

## SINGULAR PERTURBATIONS OF FORWARD-BACKWARD $p$ -PARABOLIC EQUATIONS

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ABSTRACT. In this paper we have proved the existence of entropy measure-valued solutions to forward-backward  $p$ -parabolic equations. We have obtained these solutions as singular limits of weak solutions to  $(p, q)$ -elliptic regularized boundary-value problems as  $\varepsilon \rightarrow 0+$ . When  $q > 1$  and  $q \neq 2$  we have not defined yet admissible initial and final conditions even in the form of integral inequalities.

*Dedicated to Professor David Kinderlehrer on the occasion of his 75th birthday.*

### 1. INTRODUCTION

In this paper we deal with entropy measure-valued solutions to forward-backward  $p$ -parabolic equations which are obtained as singular limits of weak solutions to elliptic boundary value problems. Singular limits of weak solutions to anisotropic elliptic boundary value problems were studied in [1], [2], [3]. The main difference with the present case is that first order terms were not involved there.

It is worth to mention that the presence of  $p$ -Laplacian can make the problem very complicated. For example, gradient Young measures occur in representation of a solution to  $p(x, t)$ -wave equation [4],  $p(x, t) > 1$ , nonlinear evolution equation [5] and forward backward-parabolic equations [6], [7].

In this paper we deal with singular limits of weak solutions to elliptic boundary value problem  $\Pi_{p,q,\varepsilon}$  with anisotropic  $(p, q)$ -Laplacian and quasi-linear first order terms where  $p, q > 1$ . In the limit case as  $\varepsilon \rightarrow +0$  we formally obtain forward-backward  $p$ -parabolic equations. In two recent papers [8], [9] two cases ( $p = 2, q = 2$ ) and, correspondingly, ( $p > 1, q = 2$ ) were analyzed. In the first case the existence and the uniqueness of entropy solutions were proved. In the second case we have only constructed entropy measure-valued solutions with the help of methods invented in [10]–[16].

It is known that in the hyperbolic case there is the equivalence of entropy solutions for different types of approximation [17], [18] only for the Cauchy problem, but nothing is known how  $q$ -Laplacian influences on vanishing viscosity method applied to the initial boundary value problem.

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In comparison with [9], when  $q > 1$  and  $q \neq 2$  we have not formulated entropy boundary conditions in the form of integral inequalities. Therefore, it is still an open question.

This paper is organized as follows. Section 2 is devoted to auxiliary functions and definitions. In section 3 we have formulated problem  $\Pi_{p,q,\varepsilon}$ . Here we have defined an entropy measure-valued solution obtained as a singular limit of weak solutions  $u_{p,q,\varepsilon}$  as  $\varepsilon \rightarrow 0+$ . In sections 4 and 5 we have proved results announced in section 3.

2. AUXILIARY DEFINITIONS AND FUNCTIONS

2.1. Boundary entropy-entropy flux triples.

**Definition 1.** A triple of functions  $(H, Q, \Phi)$  is called a boundary entropy-entropy flux triple if  $H, Q \in C^2(\mathbb{R}^2)$ ,  $\Phi \in C^2(\mathbb{R}^2; \mathbb{R}^d)$  and for any  $(z, k) \in \mathbb{R}^2$ :

$$\begin{aligned} \partial_1 Q(z, k) &= a'(z)\partial_1 H(z, k), \quad \partial_1 \Phi_j(z, k) = \varphi'_j(z)\partial_1 H(z, k), \quad \partial_1^2 H(z, k) \geq 0, \quad Q(z, z) = \\ \partial_1 Q(z, z) &= \Phi_j(z, z) = \partial_1 \Phi_j(z, z) = H(z, z) = \partial_1 H(z, z) = 0, \quad j = 1, \dots, d, \quad \Phi(z, k) = \\ &= (\Phi_1(z, k), \dots, \Phi_d(z, k)), \end{aligned}$$

where  $\partial_1$  means differentiation with respect to the first variable.

Note that function  $a$  and vector function  $\varphi = (\varphi_1, \dots, \varphi_d)$  are defined by Condition 1 formulated in subsection 3.1.

**Example 1.** We consider the following class of boundary entropy-entropy flux triples  $(H_\delta, Q_\delta, \Phi_\delta)$ :

$$\begin{aligned} H_\delta(z, k) &= \sqrt{(z - k)^2 + \delta^2} - \delta, \\ Q_\delta(z, k) &= \int_k^z a'(\lambda)\partial_1 H_\delta(\lambda, k) d\lambda, \\ \Phi_\delta(z, k) &= \int_k^z \varphi'(\lambda)\partial_1 H_\delta(\lambda, k) d\lambda. \end{aligned}$$

Passing to the limit as  $\delta \rightarrow 0+$ , we obtain

$$\begin{aligned} H_0(z, k) &= |z - k|, \quad Q_0(z, k) = \text{sign}(z - k)(a(z) - a(k)), \\ \Phi_0(z, k) &= \text{sign}(z - k)(\varphi(z) - \varphi(k)). \end{aligned}$$

**Example 2.** We introduce boundary entropy-entropy flux triple  $(H_{\delta,w}, Q_{\delta,w}, \Phi_{\delta,w})$  by the rule:

$$\begin{aligned} H_{\delta,w}(z, k) &= ((H_0(z, k) + H_0(z, w) - H_0(k, w))^2 + \delta^2)^{1/2} - \delta, \\ Q_{\delta,w}(z, k) &= \int_k^z a'(\lambda)\partial_\lambda H_{\delta,w}(\lambda, k) d\lambda, \\ \Phi_{\delta,w}(z, k) &= \int_k^z \varphi'(\lambda)\partial_\lambda H_{\delta,w}(\lambda, k) d\lambda, \quad \forall (k, z, w) \in \mathbb{R}^3, \end{aligned}$$

where  $H_0(z, k) + H_0(z, w) - H_0(k, w) = 2\text{dist}(z, \mathcal{I}[w, k])$ , where  $\mathcal{I}[w, k]$  is an interval with its endpoints  $w$  and  $k$ . In the limit as  $\delta \rightarrow 0+$  we get

$$\begin{aligned} H_{0,w}(z, k) &= H_0(z, k) + H_0(z, w) - H_0(k, w), \\ Q_{0,w}(z, k) &= Q_0(z, k) + Q_0(z, w) - Q_0(k, w), \\ \Phi_{0,w}(z, k) &= \Phi_0(z, k) + \Phi_0(z, w) - \Phi_0(k, w). \end{aligned}$$

It is important to note that

$$Q_{0,w}(z, k) = \begin{cases} 2(a(k) - a(z)) & \text{if } z \leq k \leq w, \\ 2(a(w) - a(z)) & \text{if } z \leq w \leq k, \\ 0 & \text{if } w \leq z \leq k, \\ 0 & \text{if } k \leq z \leq w, \\ 2(a(z) - a(w)) & \text{if } k \leq w \leq z, \\ 2(a(z) - a(k)) & \text{if } w \leq k \leq z. \end{cases}$$

**2.2. Essential limits of bounded variation functions.**

Note that the variation  $V(f, (0, T))$  of arbitrary functions  $f \in BV((0, T))$  has the form

$$(2.1) \quad V(f, (0, T)) = \sup \left\{ \int_0^T \alpha'(t) f(t) dt : \alpha \in C_0^1((0, T)), \|\alpha\|_{L^\infty((0, T))} \leq 1 \right\}.$$

