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NONLOCAL NEUMANN PROBLEM WITH CRITICAL EXPONENT FROM THE POINT OF VIEW OF THE TRACE

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ABSTRACT. In this work, we are concerned with questions of existence and multiplicity of solutions for a nonlocal and non-homogeneous Neumann boundary value problems involving the $p(x)$-Laplace operator and critical growth, from the point of view of the trace, via a truncation argument on generalized Lebesgue-Sobolev spaces.

Dedicated to Prof. David Kinderlehrer for his relevant mathematical achievements.

1. INTRODUCTION

In this work, we are going to study questions of existence and multiplicity of solutions for the nonlocal and non-homogeneous equation, under Neumann boundary condition involving critical growth from the point of view of the trace, given by

(1.1)
$$\begin{aligned} M\left(\psi(u)\right)\left(-div(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u\right) &= \lambda f(x,u)\left[\int_\Omega F(x,u)dx\right]^r - |u|^{h(x)-2}u \quad \text{in } \Omega,\\ M\left(\psi(u)\right)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} &= \gamma |u|^{q(x)-2}u \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain of $\mathbb{R}^N$, $\psi(u) = \displaystyle\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx$, $p, q, h \in C(\overline{\Omega})$, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, and $M : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions enjoying some conditions which will be stated later, $F(x,u) = \displaystyle\int_0^u f(x,\xi)d\xi$, $\dfrac{\partial u}{\partial \nu}$ is the outer unit normal derivative, λ, r, γ are real parameters, and $\Delta_{p(x)}$ is the $p(x)$-Laplace operator, that is,

$$\Delta_{p(x)}u = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\right), \qquad 1 < p(x) < N.$$

Problems like (1.1), involving the $p(x)$-Laplace operator, nonlinearities with non standard growth and generalized Lebesgue-Sobolev spaces have been studied by several authors, mainly under homogeneous Dirichlet boundary condition.

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Problems of this class have several interesting motivations from both physical and mathematical point of view. They arise, for instance, in the so-called model of motion of electrorheological fluids, characterized by their capability to change in drastic way the mechanical properties when influenced by an exterior electromagnetic field. See, for example, [23].

As we have said in [9], the first major discovery on electrorheological fluids is due to Willis Winslow in 1949.

These fluids have the interesting property that their viscosity depends on the electric field in the fluid. He noticed that in such fluids (for instance lithium polymetachrylate) viscosity is inversely proportional to the strength of the field. The fields induces string-like formations in the fluids, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. Electrorheological fluids have been used in robotic and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories.

For more information on this subject, the reader may consult [23] and the references therein.

Another application of such a kind of equation is related to image processing. See [24] and the references therein.

In this article, we discuss existence and multiplicity of solutions for the nonlocal Neumann problem (1.1) with critical growth on its boundary. We study the nonlocal condition for the two following important classes of functions: $M(\tau) = a + b\tau^{\eta}$ with $a \geq 0$, $b > 0$, $\eta \geq 1$ and $m_0 \leq M(\tau) \leq m_1$, where m_0 and m_1 are positive constants. Note that the original Kirchhoff term, $M(\tau) = a + b\tau$ with $a \geq 0$, $b > 0$, is included in our analysis.

We will study the problem with the following critical Sobolev exponent

$$p_*(x) = \frac{(N-1)p(x)}{N-p(x)}, \tag{1.2}$$

where p_* is a critical exponent from the point of view of the trace.

Problems in the form (1.1) are associated with the energy functional

$$\begin{aligned} J_{\lambda,\gamma}(u) \;&=\; \widehat{M}\left(\psi(u)\right) - \frac{\lambda}{r+1}\left[\int_{\Omega} F(x,u)dx\right]^{r+1} + \int_{\Omega} \frac{1}{h(x)}|u|^{h(x)}dx \\ &\quad -\gamma \int_{\partial\Omega} \frac{1}{q(x)}|u|^{q(x)}dS \end{aligned} \tag{1.3}$$

for all $u \in W^{1,p(x)}(\Omega)$, where $\widehat{M}(t) = \displaystyle\int_0^t M(\tau)d\tau$, dS denotes the boundary measure, and $W^{1,p(x)}(\Omega)$ is the generalized Lebesgue-Sobolev space whose precise definition and properties will be established in section 2.

Depending on the behavior of the functions p, q and h, the above functional is differentiable and its Fréchet-derivative is given by

$$\begin{aligned} J'_{\lambda,\gamma}(u)v \;&=\; M(\psi(u)) \int_{\Omega} \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv\right) dx + \int_{\Omega} |u|^{h(x)-2}uvdx \\ &\quad -\lambda\left[\int_{\Omega} F(x,u)dx\right]^{r} \int_{\Omega} f(x,u)vdx - \gamma \int_{\partial\Omega} |u|^{q(x)-2}uvdS \end{aligned} \tag{1.4}$$

for all $u, v \in W^{1,p(x)}(\Omega)$. So, $u \in W^{1,p(x)}(\Omega)$ is a weak solution of problem (1.1) if, and only if, u is a critical point of $J_{\lambda,\gamma}$.

