JEPE Vol 2, 2016, p. 27-36

Orthogonal ⊥

THE ROLE OF NON-NEGATIVE POLYNOMIALS FOR RANK-ONE CONVEXITY AND QUASI CONVEXITY

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ABSTRACT. We stress the relationship between the non-negativeness of polynomials and quasi convexity and rank-one convexity. In particular, we translate the celebrated theorem of Hilbert ([3]) about non-negativeness of polynomials and sums of squares, into a test for rank-one convex functions defined on 2×2-matrices. Even if the density for an integral functional is a fourth-degree, homogeneous polynomial, quasi convexity cannot be reduced to the non-negativeness of polynomials of a fixed, finite number of variables.

1. INTRODUCTION

It is well-known that the quasi convexity condition for a density $\phi(\mathbf{F}) : \mathbf{M}^{m \times N} \to \mathbf{R}$, expressed through the inequality

$$\int_Q \phi(\mathbf{F} + \nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x} \geq \phi(\mathbf{F}) \tag{1.1}$$

valid for every $\mathbf{F} \in \mathbf{M}^{m \times N}$, and every Q-periodic test field $\mathbf{u} : Q \to \mathbf{R}^m$, is the necessary and sufficient property for the integral functional

$$I(\mathbf{v}) = \int_\Omega \phi(\nabla \mathbf{v}(\mathbf{y}))\, d\mathbf{y} \tag{1.2}$$

to be (sequentially) weak lower semicontinuous ([7]). This, in turn, is one of the important ingredients of the Direct Method of the Calculus of Variations to show existence of minimizers ([2], [8]) for integral functionals like the one in (1.2). Q is the unit cube in $\mathbf{R}^N$, while Ω is a general, bounded, regular domain in $\mathbf{R}^N$.

This quasi convexity condition is hard to understand. Intimately related concepts, like polyconvexity and rank-one convexity, were introduced and examined throughout the years. See [1], [2]. In particular, a main open question that remains to be answered is the equivalence of rank-one convexity and quasi convexity. It was shown not to be the case for $m \geq 3$ in [10], but still remains unsolved for the case $m = 2$.

2010 *Mathematics Subject Classification.* Primary: 49J45, 49J10; Secondary: 14P99.

Key words and phrases. Rank-one convexity, quasi convexity, non-negative polynomials.

Received 20/06/2016, Accepted 20/06/2016.

Research supported by National Funds through Fundação para a Ciência e a Tecnologia by UID/MAT/04674/2013 - CIMA (L. Bandeira).

Research supported under grant number MTM2013-47053-P of the MINECO - Spain (P. Pedregal).

If we want to stress the dependence of the inequality (1.1) both on $\mathbf{F}$ and on $\mathbf{u}$, we would write

$$\Phi(\mathbf{F}, \mathbf{u}) \equiv \int_Q [\phi(\mathbf{F} + \nabla \mathbf{u}(\mathbf{x})) - \phi(\mathbf{F})]\, d\mathbf{x} \geq 0.$$

Put it in this form, we see that a certain functional depending on $(\mathbf{F}, \mathbf{u})$ should be non-negative. To clarify the dependence on the field $\mathbf{u}$, we can write, taking advantage of periodicity,

$$\mathbf{u}(\mathbf{x}) = \frac{1}{2\pi} \sum_{\mathbf{n} \in \mathbf{Z}^N} \sin(2\pi \mathbf{n} \cdot \mathbf{x})\, \mathbf{a_n}, \quad \mathbf{a_n} \in \mathbf{R}^m,$$

$$\nabla \mathbf{u}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbf{Z}^N} \cos(2\pi \mathbf{n} \cdot \mathbf{x})\, \mathbf{a_n} \otimes \mathbf{n},$$

and so

$$\Phi(\mathbf{F}, \{\mathbf{a_n}\}) = \int_Q \left[\phi\left(\mathbf{F} + \sum_{\mathbf{n} \in \mathbf{Z}^N} \cos(2\pi \mathbf{n} \cdot \mathbf{x})\, \mathbf{a_n} \otimes \mathbf{n} \right) - \phi(\mathbf{F}) \right] d\mathbf{x}.$$

If we further restrict the nature of ϕ, to be a polynomial of a certain degree, then Φ itself will be a polynomial of the same degree on a certain number of variables (possibly infinite), that must be non-negative. We, hence, see that the issue of the non-negativeness of polynomials might have some relevance for quasi convexity.