Without limiting the generality of the foregoing, in formula (2.1) we can replace the condition  $\|\alpha\|_{L^\infty((0, T))} \leq 1$  by the condition  $0 \leq \alpha(t) \leq 1$  for a.e.  $t \in (0, T)$ . Let  $f(t) \in L^1((0, T))$  and

$$\begin{aligned} \operatorname{essliminf}_{t \rightarrow 0+} f(t) &:= \sup_{\epsilon > 0} \operatorname{essinf}_{(0, \epsilon)} f(t), & \operatorname{esslimsup}_{t \rightarrow 0+} f(t) &:= \inf_{\epsilon > 0} \operatorname{esssup}_{(0, \epsilon)} f(t), \\ \operatorname{essliminf}_{t \rightarrow T-} f(t) &:= \sup_{\epsilon > 0} \operatorname{essinf}_{(T-\epsilon, T)} f(t), & \operatorname{esslimsup}_{t \rightarrow T-} f(t) &:= \inf_{\epsilon > 0} \operatorname{esssup}_{(T-\epsilon, T)} f(t). \end{aligned}$$

Also we assume that  $f(t) \in BV((0, T))$ . This implies the existence of essential limits (see [19, Theorem 9.89],[20, Theorem 3.28])

$$(2.2) \quad \operatorname{esslim}_{t \rightarrow 0+} f(t) := \operatorname{essliminf}_{t \rightarrow 0+} f(t) = \operatorname{esslimsup}_{t \rightarrow 0+} f(t) = \lim_{\rho \rightarrow 0+} \frac{1}{\rho} \int_0^\rho f(t) dt,$$

$$(2.3) \quad \operatorname{esslim}_{t \rightarrow T-} f(t) := \operatorname{essliminf}_{t \rightarrow T-} f(t) = \operatorname{esslimsup}_{t \rightarrow T-} f(t) = \lim_{\rho \rightarrow 0+} \frac{1}{\rho} \int_{T-\rho}^T f(t) dt.$$

Definition 1, Examples 1, 2 and essential limits (2.2), (2.3) are used in Definition 3 of an entropy measure-valued solution to equation (3.6).

3. SINGULAR LIMIT  $\lim_{\epsilon \rightarrow 0+} u_{p,q,\epsilon}$

**3.1. Existence of solutions to problem  $\Pi_{p,q,\epsilon}$ .**

In this subsection we deal with the elliptic regularization (3.1) of forward-backward  $p$ -parabolic equation (3.6) where an anisotropic  $p, q$ -Laplacian is used. Moreover, we are going to construct a singular limit of weak solutions  $u_{p,q,\epsilon}$  for non-homogeneous Dirichlet problem  $\Pi_{p,q,\epsilon}$  as  $\epsilon \rightarrow 0+$ .

**Problem  $\Pi_{p,q,\epsilon}$ .** For given initial and final conditions  $u_0, u_T \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$  the problem is to find unknown function  $u_{p,q,\epsilon} : G_T \rightarrow \mathbb{R}$  which satisfies the boundary value problem:

$$(3.1) \quad \partial_t a(u_{p,q,\epsilon}) + \operatorname{div} \varphi(u_{p,q,\epsilon}) = \operatorname{div} (|\nabla u_{p,q,\epsilon}|^{p-2} \nabla u_{p,q,\epsilon}) + \epsilon \partial_t (|\partial_t u_{p,q,\epsilon}|^{q-2} \partial_t u_{p,q,\epsilon}), \quad (\mathbf{x}, t) \in G_T,$$

$$(3.2) \quad u_{p,q,\varepsilon}|_{\Gamma_0} = u_0(\mathbf{x}), \quad u_{p,q,\varepsilon}|_{\Gamma_T} = u_T(\mathbf{x}), \quad u_{p,q,\varepsilon}|_{\Gamma_l} = 0,$$

in a weak sense, see Definition 2.

Here we assume that  $(\mathbf{x}, t) \in G_T = \Omega \times (0, T)$ ,  $\mathbf{x} \in \Omega \subset \mathbb{R}^d$ ,  $\Gamma_0 = \Omega \times \{t = 0\}$ ,  $\Gamma_T = \Omega \times \{t = T\}$ ,  $\Gamma_l = \partial\Omega \times [0, T]$ ,  $p > 1$ ,  $q > 1$ ,  $\varepsilon > 0$ . Moreover, there is no dependence between two parameters  $p$  and  $q$ .

Here functions  $a$  and  $\varphi$  satisfy the following condition.

**Condition 1.** Let  $a \in C^2(\mathbb{R})$ ,  $a(0) = 0$ ,  $\varphi(\mathbf{0}) = (\varphi_1(z), \dots, \varphi_d(z))$ ,  $z \in \mathbb{R}$ ,  $\varphi_j \in C^2(\mathbb{R})$ ,  $j = 1, \dots, d$ ,  $\varphi(\mathbf{0}) = 0$ . Function  $a(z)$  is not monotone and  $a'(z)$  does not equal to zero identically on intervals of positive measure.

**Definition 2.** Function  $u_{p,q,\varepsilon} \in L^\infty(G_T) \cap L^p(0, T; W_0^{1,p}(\Omega)) \cap L^q(\Omega; W^{1,q}((0, T)))$  is called a weak solution to problem  $\Pi_{p,q,\varepsilon}$  if the following assertions hold.

(EL.1) Let  $\hat{u} \in L^\infty(G_T) \cap L^p(0, T; W_0^{1,p}(\Omega)) \cap L^q(\Omega; W^{1,q}((0, T)))$  be an extension of functions  $u_0$  and  $u_T$  into  $G_T$ . Therefore,  $u_{p,q,\varepsilon} - \hat{u} \in L^\infty(G_T) \cap L^p(0, T; W_0^{1,p}(\Omega)) \cap L^q(\Omega; W_0^{1,q}((0, T)))$ .

(EL.2) The following equality holds

$$(3.3a) \quad \int_{G_T} \left( -a(u_{p,q,\varepsilon}) \partial_t \phi - \varphi(u_{p,q,\varepsilon}) \cdot \nabla \phi + |\nabla u_{p,q,\varepsilon}|^{p-2} \nabla u_{p,q,\varepsilon} \cdot \nabla \phi + \varepsilon |\partial_t u_{p,q,\varepsilon}|^{q-2} \partial_t u_{p,q,\varepsilon} \partial_t \phi \right) d\mathbf{x} dt = 0$$

for every  $\phi \in L^\infty(G_T) \cap L^p(0, T; W_0^{1,p}(\Omega)) \cap L^q(\Omega; W_0^{1,q}((0, T)))$ .

**Remark 1.** We can reformulate (3.3a) in the following way:

$$(3.3b) \quad \int_{G_T} \left( \partial_t a(u_{p,q,\varepsilon}) \phi + \operatorname{div} \varphi(u_{p,q,\varepsilon}) \phi + |\nabla u_{p,q,\varepsilon}|^{p-2} \nabla u_{p,q,\varepsilon} \cdot \nabla \phi + \varepsilon |\partial_t u_{p,q,\varepsilon}|^{q-2} \partial_t u_{p,q,\varepsilon} \partial_t \phi \right) d\mathbf{x} dt = 0.$$

**Proposition 1.** Under Condition 1, problem  $\Pi_{p,q,\varepsilon}$  has at least one weak solution  $u_{p,q,\varepsilon}$  for all  $u_0, u_T \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ . Moreover, maximum principle

$$(3.4) \quad \|u_{p,q,\varepsilon}\|_{L^\infty(G_T)} \leq M = \max \left( \|u_0\|_{L^\infty(\Omega)}, \|u_T\|_{L^\infty(\Omega)} \right).$$

and energy estimate

$$(3.5) \quad \int_{G_T} \left( |\nabla u_{p,q,\varepsilon}|^p + \varepsilon |\partial_t u_{p,q,\varepsilon}|^q \right) d\mathbf{x} dt \leq C_{(4.20)}$$

hold. The constant  $C_{(4.20)}$  is defined in inequality (4.20) and does not depend on  $\varepsilon > 0$ .

