ and $\varepsilon > 0$ small enough, we get (I_{δ_0}) holds again for a suitable δ_0.
(I_{u_0}) Let us fix $\overline{v} \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N) \cap C_c(\mathbb{R}^N)$ with $\overline{v} \neq 0$; thus, taken $t > 0$, by (1.11) in Remark

3 and $\mu > \frac{p}{p-1}$ we get

$$I(t\,\overline{v}) \le \frac{p-1}{p}\, t^{\frac{p}{p-1}}\, \|\overline{v}\|^{\frac{p}{p-1}}_{\mathcal{D}^{2,\frac{p}{p-1}}_{\rho}} - C\, t^{\mu} \int_{\mathrm{supp}(\overline{v})} \sigma(x)|\overline{v}|^{\mu}\, dx + C_1 \int_{\mathrm{supp}(\overline{v})} \sigma(x)\, dx.$$

Let us observe that by (σ_1) it is

$$\int_{\mathrm{supp}(\overline{v})} \sigma(x)\, dx \le C_{\sigma}\ \mathrm{meas}(\mathrm{supp}(\overline{v})) < +\infty;$$

moreover, since $\mu < \left(\frac{p}{p-1}\right)^{**}$, by exploiting $L^{\left(\frac{p}{p-1}\right)^{**}}(\mathrm{supp}(\overline{v})) \hookrightarrow L^{\mu}(\mathrm{supp}(\overline{v}))$ and (2.1) we get

$$\int_{\mathrm{supp}(\overline{v})} \sigma(x)|\overline{v}|^{\mu}\, dx \le C_{\sigma}\, c \left(\int_{\mathrm{supp}(\overline{v})} |\overline{v}|^{\left(\frac{p}{p-1}\right)^{**}}\, dx \right)^{\frac{\mu}{\left(\frac{p}{p-1}\right)^{**}}} \le C_{\sigma}\, C\, \|u\|^{\mu}_{\mathcal{D}^{2,\frac{p}{p-1}}_{\rho}} < +\infty.$$

Therefore, as $t \to +\infty$ it follows $I(t\,\overline{v}) \to -\infty$ and, taken $u_0 = t\,\overline{v}$ with t sufficiently large we conclude that (I_{u_0}) is satisfied. □

Indeed, we are able to prove that the functional I satisfies the following lemma.

Lemma 3. *Under assumptions (ρ_1), (σ_1), $(\rho\sigma_1)$ (resp. $(\rho\sigma_2)$), (f_1) (resp. (f_2)), (f_3) and (f_4), the functional I satisfies*

$(I'_{\mathcal{U}_0})$ *for every finite-dimensional subspace $\mathcal{U}_0 \subset \mathcal{D}^{2,\frac{p}{p-1}}_{\rho}(\mathbb{R}^N)$ there esists a $R = R(\mathcal{U}_0)$ such that $I(u) \le 0$ for every $u \in \mathcal{U}_0 \setminus B_{R(\mathcal{U}_0)}$.*

Proof. Now, fix $u \in \mathcal{D}^{2,\frac{p}{p-1}}_{\rho}(\mathbb{R}^N)$ and $s_0 > 0$ with $|s_0| \le 1$. Denote $\Omega_{u,s_0} = \{x \in \mathbb{R}^N : |u(x)| \ge s_0\}$. By (σ_1), (1.10) in Remark 3 and $\mu > \frac{p}{p-1}$, we have that

$$\begin{aligned}
I(u) &\le \frac{p-1}{p}\|u\|^{\frac{p}{p-1}}_{\mathcal{D}^{2,\frac{p}{p-1}}_{\rho}} - C \int_{\Omega_{u,s_0}} \sigma(x)|u|^{\mu}\, dx \\
&= \frac{p-1}{p}\|u\|^{\frac{p}{p-1}}_{\mathcal{D}^{2,\frac{p}{p-1}}_{\rho}} - C \int_{\mathbb{R}^N} \sigma(x)|u|^{\mu}\, dx + C \int_{\mathbb{R}^N\setminus\Omega_{u,s_0}} \sigma(x)|u|^{\mu}\, dx \\
&\le \frac{p-1}{p}\|u\|^{\frac{p}{p-1}}_{\mathcal{D}^{2,\frac{p}{p-1}}_{\rho}} - C \int_{\mathbb{R}^N} \sigma(x)|u|^{\mu}\, dx + C\, C_{\sigma} \int_{\mathbb{R}^N\setminus\Omega_{u,s_0}} |u|^{\mu}\, dx \\
&\le \frac{p-1}{p}\|u\|^{\frac{p}{p-1}}_{\mathcal{D}^{2,\frac{p}{p-1}}_{\rho}} - C \int_{\mathbb{R}^N} \sigma(x)|u|^{\mu}\, dx + C\, C_{\sigma} \int_{\mathbb{R}^N\setminus\Omega_{u,s_0}} |u|^{\frac{p}{p-1}}\, dx \\
&\le \frac{p-1}{p}\|u\|^{\frac{p}{p-1}}_{\mathcal{D}^{2,\frac{p}{p-1}}_{\rho}} - C \int_{\mathbb{R}^N} \sigma(x)|u|^{\mu}\, dx + C\, C_{\sigma}\, |u|^{\frac{p}{p-1}}_{\frac{p}{p-1}}.
\end{aligned}$$

Then, by (2.1) and (ρ_1) it follows

$$I(u) \le \left(\frac{p-1}{p} + C\, C_{\sigma}\, c\right) \|u\|^{\frac{p}{p-1}}_{\mathcal{D}^{2,\frac{p}{p-1}}_{\rho}} - C \int_{\mathbb{R}^N} \sigma(x)|u|^{\mu}\, dx. \tag{3.1}$$