As a matter of fact, the non-negativeness of polynomials is a very old subject but still quite alive. It is one main field of research in Real Algebraic Geometry with many applications in different areas within Mathematics and outside Mathematics. See for instance the nice, recent account [5]. Indeed, the issue of the non-negativeness of polynomials and rational functions was the subject of Hilbert's 17th problem ([4]). This problem was motivated by his celebrated theorem on non-negative forms ([3]), and sum-of-squares criteria. Today, it is known that the problem of deciding the non-negativeness of a multivariate polynomial (even quartic) is a NP-hard problem ([9]), but there is an increasing body of knowledge about this important problem in various contexts and circumstances ([5]).

Our main result, however, deals with rank-one convexity which is a necessary condition for quasi convexity. It is usually formulated by requiring that the sections

$$t \mapsto \phi(\mathbf{F} + t\mathbf{a} \otimes \mathbf{n})$$

be convex for arbitrary matrices $\mathbf{F}$, and vectors $\mathbf{a}$, $\mathbf{n}$. If ϕ is smooth, rank-one convexity can, equivalently, be formulated in the form of the so-called Legendre-Hadamard condition

$$\nabla^2 \phi(\mathbf{F}) : (\mathbf{a} \otimes \mathbf{n}) \otimes (\mathbf{a} \otimes \mathbf{n}) \geq 0$$

again for arbitrary matrices $\mathbf{F}$, and vectors $\mathbf{a}$, $\mathbf{n}$.

Our main result can be formulated in the following terms. Consider $\phi : \mathbf{M}^{2\times 2} \to \mathbf{R}$, a smooth ($\mathcal{C}^2$) function, and put

$$\phi^-(\mathbf{F}) = \sup_{\mathbf{G}} \{-\nabla^2 \phi(\mathbf{F}) : \mathbf{G} \otimes \mathbf{G} : \det \mathbf{G} = -1\},$$

$$\phi^+(\mathbf{F}) = \inf_{\mathbf{G}} \{\nabla^2 \phi(\mathbf{F}) : \mathbf{G} \otimes \mathbf{G} : \det \mathbf{G} = 1\}.$$

Theorem 1. *Such ϕ is rank-one convex if and only if*

$$\phi^-(\mathbf{F}) \leq \phi^+(\mathbf{F})$$

for each matrix $\mathbf{F}$.

By making use of Hilbert's theorem about non-negative polynomials and sums of squares, which we briefly recall in Section 2, we are able to translate it into a test for rank-one convexity for smooth densities defined on $\mathbf{M}^{2\times 2}$. The connection between these two areas is explained in Section 3 through the investigation of quadratic forms. We then extend the main such fact to general, smooth densities (Section 4), and prove our Theorem 1 above. Finally, in Section 5, we focus on quasi convexity by assessing to what extent these ideas could lead somewhere.

2. Non-negative polynomials

The subject of non-negative polynomials has known a considerable expansion ever since the pioneering work of D. Hilbert ([3]). For theoretical as well as practical reasons, it is important to be able to decide when polynomials in several variables are non-negative. We will see one such situation in this contribution.

A real polynomial $p(\mathbf{x})$ in n variables $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is said to be non-negative if $p(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbf{R}^n$. From a representation like

$$p(\mathbf{x}) = \sum_i p_i(\mathbf{x})^2, \quad \text{each } p_i\text{, a polynomial,} \tag{2.1}$$

one can immediately conclude that p is indeed non-negative. Because of this reason, and lacking other criteria, the sum-of-squares test became the main focus of attention to decide the non-negativity of polynomials. Hilbert ([3]) classified all situations in which this test is valid, i.e., those situations in which non-negativity of polynomials is equivalent to being decomposable as a sum of squares. To formulate such important result in more precise terms, we will talk about "forms" (like quadratic forms), as being the corresponding homogeneous representation of any polynomial, by introducing an additional variable, and dividing all monomials by a suitable power of such new variable, according to the simple rule

$$\tilde{p}(\tilde{\mathbf{x}}) = x_{n+1}^d p(\mathbf{x}/x_{n+1}), \quad \tilde{\mathbf{x}} = (\mathbf{x}, x_{n+1}),$$

where d is the degree of p. Assume that p is a polynomial of degree d in n variables, with associated form $\tilde{p}$. The result of Hilbert is:

Theorem 2 ([3]). *Non-negative forms are the same as sums-of squares, in the following three cases:*

(1) $n = 1$*: polynomials of arbitrary degree in one variable;*
(2) $d = 2$*: quadratic forms in any number of variables;*
(3) $d = 4$, $n = 2$*: quartic forms in three variables, or quartic polynomials in two variables.*

In all other cases, there are non-negative forms which are not sums of squares.

Hilbert later, and motivated by his result in [3], proposed his 17th problem in the famous list [4]:

> Does every non-negative polynomial have a representation as a sum of squares of "rational" functions?