### 3.2. Forward-backward $p$ -parabolic equation.

In the limit as  $\varepsilon \rightarrow 0+$  equation (3.1) would be forward-backward  $p$ -parabolic equation

$$(3.6) \quad \partial_t a(u) + \operatorname{div} \varphi(u) = \operatorname{div} \left( |\nabla u|^{p-2} \nabla u \right), \quad (\mathbf{x}, t) \in G_T.$$

To formulate boundary conditions in the form of integral inequalities (see Remark 2), we need boundary entropy-entropy flux triples  $(H, Q, \Phi)$  (see section 2).

**Definition 3.** An entropy measure-valued solution to equation (3.6) is a measurable function  $u : G_T \rightarrow \mathbb{R}$  and a gradient Young measure  $\nu_{\mathbf{x},t}$  satisfying the following conditions:

(FB.1) (Regularity)  $u \in L^\infty(G_T) \cap L^p(0, T; W_0^{1,p}(\Omega))$ ,

$$(3.7) \quad \|u\|_{L^\infty(G_T)} \leq M;$$

(FB.2) (Gradient Young measure) Here  $\nu = \nu_{\mathbf{x},t}$  is a probability measure for a.e.  $(\mathbf{x}, t) \in G_T$ , and  $\langle \nu_{\mathbf{x},t}, \mathbf{S}_p(\cdot) \rangle$  is defined as a dual pairing of vector function  $\mathbf{S}_p(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2} \boldsymbol{\xi}$  with a gradient Young measure  $\nu_{\mathbf{x},t}$ , i.e.

$$(3.8) \quad \langle \nu_{\mathbf{x},t}, \mathbf{S}_p(\cdot) \rangle := \int_{\mathbb{R}^d} \mathbf{S}_p(\boldsymbol{\xi}) \, d\nu_{\mathbf{x},t}(\boldsymbol{\xi});$$

(FB.3) (Entropy solution) Integral inequality

$$(3.9) \quad \int_{G_T} \left( Q(u, k) \partial_t \gamma + \Phi(u, k) \cdot \nabla \gamma - \partial_1 H(u, k) \langle \nu_{\mathbf{x},t}, \mathbf{S}_p(\cdot) \rangle \cdot \nabla \gamma - \partial_1^2 H(u, k) |\nabla u|^p \gamma \right) d\mathbf{x} dt \geq 0$$

holds for every boundary entropy-entropy flux triple  $(H, Q, \Phi)$  and  $\gamma \in C_0^\infty(G_T)$ ,  $\gamma \geq 0$ ,  $k \in \mathbb{R}$ ;

(FB.4) (Incomplete boundary conditions) Essential limits

$$(3.10) \quad \operatorname{esslim}_{t \rightarrow 0^+} \int_{\Omega} (Q_0(u(\mathbf{x}, t), k) + Q_0(u(\mathbf{x}, t), u_0(\mathbf{x})) - Q_0(k, u_0(\mathbf{x}))) \beta(\mathbf{x}) \, d\mathbf{x},$$

$$(3.11) \quad \operatorname{esslim}_{t \rightarrow T^-} \int_{\Omega} (Q_0(u(\mathbf{x}, t), k) + Q_0(u(\mathbf{x}, t), u_T(\mathbf{x})) - Q_0(k, u_T(\mathbf{x}))) \beta(\mathbf{x}) \, d\mathbf{x}$$

exist for every  $\beta \in L^1(\Omega)$ ,  $\beta \geq 0$  a.e. in  $\Omega$ ,  $Q_0(z, k) = \operatorname{sign}(z - k)(a(z) - a(k))$ , (see Example 1 in section 2).

**Remark 2.** We have the following representation

$$(3.12) \quad Q_0(u(\mathbf{x}, t), k) + Q_0(u(\mathbf{x}, t), u_0(\mathbf{x})) - Q_0(k, u_0(\mathbf{x})) = \begin{cases} 2(a(k) - a(u(\mathbf{x}, t))) & \text{if } u(\mathbf{x}, t) \leq k \leq u_0(\mathbf{x}), \\ 2(a(u_0(\mathbf{x})) - a(u(\mathbf{x}, t))) & \text{if } u(\mathbf{x}, t) \leq u_0(\mathbf{x}) \leq k, \\ 0 & \text{if } u_0(\mathbf{x}) \leq u(\mathbf{x}, t) \leq k, \\ 0 & \text{if } k \leq u(\mathbf{x}, t) \leq u_0(\mathbf{x}), \\ 2(a(u(\mathbf{x}, t)) - a(u_0(\mathbf{x}))) & \text{if } k \leq u_0(\mathbf{x}) \leq u(\mathbf{x}, t), \\ 2(a(u(\mathbf{x}, t)) - a(k)) & \text{if } u_0(\mathbf{x}) \leq k \leq u(\mathbf{x}, t). \end{cases}$$

We have not proved yet that  $u = \lim_{\varepsilon \rightarrow 0^+} u_{p,q,\varepsilon}$  ( $q \neq 2$ ) satisfies the following boundary conditions

$$(3.13) \quad \operatorname{esslim}_{t \rightarrow 0^+} \int_{\Omega} (Q_0(u(\mathbf{x}, t), k) + Q_0(u(\mathbf{x}, t), u_0(\mathbf{x})) - Q_0(k, u_0(\mathbf{x}))) \beta(\mathbf{x}) \, d\mathbf{x} \leq 0,$$

$$(3.14) \quad \operatorname{esslim}_{t \rightarrow T^-} \int_{\Omega} (Q_0(u(\mathbf{x}, t), k) + Q_0(u(\mathbf{x}, t), u_T(\mathbf{x})) - Q_0(k, u_T(\mathbf{x}))) \beta(\mathbf{x}) \, d\mathbf{x} \geq 0.$$

In [9] it was proved that  $\lim_{\varepsilon \rightarrow 0^+} u_{p,2,\varepsilon}$  satisfies boundary conditions (3.13), (3.14). Moreover, if we assume that function  $a(z)$  is increasing and a solution  $u$  has a trace on the boundary

$\partial G_T$ , it follows from formula (3.12) and integral inequalities (3.13) and (3.14) that

$$(3.15) \quad u|_{\Gamma_0} = u_0.$$

Under Condition 1, the latter equality may be violated.

In the following theorem it is asserted that an entropy measure-valued solution of equation (3.6) is a limit point of the set  $\{u_{p,q,\varepsilon}\}_{\varepsilon>0}$ .

**Theorem 1.** *Under Condition 1, there exists an entropy measure-valued solution to equation (3.6) such that*

$$(3.16) \quad u = \lim_{\varepsilon \rightarrow 0^+} u_{p,q,\varepsilon}$$

in  $L^p(0, T; L^p(\Omega))$ . Furthermore, gradient Young measure  $\nu_{\mathbf{x},t}$  is associated with the set  $\{\nabla u_{p,q,\varepsilon}\}_{\varepsilon>0}$ .