Let $\mathcal{U}_0$ be a finite dimensional subspace of $\mathcal{D}^{2,\frac{p}{p-1}}_{\rho}(\mathbb{R}^N)$. Clearly the term $\left(\int_{\mathbb{R}^N} \sigma(x)|u|^{\mu}\, dx\right)^{\frac{1}{\mu}}$ is a norm in $\mathcal{D}^{2,\frac{p}{p-1}}_{\rho}(\mathbb{R}^N)$, hence by (3.1) and the equivalence of all norms in $\mathcal{U}_0$, there exists a positive constant $R = R(\mathcal{U}_0)$ such that

$$I(u) \le 0 \quad \text{if } u \in \mathcal{U}_0,\ \|u\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} \ge R$$

and $(I'_{\mathcal{U}_0})$ is proved. □

Since by Lemma 2 the functional I has the Mountain pass geometry, hence there exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ at level c_{MP}, namely such that

$$I(u_n) \to c_{MP} \tag{3.2}$$

and

$$dI(u_n) \to 0 \quad \text{in } \mathcal{D}_\rho^{-2,\frac{p}{p-1}}(\mathbb{R}^N), \tag{3.3}$$

where c_{MP} is the minimax level of the Mountain Pass Theorem (see [2]) applied to I, that is

$$c_{MP} = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} I(\eta(t))$$

with

$$\Gamma = \left\{\eta \in C([0,1], \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)) :\ \eta(0) = 0 \quad \text{and} \quad I(\eta(t)) \le 0\right\}.$$

At this point, by exploiting Proposition 3, we can prove that I satisfies (PS) condition at level c_{MP}.

Lemma 4. *Suppose that ρ and σ satisfy (ρ_1), (σ_1), (σ_2), $(\rho\sigma_1)$ (resp. $(\rho\sigma_2)$) and f verifies (f_1) (resp. (f_2)), (f_3) and (f_4). Then, every Palais-Smale sequence $\{u_n\}$ in $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ at level c_{MP} converges in $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$, up to subsequences.*

Proof. Let $\{u_n\}$ be a (PS) sequence at level c_{MP}, namely satisfying (3.2) and (3.3). Then by (2.2), (σ_1) and (f_4) it follows

$$\begin{aligned} c\,(1 + \|u_n\|_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}}) &\ge \mu I(u_n) - dI(u_n)[u_n] \\ &= \left(\mu\frac{p-1}{p} - 1\right) \|u_n\|^{\frac{p}{p-1}}_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}} + \int_{\mathbb{R}^N} \sigma(x)\ (f(x,u_n)u_n - \mu F(x,u_n))\,dx \\ &\ge \left(\mu\frac{p-1}{p} - 1\right) \|u_n\|^{\frac{p}{p-1}}_{\mathcal{D}_\rho^{2,\frac{p}{p-1}}}, \end{aligned}$$

for a suitable positive constant c. Hence, $\{u_n\}$ is bounded in $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N))$. So, there exists $u \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N))$ such that, up to subsequences, $u_n \rightharpoonup u$ weakly in $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ and, from Proposition 3,

$$dI_{\sigma,F}(u_n) \to dI_{\sigma,F}(u) \text{ in } \mathcal{D}_\rho^{-2,\frac{p}{p-1}}(\mathbb{R}^N). \tag{3.4}$$

Thus, by (3.3) we get

$$\begin{aligned} o_n(1) &= (dI(u_n) - dI(u))[u_n - u] \\ &= (dI_\rho(u_n) - dI_\rho(u))[u_n - u] - (dI_{\sigma,F}(u_n) - dI_{\sigma,F}(u))[u_n - u] \end{aligned}$$

as $n \to +\infty$ which together with (3.4) implies

$$(dI_\rho(u_n) - dI_\rho(u))[u_n - u] \to 0 \quad \text{if } n \to +\infty,$$

i.e.,

$$\int_{\mathbb{R}^N}((-\Delta u_n)^{\frac{1}{p-1}} - (-\Delta u)^{\frac{1}{p-1}})(-\Delta u_n - (-\Delta u))dx + \int_{\mathbb{R}^N}\rho(x)(u_n^{\frac{1}{p-1}} - u^{\frac{1}{p-1}})(u_n - u)dx \to 0. \tag{3.5}$$

Now, let us remark that, as $\frac{1}{p-1} \geq 1$, it is not difficult to prove that a constant $C > 0$ exists such that

$$(s^{\frac{1}{p-1}} - t^{\frac{1}{p-1}})\ (s-t)\ \geq\ C|s-t|^{\frac{p}{p-1}} \quad \text{for all } s,\, t \in \mathbb{R};$$

whence the conclusion follows from (3.5) and (ρ_1). □

At this point, we can prove Theorem 1.

Proof of Theorem 1. By Proposition 1 we get the functional $I \in C^1(\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N))$ and by Lemma 2 it has the Mountain Pass geometry. Moreover, Lemma 4 implies that I satisfies the Palais–Smale condition in $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$. Whence, the classical Mountain Pass Theorem applies (see [2, Theorem 2.1]) and a critical point $\overline{u}$ of I in $\mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ hence a weak solution $\overline{u} \in \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ of problem (1.3) exists.
Furthermore, if also condition (f_5) holds, the functional I is even. Then by (I_0) and (I_{δ_0}) in Lemma 2, Lemma 3 and Lemma 4 we get the hypotheses of the Symmetric version of Mountain Pass Theorem are satisfied (see [2, Corollary 2.9]) and I has an unbounded sequence $\{\overline{u}_n\} \subset \mathcal{D}_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ of critical points. □

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