Equivalently, given $p(\mathbf{x})$, he was asking about the existence of a polynomial $q(\mathbf{x})$ so that $q(\mathbf{x})^2 p(\mathbf{x})$ is a sum of squares of polynomials. Artin proved in 1927 that this is so.

The recent development of this area is astonishing. It has become quite specialized. See [5], and references therein.

3. The fundamental fact

We pretend to find some interesting application of case (3) in Hilbert's result, and relate it to rank-one convexity. We will restrict attention to 2×2-matrices, and densities defined on them.

Consider a (constant) quadratic form associated with the symmetric 4×4-matrix $\mathbf{Q}$. We will use henceforth the identification

$$\mathbf{F} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \mapsto \mathbf{F} = (F_{11}, F_{12}, F_{21}, F_{22}), \tag{3.1}$$

so that $\mathbf{Q}$ is understood as a quadratic form acting on four-component vectors. Put

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and note that $\mathbf{D} : \mathbf{F} \otimes \mathbf{F} = 2 \det \mathbf{F}$ if $\mathbf{F}$ is a 2×2-matrix understood through the identification (3.1). Given a 4×4-symmetric matrix $\mathbf{Q}$, we refer to it as being rank-one convex if the associated quadratic form

$$\phi(\mathbf{F}) = \mathbf{Q} : \mathbf{F} \otimes \mathbf{F}$$

is a rank-one convex function. Our main lemma follows.

Lemma 1. *The quadratic form $\mathbf{Q}$ is rank-one convex if and only if there is a number α such that $\mathbf{Q} = \mathbf{S} + \alpha \mathbf{D}$, and $\mathbf{S}$ is non-negative definite.*

Though we use Hilbert's theorem to prove this lemma, it was already shown in an elementary way by P. Marcellini in [6].

Proof. The "if" part is immediate. Indeed, if $\mathbf{Q} = \mathbf{S} + \alpha \mathbf{D}$ with $\mathbf{S}$ non-negative, then for a rank-one matrix $\mathbf{G}$

$$\mathbf{Q} : \mathbf{G} \otimes \mathbf{G} = \mathbf{S} : \mathbf{G} \otimes \mathbf{G} + \alpha \mathbf{D} : \mathbf{G} \otimes \mathbf{G} = \mathbf{S} : \mathbf{G} \otimes \mathbf{G} + 2\alpha \det \mathbf{G} = \mathbf{S} : \mathbf{G} \otimes \mathbf{G} \geq 0.$$

The remarkable fact is the converse.

Suppose $\mathbf{Q}$ is rank-one convex, i.e.

$$\mathbf{Q} : (\mathbf{x} \otimes \mathbf{y}) \otimes (\mathbf{x} \otimes \mathbf{y}) \geq 0 \tag{3.2}$$

for arbitrary vectors $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$. Condition (3.2) is a short-hand form of the typical rank-one convex condition

$$\sum_{i,j,k,l=1,2} Q_{ijkl} x_i y_j x_k y_l \geq 0$$

under the identification (3.1). Because of homogeneity, we can equivalently put

$$\mathbf{Q} : (\tilde{\mathbf{x}} \otimes \tilde{\mathbf{y}}) \otimes (\tilde{\mathbf{x}} \otimes \tilde{\mathbf{y}}) \geq 0$$

for $\tilde{\mathbf{x}} = (x, 1)$, $x = x_1/x_2$, and likewise for $\tilde{\mathbf{y}} = (y, 1)$. In this way, this last inequality is telling us that the fourth-degree polynomial

$$P_4(x, y) = \mathbf{Q} : [(x, 1) \otimes (y, 1)] \otimes [(x, 1) \otimes (y, 1)]$$

in the two variables (x, y) is non-negative. We can also write

$$P_4(x, y) = \mathbf{Q} : \mathbf{X} \otimes \mathbf{X}, \quad \mathbf{X} = (xy, x, y, 1).$$

By Hilbert's theorem, we can find, at least, one representation of P_4 as a sum of squares. But because $\mathbf{D} : \mathbf{X} \otimes \mathbf{X} = 0$, all possible representations of P_4 are of the form

$$P_4(x, y) = (\mathbf{Q} - \alpha\mathbf{D}) : \mathbf{X} \otimes \mathbf{X}$$

for $\alpha \in \mathbf{R}$. This can be checked more explicitly if we write

$$P_4(x, y) = \begin{pmatrix} xy & x & y & 1 \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{pmatrix} \begin{pmatrix} xy \\ x \\ y \\ 1 \end{pmatrix},$$

perform the multiplication with some care, and take into account the symmetry of $\mathbf{Q}$. Note how the coefficient corresponding to the monomial xy is obtained through the entries Q_{14}, Q_{23}, Q_{32}, Q_{41}, but there is no further ambiguity or freedom. Hilbert's theorem implies then that there should be at least one real number α, such that

$$(\mathbf{Q} - \alpha\mathbf{D}) : \mathbf{X} \otimes \mathbf{X} = (\mathbf{C}\mathbf{X}) \otimes (\mathbf{C}\mathbf{X}), \quad (\mathbf{Q} - \alpha\mathbf{D}) = \mathbf{C}^T\mathbf{C},$$

for a certain matrix $\mathbf{C}$. Hence $\mathbf{Q} - \alpha\mathbf{D}$ is non-negative definite. This proves the claim. □