#### 4. PROOF OF PROPOSITION 1

To establish the existence of solution  $u_{p,q,\varepsilon}$  to problem  $\Pi_{p,q,\varepsilon}$ , we use well-known results for elliptic equations [12], [13].

We are going to deduce estimates (3.4), (3.5). Let us introduce the function

$$(4.1) \quad u_{p,q,\varepsilon}^M = \max(u_{p,q,\varepsilon} - M, 0) = \begin{cases} u_{p,q,\varepsilon} - M & \text{if } u_{p,q,\varepsilon} > M \\ 0 & \text{if } u_{p,q,\varepsilon} \leq M \end{cases}, \quad u_{p,q,\varepsilon}^M|_{\partial G_T} = 0,$$

$$(4.2) \quad \nabla u_{p,q,\varepsilon}^M = \begin{cases} \nabla u_{p,q,\varepsilon} & \text{if } u_{p,q,\varepsilon} > M, \\ 0 & \text{if } u_{p,q,\varepsilon} \leq M, \end{cases} \quad \partial_t u_{p,q,\varepsilon}^M = \begin{cases} \partial_t u_{p,q,\varepsilon} & \text{if } u_{p,q,\varepsilon} > M, \\ 0 & \text{if } u_{p,q,\varepsilon} \leq M. \end{cases}$$

Putting  $\phi = u_{p,q,\varepsilon}^M$  in (3.3a), we derive

$$(4.3) \quad \int_{G_T} \left( |\nabla u_{p,q,\varepsilon}^M|^p + \varepsilon |\partial_t u_{p,q,\varepsilon}^M|^q \right) d\mathbf{x}dt = I_1 + I_2,$$

where

$$(4.4) \quad I_1 := \int_{G_T} a(u_{p,q,\varepsilon}) \partial_t u_{p,q,\varepsilon}^M d\mathbf{x}dt, \quad I_2 := \int_{G_T} \varphi(u_{p,q,\varepsilon}) \cdot \nabla u_{p,q,\varepsilon}^M d\mathbf{x}dt.$$

Taking into account the properties of functions  $u_{p,q,\varepsilon}^M$ ,  $\nabla u_{p,q,\varepsilon}^M$ ,  $\partial_t u_{p,q,\varepsilon}^M$ , we have

$$(4.5) \quad I_1 = \int_{G_T} a(u_{p,q,\varepsilon}^M + M) \partial_t u_{p,q,\varepsilon}^M d\mathbf{x}dt = \int_{G_T} \partial_t \left( \int_0^{u_{p,q,\varepsilon}^M} a(\lambda + M) d\lambda \right) d\mathbf{x}dt = \int_{\Omega} \left( \int_0^{u_{p,q,\varepsilon}^M} a(\lambda + M) d\lambda \right) d\mathbf{x} \Big|_{t=0}^{t=T} = 0,$$

$$(4.6) \quad I_2 = \int_{G_T} \varphi(u_{p,q,\varepsilon}^M + M) \cdot \nabla u_{p,q,\varepsilon}^M d\mathbf{x}dt = \int_{G_T} \operatorname{div} \left( \int_0^{u_{p,q,\varepsilon}^M} \varphi(\lambda + M) d\lambda \right) d\mathbf{x}dt = 0.$$

Hence, according to (4.3) we get

$$(4.7) \quad \int_{G_T} \left( |\nabla u_{p,q,\varepsilon}^M|^p + \varepsilon |\partial_t u_{p,q,\varepsilon}^M|^q \right) d\mathbf{x}dt = 0$$

and

$$(4.8) \quad u_{p,q,\varepsilon}^M = 0 \implies u_{p,q,\varepsilon} \leq M.$$

Analogously, we obtain

$$(4.9) \quad -u_{p,q,\varepsilon} \leq M \text{ and } |u_{p,q,\varepsilon}| \leq M.$$

To derive estimate (3.5), we consider an extension  $\widehat{u}$  of functions  $u_0, u_T$  into  $G_T$  such that

$$(4.10) \quad \widehat{u} \in L^\infty(G_T) \cap L^p(0, T; W_0^{1,p}(\Omega)) \cap L^q(\Omega; W^{1,q}((0, T))), \quad (u_{p,q,\varepsilon} - \widehat{u})|_{\partial G_T} = 0.$$

In equation (3.3a) we put  $\phi = u_{p,q,\varepsilon} - \widehat{u}$ :

$$(4.11) \quad - \int_{G_T} \partial_t(u_{p,q,\varepsilon} - \widehat{u})a(u_{p,q,\varepsilon}) \, d\mathbf{x}dt \\ + \int_{G_T} \nabla(u_{p,q,\varepsilon} - \widehat{u}) \cdot \nabla u_{p,q,\varepsilon} |\nabla u_{p,q,\varepsilon}|^{p-2} \, d\mathbf{x}dt = \\ \int_{G_T} \nabla(u_{p,q,\varepsilon} - \widehat{u}) \cdot \varphi(u_{p,q,\varepsilon}) \, d\mathbf{x}dt - \varepsilon \int_{G_T} \partial_t(u_{p,q,\varepsilon} - \widehat{u}) \partial_t u_{p,q,\varepsilon} |\partial_t u_{p,q,\varepsilon}|^{q-2} \, d\mathbf{x}dt.$$

This reads in the following way

$$(4.12) \quad \int_{G_T} (|\nabla u_{p,q,\varepsilon}|^p + \varepsilon |\partial_t u_{p,q,\varepsilon}|^q) \, d\mathbf{x}dt = J_1 + J_2 + J_3 + J_4,$$

where

$$(4.13) \quad J_1 := \int_{G_T} \left( |\nabla u_{p,q,\varepsilon}|^{p-2} \nabla u_{p,q,\varepsilon} \cdot \nabla \widehat{u} + \varepsilon |\partial_t u_{p,q,\varepsilon}|^{q-2} \partial_t u_{p,q,\varepsilon} \partial_t \widehat{u} \right) \, d\mathbf{x}dt,$$

$$(4.14) \quad J_2 := \int_{G_T} a(u_{p,q,\varepsilon}) \partial_t u_{p,q,\varepsilon} \, d\mathbf{x}dt,$$

$$(4.15) \quad J_3 := - \int_{G_T} a(u_{p,q,\varepsilon}) \partial_t \widehat{u} \, d\mathbf{x}dt, \quad J_4 := \int_{G_T} \varphi(u_{p,q,\varepsilon}) \cdot (\nabla u_{p,q,\varepsilon} - \nabla \widehat{u}) \, d\mathbf{x}dt.$$