The "if" part of this lemma holds for more general situations where dimensions of matrices are larger. The crucial strength, however, is the "only if" part which is only valid for 2×2-matrices.

4. Some consequences and some examples

The following is a classical result for quadratic forms.

Theorem 3 ([2]). *Let $\phi(\mathbf{F}) = \mathbf{F}^T\mathbf{A}\mathbf{F} = \mathbf{A} : \mathbf{F} \otimes \mathbf{F}$ be a quadratic form. Then*

(1) *ϕ is rank-one convex iff ϕ is quasi convex.*
(2) *if one of the two dimensions is 2, then*

$$\phi, \text{ polyconvex} \iff \phi, \text{ quasi convex} \iff \phi, \text{ rank-one convex.}$$

(3) *if both dimensions are greater than 3, in general rank-one convexity does not imply polyconvexity.*

Through Lemma 1, the second statement of this theorem admits some improvement in the case in which both dimensions are 2.

Corollary 1. *Every rank-one convex quadratic form on 2×2-matrices is the sum of a convex quadratic form, and a multiple of the determinant (and so it is polyconvex).*

This main fact can be used directly for non-quadratic, smooth functions.

Corollary 2. *A smooth ($\mathcal{C}^2$) function $\phi : \mathbf{M}^{2\times 2} \to \mathbf{R}$ is rank-one convex if and only if there is a scalar function $\alpha : \mathbf{M}^{2\times 2} \to \mathbf{R}$ and a symmetric, non-negative definite matrix field $\mathbf{S} : \mathbf{M}^{2\times 2} \to \mathbf{M}^{4\times 4}$ such that*

$$\nabla^2\phi(\mathbf{F}) = \mathbf{S}(\mathbf{F}) + \alpha(\mathbf{F})\mathbf{D}.$$

We can further explore the condition for a matrix $\mathbf{Q}$ to enjoy the property that there is some real α so that $\mathbf{Q} - \alpha\mathbf{D}$ is non-negative definite.

Proposition 1. *There is some real number α such that $\mathbf{Q}-\alpha\mathbf{D}$ is non-negative definite if and only if*

$$\sup_{\mathbf{G}}\{-\mathbf{Q}:\mathbf{G}\otimes\mathbf{G}:\det\,\mathbf{G}=-1\}\in\mathbf{R}\le\inf_{\mathbf{G}}\{\mathbf{Q}:\mathbf{G}\otimes\mathbf{G}:\det\,\mathbf{G}=1\}\in\mathbf{R}.$$

Proof. The proof is easy if we use homogeneity in the condition

$$0\le(\mathbf{Q}-\alpha\mathbf{D}):\mathbf{G}\otimes\mathbf{G}=\mathbf{Q}:\mathbf{G}\otimes\mathbf{G}-2\,\alpha\,\det\mathbf{G}$$

for every matrix $\mathbf{G}$. □

The proof of Theorem 1 is the result of putting together Corollary 2, and this las proposition.

As an illustration, we reexamine two typical examples. They can be found in [2]. The first one is

$$\phi(\mathbf{F})=|\mathbf{F}|^4-2(\det\mathbf{F})^2.$$

It is elementary to find that

$$\frac{1}{4}\nabla^2\phi(\mathbf{F}):\mathbf{G}\otimes\mathbf{G}=2(\mathbf{F}:\mathbf{G})^2+|\mathbf{F}|^2|\mathbf{G}|^2-(\mathbf{DF}:\mathbf{G})^2-2\det\mathbf{F}\,\det\mathbf{G}.$$

By the Cauchy-Schwarz classical inequality, we realize that

$$\phi^-(\mathbf{F})=\sup_{\mathbf{G}}\{-\nabla^2\phi(\mathbf{F}):\mathbf{G}\otimes\mathbf{G}:\det\,\mathbf{G}=-1\}\le-8\det\mathbf{F},$$
$$\phi^+(\mathbf{F})=\inf_{\mathbf{G}}\{\nabla^2\phi(\mathbf{F}):\mathbf{G}\otimes\mathbf{G}:\det\,\mathbf{G}=1\}\ge-8\det\mathbf{F},$$

and so ϕ is indeed rank-one convex. The second one is

$$\phi(\mathbf{F})=|\mathbf{F}|^4-\frac{4}{\sqrt{3}}|\mathbf{F}|^2\det\mathbf{F}. \tag{4.1}$$

It is also straightforward to check that

$$\frac{1}{4}\nabla^2\phi(\mathbf{F})=2\mathbf{F}\otimes\mathbf{F}+|\mathbf{F}|^2\mathbf{1}-\frac{2}{\sqrt{3}}\det\mathbf{F}\,\mathbf{1}-\frac{4}{\sqrt{3}}\mathbf{F}\otimes\mathbf{DF}-\frac{1}{\sqrt{3}}|\mathbf{F}|^2\mathbf{D},$$

where $\mathbf{1}$ stands for the identity matrix of size 4×4.

Corollary 2 enables us to change the last term to an arbitrary contribution of the form $\alpha(\mathbf{F})\mathbf{D}$ in order to produce a non-negative definite matrix

$$2\mathbf{F}\otimes\mathbf{F}+|\mathbf{F}|^2\mathbf{1}-\frac{2}{\sqrt{3}}\det\mathbf{F}\,\mathbf{1}-\frac{4}{\sqrt{3}}\mathbf{F}\otimes\mathbf{DF}-\frac{1}{\sqrt{3}}|\mathbf{F}|^2\mathbf{D}+\alpha(\mathbf{F})\mathbf{D}. \tag{4.2}$$

If we set $\alpha(\mathbf{F})$ to the form $(\alpha-4/\sqrt{3})|\mathbf{F}|^2$, for α a constant, a few careful calculations yield that the eigenvalues of (4.2) are

$$\lambda=\left(2\pm\frac{\alpha}{2}\right)|\mathbf{F}|^2-\frac{4}{\sqrt{3}}\det(\mathbf{F})$$
$$\lambda=4|\mathbf{F}|^2-4\sqrt{3}\,\det(\mathbf{F})\pm\frac{1}{2}\sqrt{\left[\left(\alpha+\frac{8}{\sqrt{3}}\right)^2+16\right]|\mathbf{F}|^4-16\left(\alpha+\frac{8}{\sqrt{3}}\right)|\mathbf{F}|^2\det(\mathbf{F})}.$$

For the choice $\alpha=-2/\sqrt{3}$, it turns out that these eigenvalues are non-negative: for the first pair of eigenvalues, it is clear that the minimum is 0, and that it is attained for $\mathbf{F}=\mathbf{0}$, while for the second pair, some elementary computations lead also to 0 as the minimum, attained when $\det(\mathbf{F})=(\sqrt{3}/4)|\mathbf{F}|^2$.

5. Quasi convexity

Let $\phi : \mathbf{M}^{2\times 2} \to \mathbf{R}$ be a density. As stated in the Introduction, the quasi convexity condition amounts to having

$$\int_Q \phi(\mathbf{F} + \nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x} \ge \phi(\mathbf{F})$$

for every matrix $\mathbf{F}$, and every periodic mapping $\mathbf{u} : Q \to \mathbf{R}^2$. Q is the unit cube in $\mathbf{R}^2$. Put

$$\Phi(\mathbf{F}, \mathbf{u}) \equiv \int_Q \phi(\mathbf{F} + \nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x} - \phi(\mathbf{F}).$$

Quasi convexity takes place if the functional Φ is always non-negative. Variable $\mathbf{F}$ does not require a particular analysis as it is a finite-dimensional variable, but $\mathbf{u}$ does. In fact, information on how ϕ behaves on sums of the form $\mathbf{F} + \mathbf{G}$, $\mathbf{G} = \nabla \mathbf{u}$, might be helpful in saying something relevant.

To see this issue more clearly, let us review the quadratic case in which we take

$$\phi(\mathbf{F}) = P(\mathbf{F}) = (1/2)\mathbf{F}^T \mathbb{A} \mathbf{F}, \tag{5.1}$$

a quadratic form for a 4×4-, symmetric matrix $\mathbb{A}$. $\mathbf{F}$ is identified with a four-component vector through (3.1). We can write

$$\mathbf{P}(\mathbf{G}; \mathbf{F}) \equiv P(\mathbf{F} + \mathbf{G}) - P(\mathbf{F}) = \frac{1}{2}(\mathbf{F}^T + \mathbf{G}^T)\mathbb{A}(\mathbf{F} + \mathbf{G}) - \frac{1}{2}\mathbf{F}^T \mathbb{A} \mathbf{F}.$$