Applying the Young inequality

$$yz \leq \frac{\kappa^{r'}}{r'} y^{r'} + \frac{\kappa^{-r}}{r} z^r, \quad 1 < r < \infty, \quad r' = \frac{r}{r-1}, \quad y, z \geq 0, \quad \kappa \in (0, 1],$$

in several cases, we derive

$$(4.16) \quad |J_1| \leq \int_{G_T} \left( \frac{\delta^{p'}}{p'} |\nabla u_{p,q,\varepsilon}|^p + \frac{\delta^{-p}}{p} |\nabla \widehat{u}|^p + \frac{\varepsilon \delta_1^{q'}}{q'} |\partial_t u_{p,q,\varepsilon}|^q + \frac{\varepsilon \delta_1^{-q}}{q} |\partial_t \widehat{u}|^q \right) \, d\mathbf{x}dt,$$

$$(4.17) \quad |J_3| \leq \int_{G_T} \left( \frac{1}{q} |\partial_t \widehat{u}|^q + \frac{1}{q'} |a(u_{p,q,\varepsilon})|^{q'} \right) \, d\mathbf{x}dt,$$

$$(4.18) \quad |J_4| \leq \int_{G_T} \left( \frac{\delta_2^p}{p} |\nabla u_{p,q,\varepsilon}|^p + \frac{\delta_2^{-p'}}{p'} |\varphi(u_{p,q,\varepsilon})|^{p'} + \frac{1}{p} |\nabla \widehat{u}|^p + \frac{1}{p'} |\varphi(u_{p,q,\varepsilon})|^{p'} \right) \, d\mathbf{x}dt,$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ . We evaluate the term  $J_2$  in the following way

$$(4.19) \quad |J_2| = \left| \int_{\Omega} \left( \int_0^{u_{p,q,\varepsilon}(\mathbf{x},t)} a(\lambda) d\lambda \right) d\mathbf{x} \right|_{t=0}^{t=T} \leq 2 \sup_{|z| \leq M} \left| \int_0^z a(\lambda) d\lambda \right| |\Omega|.$$

Gathering last estimates and choosing  $\delta$ ,  $\delta_1$  and  $\delta_2$  appropriately small and taking into account (3.4), we find that

$$(4.20) \quad \int_{G_T} (|\nabla u_{p,q,\varepsilon}|^p + \varepsilon |\partial_t u_{p,q,\varepsilon}|^q) d\mathbf{x}dt \leq C(p, q) \int_{G_T} (|\partial_t \hat{u}|^q + |\nabla \hat{u}|^p + |a(u_{p,q,\varepsilon})|^{q'} + |\varphi(u_{p,q,\varepsilon})|^{p'}) d\mathbf{x}dt + \sup_{|z| \leq M} \left| \int_0^z a(\lambda) d\lambda \right| |\Omega| \leq C(p, q) \int_{G_T} (|\partial_t \hat{u}|^q + |\nabla \hat{u}|^p) d\mathbf{x}dt + C(p, q, |\Omega|, T) \left( \sup_{|z| \leq M} |a(z)|^{q'} + \sup_{|z| \leq M} |\varphi(z)|^{p'} + \sup_{|z| \leq M} \left| \int_0^z a(\lambda) d\lambda \right| \right) =: C_{(4.20)}.$$

Estimates (3.4), (3.5) imply that the operator  $L$  defined by the formula

$$L(u_{p,q,\varepsilon}, \vartheta) := \int_{G_T} \left( -a(u_{p,q,\varepsilon}) \partial_t(\vartheta - \hat{u}) - \varphi(u_{p,q,\varepsilon}) \cdot \nabla(\vartheta - \hat{u}) + |\nabla u_{p,q,\varepsilon}|^{p-2} \nabla u_{p,q,\varepsilon} \cdot \nabla(\vartheta - \hat{u}) + \varepsilon |\partial_t u_{p,q,\varepsilon}|^{q-2} \partial_t u_{p,q,\varepsilon} \partial_t(\vartheta - \hat{u}) \right) d\mathbf{x}dt$$

is coercive in the space  $L^\infty(G_T) \cap L^p(0, T; W_0^{1,p}(\Omega)) \cap L^q(\Omega; W^{1,q}(0, T))$ , that is

$$L(u_{p,q,\varepsilon}, u_{p,q,\varepsilon}) = \int_{G_T} (|\nabla u_{p,q,\varepsilon}|^p + \varepsilon |\partial_t u_{p,q,\varepsilon}|^q) d\mathbf{x}dt - J_1 - J_2 - J_3 - J_4 \geq \min \left( \left(1 - \frac{\delta^{p'}}{p'} - \frac{\delta_2^p}{p}\right), \varepsilon \left(1 - \frac{\delta_1^{q'}}{q'}\right) \right) \left( \int_{G_T} |\nabla u_{p,q,\varepsilon}|^p d\mathbf{x}dt + \int_{G_T} |\partial_t u_{p,q,\varepsilon}|^q d\mathbf{x}dt \right) - C_{(4.21)},$$

where  $J_i$  are defined in (4.13)–(4.15) and

$$(4.21) \quad C_{(4.21)} = \frac{1 + \delta^{-p}}{p} \int_{G_T} |\nabla \hat{u}|^p d\mathbf{x}dt + \frac{1 + \varepsilon \delta_1^{-q}}{q} \int_{G_T} |\partial_t \hat{u}|^q d\mathbf{x}dt + |G_T| \left( \frac{1}{q'} \sup_{|z| \leq M} |a(z)|^{q'} + \frac{1 + \delta_2^{-p'}}{p'} \sup_{|z| \leq M} |\varphi(z)|^p \right) + 2 |\Omega| \sup_{|z| \leq M} \left| \int_0^z a(\lambda) d\lambda \right|.$$

Therefore, according to well known results (see [12, Theorem 9.2, Ch. IV], [13, Theorem 8.5]) we conclude that problem  $\Pi_{p,q,\varepsilon}$  has at least one weak solution  $u_{p,q,\varepsilon}$ .

### 5. PROOF OF THEOREM 1

In this section we represent entropy measure-valued solutions in the form (3.16). The main difficulty is that we have only compactness result on  $\{u_{p,q,\varepsilon}\}_{\varepsilon>0}$  in  $L^p(G_T)$  and only uniform boundedness of  $\{\nabla u_{p,q,\varepsilon}\}_{\varepsilon>0}$  in  $L^p(G_T; \mathbb{R}^d)$ .

### 5.1. Relative compactness of $\{u_{p,q,\varepsilon}\}_{\varepsilon>0}$ .

We are going to prove that an entropy measure-valued solution  $u$  of equation (3.6) is represented as a limit point in  $L^p(0, T; L^p(\Omega))$  of the set  $\{u_{p,q,\varepsilon}\}_{\varepsilon>0}$  as  $\varepsilon \rightarrow +0$ . Therefore, a gradient Young measure  $\nu_{\mathbf{x},t}$  is associated with  $\{\nabla u_{p,q,\varepsilon}\}_{\varepsilon>0}$ .