Because the variable $\mathbf{G}$ stands for the gradient $\nabla \mathbf{u}(\mathbf{x})$, and a subsequence integration over the unit cube Q is to be performed, we immediately see that periodicity leads to the vanishing of the two integrals

$$\int_Q \nabla \mathbf{u}(\mathbf{x})^T \mathbb{A} \mathbf{F}\, d\mathbf{x}, \quad \int_Q \mathbf{F}^T \mathbb{A} \nabla \mathbf{u}(\mathbf{x})\, d\mathbf{x}.$$

Hence,

$$\int_Q \mathbf{P}(\nabla \mathbf{u}(\mathbf{x}); \mathbf{F})\, d\mathbf{x} = \frac{1}{2} \int_Q \nabla \mathbf{u}(\mathbf{x})^T \mathbb{A} \nabla \mathbf{u}(\mathbf{x})\, d\mathbf{x}.$$

If, as we did earlier,

$$\mathbf{u}(\mathbf{x}) = \frac{1}{2\pi} \sum_{\mathbf{n} \in \mathbf{Z}^N} \sin(2\pi \mathbf{n} \cdot \mathbf{x})\, \mathbf{a_n}, \quad \mathbf{a_n} \in \mathbf{R}^m,$$

$$\nabla \mathbf{u}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbf{Z}^N} \cos(2\pi \mathbf{n} \cdot \mathbf{x})\, \mathbf{a_n} \otimes \mathbf{n}, \tag{5.2}$$

we find

$$\int_Q \mathbf{P}(\nabla \mathbf{u}(\mathbf{x}); \mathbf{F})\, d\mathbf{x} = \frac{1}{2} \int_Q \sum_{\mathbf{n}, \mathbf{m} \in \mathbf{Z}^N} \int_Q \cos(2\pi \mathbf{n} \cdot \mathbf{x}) \cos(2\pi \mathbf{m} \cdot \mathbf{x})\, d\mathbf{x}\ (\mathbf{a_n} \otimes \mathbf{n})^T \mathbb{A} (\mathbf{a_m} \otimes \mathbf{m}).$$

But the integrals

$$\int_Q \cos(2\pi \mathbf{n} \cdot \mathbf{x}) \cos(2\pi \mathbf{m} \cdot \mathbf{x})\, d\mathbf{x}$$

vanish always except when $\mathbf{n} = \pm\mathbf{m}$, in which case the value is strictly positive. We conclude then that the quasi convexity condition

$$\int_Q \mathbf{P}(\nabla\mathbf{u}(\mathbf{x}); \mathbf{F})\, d\mathbf{x} \geq 0$$

is equivalent to

$$\sum_{\mathbf{n}\in\mathbf{Z}^N} (\mathbf{a_n} \otimes \mathbf{n})^T \mathbb{A}(\mathbf{a_n} \otimes \mathbf{n}) \geq 0.$$

The arbitrariness of the full family of coefficients $\{\mathbf{a_n}\}$ implies that the non-negativeness of the sum can only occur when each term is non-negative

$$(\mathbf{a} \otimes \mathbf{n})^T \mathbb{A}(\mathbf{a} \otimes \mathbf{n}) \geq 0$$

for all $\mathbf{a} \in \mathbf{R}^m$, $\mathbf{n} \in \mathbf{R}^N$. This is exactly the rank-one convex condition, and so, for a quadratic density as the one in (5.1), rank-one convexity is equivalent to quasi convexity.

We would like to explore how far this viewpoint might take us for a homogeneous, four-degree polynomial. More specifically, consider a fully symmetric, constant fourth-order tensor $\mathbf{T} : (\mathbf{R}^{2\times2})^2 \to \mathbf{R}$, acting on matrices, and take $P(\mathbf{X}) = \mathbf{T}(\mathbf{X}, \mathbf{X}, \mathbf{X}, \mathbf{X})$ for $\mathbf{X} \in \mathbf{R}^{2\times2}$, a homogeneous fourth degree polynomial in the entries of $\mathbf{X}$. Then, for

$$\mathbf{P}(\mathbf{G}; \mathbf{F}) \equiv P(\mathbf{F} + \mathbf{G}) - P(\mathbf{F})$$

we can write

$$\mathbf{P}(\mathbf{G}; \mathbf{F}) = \mathbf{T}(\mathbf{F} + \mathbf{G}, \mathbf{F} + \mathbf{G}, \mathbf{F} + \mathbf{G}, \mathbf{F} + \mathbf{G}) - \mathbf{T}(\mathbf{F}, \mathbf{F}, \mathbf{F}, \mathbf{F}),$$

that is

$$\mathbf{P}(\mathbf{G}; \mathbf{F}) = \mathbf{T}(\mathbf{G}, \mathbf{G}, \mathbf{G}, \mathbf{G}) + 4\mathbf{T}(\mathbf{G}, \mathbf{G}, \mathbf{G}, \mathbf{F}) + 6\mathbf{T}(\mathbf{G}, \mathbf{G}, \mathbf{F}, \mathbf{F}) + 4\mathbf{T}(\mathbf{G}, \mathbf{F}, \mathbf{F}, \mathbf{F}). \tag{5.3}$$

As indicated above, variable $\mathbf{G}$ stands for the gradient $\nabla\mathbf{u}(\mathbf{x})$ of a smooth, Q-periodic mapping, and we are interested in examining the sign of the functional

$$\Phi(\mathbf{F}, \mathbf{u}) \equiv \int_Q \mathbf{P}(\nabla\mathbf{u}(\mathbf{x}); \mathbf{F})\, d\mathbf{x}.$$

Due to the periodic boundary conditions on $\mathbf{u}$, the integral of the last term in (5.3) drops out, and we are left with

$$\begin{aligned} \Phi(\mathbf{F}, \mathbf{u}) = \int_Q & [\mathbf{T}(\nabla\mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x})) \\ & 4\mathbf{T}(\nabla\mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}), \mathbf{F}) + 6\mathbf{T}(\nabla\mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}), \mathbf{F}, \mathbf{F})]\, d\mathbf{x}. \end{aligned} \tag{5.4}$$

It is easy to write down necessary conditions for quasi convexity by simply selecting particular groups of terms in (5.2).

(1) For three terms, we can take $\mathbf{n}_3 = \mathbf{n}_1 \pm \mathbf{n}_2$, $\{\mathbf{n}_1, \mathbf{n}_2\}$, independent, $\mathbf{a}_i \in \mathbf{R}^m$,

$$\begin{aligned} \nabla\mathbf{u}(\mathbf{x}) = & \cos(2\pi\mathbf{n}_1 \cdot \mathbf{x})\, \mathbf{a}_1 \otimes \mathbf{n}_1 + \cos(2\pi\mathbf{n}_2 \cdot \mathbf{x})\, \mathbf{a}_2 \otimes \mathbf{n}_2 \\ & + \cos(2\pi\mathbf{n}_3 \cdot \mathbf{x})\, \mathbf{a}_3 \otimes \mathbf{n}_3, \end{aligned}$$

and derive necessary conditions by taking this gradient to the quasi convexity inequality.

(2) Similarly, for four terms, put $\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3=\mathbf{n}_4$, $\mathbf{a}_i\in\mathbf{R}^m$, and take the gradient

$$\begin{aligned}\nabla\mathbf{u}(\mathbf{x}) =&\cos(2\pi\mathbf{n}_1\cdot\mathbf{x})\,\mathbf{a}_1\otimes\mathbf{n}_1+\cos(2\pi\mathbf{n}_2\cdot\mathbf{x})\,\mathbf{a}_2\otimes\mathbf{n}_2\\&+\cos(2\pi\mathbf{n}_3\cdot\mathbf{x})\,\mathbf{a}_3\otimes\mathbf{n}_3+\cos(2\pi\mathbf{n}_4\cdot\mathbf{x})\,\mathbf{a}_4\otimes\mathbf{n}_4\end{aligned}$$

to the quasi convexity inequality.

In trying to say something interesting about sufficiency for quasi convexity, the crucial issue is whether there are basic families of gradients of the above form with a finite number of terms that do not interact with each other through the corresponding trigonometric integrals. Namely, if we put

$$\nabla\mathbf{u}=\nabla\mathbf{u}_1+\nabla\mathbf{u}_2,\quad \nabla\mathbf{u}_i=\sum_{\mathbf{n}\in\mathbf{Z}_i}\cos(2\pi\mathbf{n}\cdot\mathbf{x})\,\mathbf{a}_{\mathbf{n}}\otimes\mathbf{n},\quad i=1,2,\mathbf{Z}_i\subset\mathbf{Z}^2,\text{even},$$

and take this decomposition to (5.4), we would have terms of three kinds, according to the three terms for $\Phi(\mathbf{F},\mathbf{u})$. The first one is

$$\begin{aligned}\mathbf{T}(\nabla\mathbf{u},\nabla\mathbf{u},\mathbf{F},\mathbf{F}) =&\mathbf{T}(\nabla\mathbf{u}_1+\nabla\mathbf{u}_2,\nabla\mathbf{u}_1+\nabla\mathbf{u}_2,\mathbf{F},\mathbf{F})\\=&\mathbf{T}(\nabla\mathbf{u}_1,\nabla\mathbf{u}_1,\mathbf{F},\mathbf{F})+\mathbf{T}(\nabla\mathbf{u}_2,\nabla\mathbf{u}_2,\mathbf{F},\mathbf{F})\\&+2\mathbf{T}(\nabla\mathbf{u}_1,\nabla\mathbf{u}_2,\mathbf{F},\mathbf{F}).\end{aligned}$$