From Definition 2 it follows that

$$\int_{G_T} \partial_t w_{p,q,\varepsilon} \phi \, d\mathbf{x}dt = \int_{G_T} (\varphi(u_{p,q,\varepsilon}) \cdot \nabla \phi - |\nabla u_{p,q,\varepsilon}|^{p-2} \nabla u_{p,q,\varepsilon} \cdot \nabla \phi) \, d\mathbf{x}dt,$$

where  $w_{p,q,\varepsilon} = a(u_{p,q,\varepsilon}) - \varepsilon |\partial_t u_{p,q,\varepsilon}|^{q-2} \partial_t u_{p,q,\varepsilon}$ ,  $\phi \in L^p(0, T; W_0^{1,p}(\Omega))$ . Therefore, we conclude that  $\partial_t w_{p,q,\varepsilon} \in L^{p'}(0, T; W^{-1,p}(\Omega))$  and for every  $\phi \in L^p(0, T; W_0^{1,p}(\Omega))$  it follows that

$$(5.1) \quad \left| \int_{G_T} \partial_t w_{p,q,\varepsilon}(\mathbf{x}, t) \phi(\mathbf{x}, t) \, d\mathbf{x}dt \right| \leq C_{(5.1)} \|\phi\|_{L^p(0, T; W_0^{1,p}(\Omega))},$$

where

$$C_{(5.1)} := \sup_{|z| \leq M} |\varphi(z)| |G_T|^{\frac{p-1}{p}} + (C_{(4.20)})^{\frac{p-1}{p}}.$$

Let  $g(z) = \int_0^z (a'(\tau))^2 \, d\tau$ . We take an arbitrary function  $\psi \in W_0^{s,p}(\Omega)$ ,  $s \geq [\frac{d}{p}] + 1$ , where  $\|\psi\|_{L^\infty(\Omega)} \leq C(\Omega) \|\psi\|_{W_0^{s,p}(\Omega)}$ . We have

$$(5.2) \quad \begin{aligned} & \int_{\Omega} (g(u_{p,q,\varepsilon}(\mathbf{x}, t+h)) - g(u_{p,q,\varepsilon}(\mathbf{x}, t))) \psi(\mathbf{x}) \, d\mathbf{x} \\ &= \int_t^{t+h} \int_{\Omega} \partial_s g(u_{p,q,\varepsilon}(\mathbf{x}, s)) \psi(\mathbf{x}) \, d\mathbf{x}ds = \\ & \int_t^{t+h} \int_{\Omega} a'(u_{p,q,\varepsilon}(\mathbf{x}, s)) \partial_s a(u_{p,q,\varepsilon}(\mathbf{x}, s)) \psi(\mathbf{x}) \, d\mathbf{x}ds = \\ & \int_t^{t+h} \int_{\Omega} a'(u_{p,q,\varepsilon}(\mathbf{x}, s)) \partial_s w_{p,q,\varepsilon}(\mathbf{x}, s) \psi(\mathbf{x}) \, d\mathbf{x}ds \\ &+ \int_t^{t+h} \int_{\Omega} a'(u_{p,q,\varepsilon}(\mathbf{x}, s)) \partial_s (\varepsilon |\partial_s u_{p,q,\varepsilon}(\mathbf{x}, s)|^{q-2} \partial_s u_{p,q,\varepsilon}(\mathbf{x}, s)) \psi(\mathbf{x}) \, d\mathbf{x}ds = \\ & \int_t^{t+h} \int_{\Omega} a'(u_{p,q,\varepsilon}(\mathbf{x}, s)) \partial_s w_{p,q,\varepsilon}(\mathbf{x}, s) \psi(\mathbf{x}) \, d\mathbf{x}ds \\ & - \int_t^{t+h} \int_{\Omega} \varepsilon a''(u_{p,q,\varepsilon}(\mathbf{x}, s)) |\partial_s u_{p,q,\varepsilon}(\mathbf{x}, s)|^q \psi(\mathbf{x}) \, d\mathbf{x}ds \\ &+ \int_t^{t+h} \int_{\Omega} \partial_s (\varepsilon |\partial_s u_{p,q,\varepsilon}(\mathbf{x}, s)|^{q-2} \partial_s a(u_{p,q,\varepsilon}(\mathbf{x}, s))) \psi(\mathbf{x}) \, d\mathbf{x}ds = \\ & \int_t^{t+h} \int_{\Omega} a'(u_{p,q,\varepsilon}(\mathbf{x}, s)) \partial_s w_{p,q,\varepsilon}(\mathbf{x}, s) \psi(\mathbf{x}) \, d\mathbf{x}ds \\ & - \int_t^{t+h} \int_{\Omega} \varepsilon a''(u_{p,q,\varepsilon}(\mathbf{x}, s)) |\partial_s u_{p,q,\varepsilon}(\mathbf{x}, s)|^q \psi(\mathbf{x}) \, d\mathbf{x}ds \\ &+ \int_{\Omega} \varepsilon (|\partial_t u_{p,q,\varepsilon}(\mathbf{x}, t+h)|^{q-2} \partial_t a(u_{p,q,\varepsilon}(\mathbf{x}, t+h)) - |\partial_t u_{p,q,\varepsilon}(\mathbf{x}, t)|^{q-2} \partial_t a(u_{p,q,\varepsilon}(\mathbf{x}, t))) \psi(\mathbf{x}) \, d\mathbf{x}. \end{aligned}$$

We apply Theorem 5 from [14] to the set  $\{g(u_{p,q,\varepsilon})\}_{\varepsilon>0}$ . It follows from (5.1) that

$$(5.3) \quad \left| \int_t^{t+h} \int_{\Omega} a'(u_{p,q,\varepsilon}(\mathbf{x}, s)) \partial_s w_{p,q,\varepsilon}(\mathbf{x}, s) \psi(\mathbf{x}) \, d\mathbf{x} ds \right| \leq C_{(5.1)} \|a'(u_{p,q,\varepsilon})\psi\|_{L^p(t, t+h; W_0^{1,p}(\Omega))} \leq C_{(5.1)} \max(\mathcal{A}_1, \mathcal{A}_2) \|u_{p,q,\varepsilon}\|_{L^p(t, t+h; W_0^{1,p}(\Omega))} C(\Omega) \|\psi\|_{W_0^{s,p}(\Omega)} \rightarrow 0 \quad \text{as } h \rightarrow 0+,$$

$$(5.4) \quad \left| \int_t^{t+h} \int_{\Omega} \varepsilon a''(u_{p,q,\varepsilon}(\mathbf{x}, s)) |\partial_s u_{p,q,\varepsilon}(\mathbf{x}, s)|^q \psi(\mathbf{x}) \, d\mathbf{x} ds \right| \leq \mathcal{A}_2 \int_t^{t+h} \int_{\Omega} \varepsilon |\partial_s u_{p,q,\varepsilon}(\mathbf{x}, s)|^q \, d\mathbf{x} ds C(\Omega) \|\psi\|_{W_0^{s,p}(\Omega)} \rightarrow 0 \quad \text{as } h \rightarrow 0+,$$

$$(5.5) \quad \left| \int_{\Omega} \varepsilon (|\partial_t u_{p,q,\varepsilon}(\mathbf{x}, t+h)|^{q-2} \partial_t a(u_{p,q,\varepsilon}(\mathbf{x}, t+h)) - |\partial_t u_{p,q,\varepsilon}(\mathbf{x}, t)|^{q-2} \partial_t a(u_{p,q,\varepsilon}(\mathbf{x}, t))) \psi(\mathbf{x}) \, d\mathbf{x} \right| \leq \left\| \varepsilon (|\partial_t u_{p,q,\varepsilon}(\cdot, t+h)|^{q-2} \partial_t a(u_{p,q,\varepsilon}(\cdot, t+h)) - |\partial_t u_{p,q,\varepsilon}(\cdot, t)|^{q-2} \partial_t a(u_{p,q,\varepsilon}(\cdot, t))) \right\|_{L^{q'}(\Omega)} \times |\Omega|^{\frac{1}{q}} C(\Omega) \|\psi\|_{W_0^{s,p}(\Omega)} \rightarrow 0 \quad \text{as } h \rightarrow 0+,$$

where  $\mathcal{A}_1 = \sup_{|z|\leq M} |a'(z)|$ ,  $\mathcal{A}_2 = \sup_{|z|\leq M} |a''(z)|$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ . The limits (5.3) and (5.4) are valid due to estimates (3.5) and absolute continuity of the Lebesgue integral. The latter inequality and the corresponding limit are valid since the set  $\{\varepsilon |\partial_t u_{p,q,\varepsilon}|^{q-2} \partial_t a(u_{p,q,\varepsilon})\}_{\varepsilon>0}$  is relatively compact in  $L^{q'}(G_T)$ .