According to the above trigonometric integrals, we would like to have that the terms in $\nabla\mathbf{u}_1$ and $\nabla\mathbf{u}_2$ cannot interact with each other. This is possible as long as

$$\int_Q\cos(2\pi\mathbf{n}_1\cdot\mathbf{x})\,\cos(2\pi\mathbf{n}_2\cdot\mathbf{x})\,d\mathbf{x}=0$$

whenever $\mathbf{n}_1\in\mathbf{Z}_1$, $\mathbf{n}_2\in\mathbf{Z}_2$, and for this, it suffices to have $\mathbf{Z}_1\cap\mathbf{Z}_2=\emptyset$. Similarly, for cubic terms:

$$\begin{aligned}\mathbf{T}(\nabla\mathbf{u},\nabla\mathbf{u},\nabla\mathbf{u},\mathbf{F}) =&\mathbf{T}(\nabla\mathbf{u}_1+\nabla\mathbf{u}_2,\nabla\mathbf{u}_1+\nabla\mathbf{u}_2,\nabla\mathbf{u}_1+\nabla\mathbf{u}_2,\mathbf{F})\\=&\mathbf{T}(\nabla\mathbf{u}_1,\nabla\mathbf{u}_1,\nabla\mathbf{u}_1,\mathbf{F})+\mathbf{T}(\nabla\mathbf{u}_2,\nabla\mathbf{u}_2,\nabla\mathbf{u}_2,\mathbf{F})\\&+3\mathbf{T}(\nabla\mathbf{u}_1,\nabla\mathbf{u}_1,\nabla\mathbf{u}_2,\mathbf{F})+3\mathbf{T}(\nabla\mathbf{u}_1,\nabla\mathbf{u}_2,\nabla\mathbf{u}_2,\mathbf{F}),\end{aligned}$$

and so, we would like to have

$$\int_Q\cos(2\pi\mathbf{n}_1\cdot\mathbf{x})\,\cos(2\pi\mathbf{n}_2\cdot\mathbf{x})\,\cos(2\pi\mathbf{n}_3\cdot\mathbf{x})\,d\mathbf{x}=0$$

whenever $\mathbf{n}_1,\mathbf{n}_2\in\mathbf{Z}_1$, $\mathbf{n}_3\in\mathbf{Z}_2$, or $\mathbf{n}_1,\mathbf{n}_2\in\mathbf{Z}_2$, $\mathbf{n}_3\in\mathbf{Z}_1$. This condition forces to ensure that $\mathbf{n}_1,\mathbf{n}_2\in\mathbf{Z}_i$ implies $\mathbf{n}_1\pm\mathbf{n}_2\in\mathbf{Z}_i$, for each $i=1,2$, but then there is no way to separate a given gradient into two disjoint sums of terms not interacting through the cubic terms. The situation is even worse for fourth-degree terms, so it seems as if there is no hope of isolating a necessary and sufficient condition for quasi convexity for fourth-degree polynomials based on the non-negativity of certain polynomials of a finite number of variables.

References

[1] J.M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity* Arch. Rational Mech. Anal. **63** (1977), 337–403.

[2] Dacorogna, B. *Direct methods in the Calculus of Variations*, Springer, 2008 (second edition).

[3] D. Hilbert, *Über die Darstellung Definiter Formen als Summe von Formenquadraten* Mathematische Annalen, **32** (1888), 342–250.

[4] D. Hilbert, *Mathematische Probleme*, Lecture, Second Internat. Congr. Math. (Paris, 1900), Nachr. Ges. Wiss. Göttingen Math. Phys. KL. (1900), 253–297; English transl., *Bull. Amer. Math. Soc.* **8** (1902), 437–479; *Bull. (New Series) Amer. Math. Soc.* **37** (2000), 407–436.
[5] J. B. Laserre, *Moments, Positive Polynomials and Their Applications*, Imperial College Press, London, 2010.
[6] P. Marcellini, *Quasi convex quadratic forms in two dimensions* Appl. Math. Optim. **11** (1984), n 2, 183–189.
[7] C. B. Morrey, *Quasiconvexity and the lower semicontinuity of multiple integrals* Pacific J. Math. **2** (1952), 25–53.
[8] C. B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer 1966.
[9] J. Nie, *Discriminants and nonnegative polynomials*, J. Symb. Comp., **47** (2012), 167–191.
[10] V. Šverák, *Rank-one convexity does not imply quasiconvexity* Proc. Roy. Soc. Edinburgh Sect. **120 A** (1992), 293–300.

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