From inequalities (5.3)–(5.5) and corresponding limits the following one-sided limit follows

$$(5.6) \quad \lim_{h \rightarrow 0+} \int_0^{T-h} \|g(u_{p,q,\varepsilon}(\cdot, t+h)) - g(u_{p,q,\varepsilon}(\cdot, t))\|_{W^{-s,p}(\Omega)} \, dt = 0.$$

From estimate (3.5), boundedness of the domain  $\Omega$ , the maximum principle (3.4) and Condition 1 it follows that

$$(5.7) \quad \{g(u_{p,q,\varepsilon})\}_{\varepsilon>0} \subset L^p(0, T; W_0^{1,p}(\Omega)) \cap L^p(0, T; L^p(\Omega)).$$

From (5.6) and (5.7) (see [14, theorem 5]) it follows that the set  $\{g(u_{p,q,\varepsilon})\}_{\varepsilon>0}$  is relatively compact in  $L^p(0, T; L^p(\Omega)) = L^p(G_T)$ . By the strict monotonicity of the function  $g$  the set  $\{u_{p,q,\varepsilon}\}_{\varepsilon>0}$  is relatively compact in  $L^p(G_T)$ .

**5.2. Entropy measure-valued solution  $(u, \nu_{\mathbf{x},t})$ .**

We put  $\phi(\mathbf{x}, t) = \partial_1 H(u_{p,q,\varepsilon}(\mathbf{x}, t), k) \gamma(\mathbf{x}, t)$  in (3.3b) and integrate by parts:

$$(5.8) \quad - \int_{G_T} (Q(u_{p,q,\varepsilon}, k) \partial_t \gamma + \Phi(u_{p,q,\varepsilon}, k) \cdot \nabla \gamma - \partial_1^2 H(u_{p,q,\varepsilon}, k) (|\nabla u_{p,q,\varepsilon}|^p + \varepsilon |\partial_t u_{p,q,\varepsilon}|^q) \gamma - \partial_1 H(u_{p,q,\varepsilon}, k) |\nabla u_{p,q,\varepsilon}|^{p-2} \nabla u_{p,q,\varepsilon} \cdot \nabla \gamma - \varepsilon \partial_1 H(u_{p,q,\varepsilon}, k) |\partial_t u_{p,q,\varepsilon}|^{q-2} \partial_t u_{p,q,\varepsilon} \partial_t \gamma) \, d\mathbf{x} dt = 0,$$

where  $\gamma \in C_0^\infty(G_T)$ ,  $\gamma \geq 0$ , see (FB.3) in Definition 3. It follows from Definition 1 and estimates (3.4) and (3.5) that the term  $\varepsilon \partial_1^2 H(u_{p,q,\varepsilon}, k) |\partial_t u_{p,q,\varepsilon}|^q$  is integrable and positive almost everywhere on  $G_T$ :

$$(5.9) \quad - \int_{G_T} (Q(u_{p,q,\varepsilon}, k) \partial_t \gamma + \Phi(u_{p,q,\varepsilon}, k) \cdot \nabla \gamma - \partial_1^2 H(u_{p,q,\varepsilon}, k) |\nabla u_{p,q,\varepsilon}|^p \gamma - \partial_1 H(u_{p,q,\varepsilon}, k) |\nabla u_{p,q,\varepsilon}|^{p-2} \nabla u_{p,q,\varepsilon} \cdot \nabla \gamma - \varepsilon \partial_1 H(u_{p,q,\varepsilon}, k) |\partial_t u_{p,q,\varepsilon}|^{q-2} \partial_t u_{p,q,\varepsilon} \partial_t \gamma) \, d\mathbf{x} dt \leq 0.$$

Because of estimates (3.4) and (3.5) the set  $\{u_{p,q,\varepsilon}\}_{\varepsilon>0}$  is uniformly bounded in  $L^p(0, T; W_0^{1,p}(\Omega))$ , the sequence  $\nabla u_{p,q,\varepsilon_l}$  converges weakly to  $\nabla u$  in  $L^p(G_T; \mathbb{R}^d)$  as  $l \rightarrow \infty$  and  $\varepsilon_l \rightarrow 0+$ . Moreover, there exists a gradient Young measure  $\{\nu_{\mathbf{x},t}\}$  and we can express the following weak limit with the help of [15]:

$$\partial_1 H(u_{p,q,\varepsilon}, k) |\nabla u_{p,q,\varepsilon}(\mathbf{x}, t)|^{p-2} \nabla u_{p,q,\varepsilon}(\mathbf{x}, t) \rightharpoonup \partial_1 H(u, k) \langle \nu_{\mathbf{x},t}, S_p(\cdot) \rangle = \partial_1 H(u, k) \int_{\mathbb{R}^d} |\xi|^{p-2} \xi \, d\nu_{\mathbf{x},t}.$$

Let  $\nabla u = \int_{\mathbb{R}^d} \xi \, d\nu_{\mathbf{x},t}$ . Since

$$L(\mathbf{x}, t, \xi) = \partial_1^2 H(u(\mathbf{x}, t), k) |\xi|^p \gamma(\mathbf{x}, t)$$

is convex in  $\xi \in \mathbb{R}^d$ , we have the following result (see [16, Theorem 4.3])

$$(5.10) \quad \int_{G_T} \partial_1^2 H(u, k) |\nabla u|^p \gamma \, d\mathbf{x} dt \leq \int_{G_T} \partial_1^2 H(u, k) \langle \nu_{\mathbf{x},t}, |\cdot|^p \rangle \gamma \, d\mathbf{x} dt \leq \liminf_{\varepsilon_l \rightarrow +0} \int_{G_T} \partial_1^2 H(u_{\varepsilon_l}, k) |\nabla u_{p,q,\varepsilon_l}|^p \gamma \, d\mathbf{x} dt.$$

Moreover, we have

$$(5.11) \quad \left| \int_{G_T} \varepsilon \partial_1 H(u_{p,q,\varepsilon}, k) |\partial_t u_{p,q,\varepsilon}|^{q-2} \partial_t u_{p,q,\varepsilon} \partial_t \gamma \, d\mathbf{x} dt \right| \leq \varepsilon^{\frac{1}{q}} \left( \int_{G_T} \varepsilon |\partial_t u_{p,q,\varepsilon}|^q \, d\mathbf{x} dt \right)^{\frac{q-1}{q}} \left( \int_{G_T} |\partial_1 H(u_{p,q,\varepsilon}, k) \partial_t \gamma|^q \, d\mathbf{x} dt \right)^{\frac{1}{q}} \leq \varepsilon^{\frac{1}{q}} \left( C_{(4.20)} \right)^{\frac{q-1}{q}} \left( \int_{G_T} |\partial_t \gamma|^q \, d\mathbf{x} dt \right)^{\frac{1}{q}} \sup_{|z| \leq M} |\partial_1 H(z, k)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0+.$$

Since the set  $\{u_{p,q,\varepsilon}\}_{\varepsilon>0}$  is uniformly bounded in  $L^\infty(G_T)$  and it is relatively compact in  $L^p(0, T; L^p(\Omega))$  (see subsection 5.1), from this set we can select a subsequence  $\{u_{p,q,\varepsilon_l}\}_{l \in \mathbb{N}}$  which has a limit  $u \in L^\infty(G_T) \cap L^p(0, T; L^p(\Omega))$ . Let function  $u_{p,q,\varepsilon_l}$  satisfy inequality (5.9) when  $\varepsilon = \varepsilon_l$ . It is easy to see that the function  $u$  satisfies the inequalities

$$\|u\|_{L^\infty(G_T)} \leq M, \quad \int_{G_T} |\nabla u|^p \, d\mathbf{x} dt \leq C_{(4.20)}.$$

Therefore, a limit point of a subsequence  $\{u_{p,q,\varepsilon_l}\}_{l \in \mathbb{N}}$  is an entropy measure-valued solution to equation (3.6). Moreover, a gradient Young measure  $\nu_{\mathbf{x},t}$  is associated with a subsequence  $\{\nabla u_{p,q,\varepsilon_l}\}_{l \in \mathbb{N}}$ .

**5.3. Essential limits (3.10) and (3.11).**

We are going to prove the existence of essential limits (3.10) and (3.11) with the help of boundary entropy-entropy flux triples from Example 2 (see section 2).

We put  $\gamma(t, \mathbf{x}) = \alpha(t)\beta(\mathbf{x})$  in inequality (3.9), where  $\alpha \in C_0^\infty((0, T))$ ,  $\beta \in C_0^\infty(\Omega)$ ,  $\alpha \geq 0$ ,  $\beta \geq 0$ . Here we follow the idea represented in [11]. It can be shown that the function

$$f_{\delta,w,k,\beta}(t) = \int_{\Omega} Q_{\delta,w}(u(\mathbf{x}, t), k)\beta(\mathbf{x}) \, d\mathbf{x}$$

belongs to  $BV((0, T))$ . Moreover, similar results were proved in [8], [9]. Let  $\bigcup_i E_i = \Omega$ . We replace smooth function  $\beta(\mathbf{x})$  with a simple function  $\sum_i \beta_i \chi_{E_i}(\mathbf{x})$ . Moreover, initial data  $u_0$  can be approximated by  $\sum_i w_i \chi_{E_i}(\mathbf{x})$ . Therefore

$$\left| \int_{\Omega} Q_{0,u_0(\mathbf{x})}(u(\mathbf{x}, t), k)\beta(\mathbf{x}) \, d\mathbf{x} - \sum_i \int_{E_i} Q_{\delta,w_i}(u(\mathbf{x}, t), k)\beta_i \, d\mathbf{x} \right| \rightarrow 0$$

for a.e.  $t \in (0, T)$  as  $\sum_i w_i \chi_{E_i}(\mathbf{x}) \xrightarrow{\Omega} u_0(\mathbf{x})$ ,  $\sum_i \beta_i \chi_{E_i}(\mathbf{x}) \xrightarrow{\Omega} \beta(\mathbf{x})$ ,  $\delta \rightarrow 0+$ . The existence of (3.11) is proved in the same way.

**5.4. On Remark 2.**

It is important to note that it is impossible to deduce boundary conditions (3.13)–(3.14) with the help of methods invented in [11]. Consider the functions  $s$ ,  $\xi_K \in C_0([0, T])$ , introduced in [11]:

$$(5.12) \quad s(t) = \min\{t, \frac{T}{10}, T - t\}, \quad \xi_K(t) = 1 - \exp(-\frac{s(t)}{K}), \quad t \in [0, T].$$

Due to the presence of  $\varepsilon \partial_t (|\partial_t u_{p,q,\varepsilon}|^{q-2} \partial_t u_{p,q,\varepsilon})$  in equation (3.1), we can not even formulate the analogous lemma for  $u_{p,q,\varepsilon}$  when  $q \neq 2$ ,  $q > 1$ .

**Lemma 1.** *For any value of the positive parameter  $\varepsilon$  the inequality holds*

$$(5.13) \quad - \int_{G_T} (Q(u_{p,2,\varepsilon}, k)\partial_t \gamma + \Phi(u_{p,2,\varepsilon}, k) \cdot \nabla \gamma + \varepsilon H(u_{p,2,\varepsilon}, k)\partial_t^2 \gamma - \partial_1 H(u_{p,2,\varepsilon}, k) |\nabla u_{p,2,\varepsilon}|^{p-2} \nabla u_{p,2,\varepsilon} \cdot \nabla \gamma - \partial_1^2 H(u_{p,2,\varepsilon}, k) |\nabla u_{p,2,\varepsilon}|^p \gamma) \xi_{K_\varepsilon} \, d\mathbf{x}dt \leq 2\varepsilon \int_{G_T} H(u_{p,2,\varepsilon}, k) \xi'_{K_\varepsilon} \partial_t \gamma \, d\mathbf{x}dt + \mathcal{A}_1 \int_{\Omega} (H(u_0(\mathbf{x}), k)\gamma(\mathbf{x}, 0) + H(u_T(\mathbf{x}), k)\gamma(\mathbf{x}, T)) \, d\mathbf{x},$$

where  $(H, Q, \Phi)$  is an arbitrary boundary entropy-entropy flux triple,  $\gamma \in C_0^\infty(\Omega \times \mathbb{R})$  is non-negative test function,  $k \in \mathbb{R}$ ,  $M = \max(\|u_0\|_{L^\infty(\Omega)}, \|u_T\|_{L^\infty(\Omega)})$ ,  $\mathcal{A}_1 = \max \|a'\|_{C([-M, M])}$ ,  $K_\varepsilon = \varepsilon/\mathcal{A}_1$ .

With the help of Lemma 1 proved in [9] we deduce

$$(5.14) \quad - \int_{G_T} (Q(u, k)\partial_t \gamma + \Phi(u, k) \cdot \nabla \gamma - \partial_1 H(u, k)\langle \nu_{\mathbf{x},t}, \mathbf{S}_p(\cdot) \rangle \cdot \nabla \gamma - \partial_1^2 H(u, k) |\nabla u|^p \gamma) \, d\mathbf{x}dt \leq \mathcal{A}_1 \int_{\Omega} (H(u_0(\mathbf{x}), k)\gamma(\mathbf{x}, 0) + H(u_T(\mathbf{x}), k)\gamma(\mathbf{x}, T)) \, d\mathbf{x}.$$

In comparison with (3.9), due to  $H_{0,u_0(\mathbf{x})}(u_0(\mathbf{x}), k) = H_{0,u_T(\mathbf{x})}(u_T(\mathbf{x}), k) = 0$  a.e. in  $\Omega$ , the right hand side of inequality (5.14) enables to define signs of essential limits (3.11) and (3.12). For details see subsection 5.3, also see [11], [8], [9].

## CONCLUSION

In present paper we have proved only the existence of an entropy measure-valued solution  $(u, \nu_{\mathbf{x},t})$  to the forward-backward  $p$ -parabolic equation. Function  $u$  has been obtained in the form  $\lim_{\varepsilon \rightarrow 0^+} u_{p,q,\varepsilon}$  where  $u_{p,q,\varepsilon}$  is a weak solution to problem  $\Pi_{p,q,\varepsilon}$ . We have not derived boundary conditions in the case  $q \neq 2$ ,  $q > 1$ , see Remark 2 in section 3.

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