We use a truncation argument, the concentration-compactness principle of Lions [22], to the variable exponent spaces from the point of view of the trace, extended by Bonder and Silva [5], and an appropriate mini-max class of critical points via the classical concept and properties of the genus, to prove our main result as follows:

Theorem 1.
(*i*) Assume $q : \partial\Omega \to [1, \infty)$ and $\mathcal{A} := \{x \in \partial\Omega : q(x) = p_*(x)\} \neq \emptyset$ and $M(\tau) = a + b\tau^{\eta}$, with $a \geq 0, b > 0$ and $\eta \geq 1$. Moreover, assume the existence of functions $p(x), q(x), h(x), \beta(x) \in C_+(\overline{\Omega})$, see section 2, positive constants A_1, A_2 such that $A_1 t^{\beta(x)-1} \leq f(x,t) \leq A_2 t^{\beta(x)-1}$ for all $t \geq 0$ and for all $x \in \overline{\Omega}$, with $f(x,t) = -f(x,-t)$ for all $t \in \mathbb{R}$ and for all $x \in \overline{\Omega}$. Furthermore, $\beta^+(r+1) < p^-$ and $\dfrac{(\eta+1)(p^+)^{\eta+1}}{(p^-)^{\eta}} < h^- \leq h^+ < q^-$. Then there exists $\overline{\lambda} > 0$ such that for all $0 < \lambda < \overline{\lambda}$ there exists infinitely many solutions to (1.1) in $W^{1,p(x)}(\Omega)$.
(*ii*) Assume $q : \partial\Omega \to [1, \infty)$ and $\mathcal{A} := \{x \in \partial\Omega : q(x) = p_*(x)\} \neq \emptyset$. Moreover, assume the existence of functions $p(x), q(x), h(x), \beta(x) \in C_+(\overline{\Omega})$, see section 2, positive constants A_1, A_2 such that $A_1 t^{\beta(x)-1} \leq f(x,t) \leq A_2 t^{\beta(x)-1}$ for all $t \geq 0$ and for all $x \in \overline{\Omega}$, with $f(x,t) = -f(x,-t)$ for all $t \in \mathbb{R}$ and for all $x \in \overline{\Omega}$. Furthermore, assume there exists $0 < m_0$ and m_1 such that $m_0 \leq M(\tau) \leq m_1$, with $\dfrac{p^+ m_1}{m_0} < h^- \leq h^+ < q^-$ and $\beta^-(r+1) < p^-$. Then there exists $\widetilde{\lambda} > 0$ such that for all $0 < \lambda < \widetilde{\lambda}$ there exists infinitely many solutions to (1.1) in $W^{1,p(x)}(\Omega)$.

We should point out that the novelty in the present paper is the appearance of the integral terms, $\left[\displaystyle\int_{\Omega} F(x,u)dx\right]^r$, the critical growth on Neumann boundary conditions, the use of the $p(x)$-Laplacian and the generalized Lebesgue-Sobolev spaces.

We should point out that some ideas contained in this paper were inspired by the articles Ambrosetti and Rabinowitz [1], Azorero and Alonso [3], Bonder and Silva [4], Bonder, Saintier and Silva [[5], [6]], Corrêa and Costa [10], Corrêa and Figueiredo [[11], [12]], Fan [[14], [15]], Liang and Zhang [21], Guo and Zhao [19] and Yao [26].

This paper is organized as follows: In section 2 we present some preliminaries on the variable exponent spaces. In section 3, we give some basic notions on the Krasnoselskii's genus. In section 4, we proof of our main result.

2. Preliminaries on variable exponent spaces

First of all, we set

$$C_+(\overline{\Omega}) := \left\{z; z \in C(\overline{\Omega}), z(x) > 1 \text{ for all } x \in \overline{\Omega}\right\}$$

and for each $z \in C_+(\overline{\Omega})$ we define

$$z^+ := \max_{\overline{\Omega}} z(x) \text{ and } z^- := \min_{\overline{\Omega}} z(x).$$

We denote by $\mathcal{M}(\Omega)$ the set of real measurable functions defined on Ω.

For each $p \in C_+(\overline{\Omega})$, we define the generalized Lebesgue space by

$$L^{p(x)}(\Omega) := \left\{ u \in \mathcal{M}(\Omega); \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

We consider $L^{p(x)}(\Omega)$ endowed with the Luxemburg norm

$$|u|_{p(x)} := \inf \left\{ \mu > 0; \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad \text{for all } \ x \in \overline{\Omega}.$$

The proof of the following propositions may be found in Bonder and Silva [4], Bonder, Saintier and Silva [[5], [6]], Fan, Shen and Zhao [16], Fan and Zhang [17] and Fan and Zhao [18].

Proposition 1. (**Hölder Inequality**) If $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, then

$$\left| \int_\Omega uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)}.$$

Proposition 2. *Set* $\rho(u) := \displaystyle\int_\Omega |u(x)|^{p(x)} dx$. *For all* $u \in L^{p(x)}(\Omega)$, *we have:*

(*i*) *For* $u \neq 0$, $|u|_{p(x)} = \lambda \ \Leftrightarrow \ \rho(\frac{u}{\lambda}) = 1$;
(*ii*) $|u|_{p(x)} < 1 \ (= 1; > 1) \ \Leftrightarrow \ \rho(u) < 1 \ (= 1; > 1)$;
(*iii*) *If* $|u|_{p(x)} > 1$, *then* $|u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;
(*iv*) *If* $|u|_{p(x)} < 1$, *then* $|u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;
(*v*) $\lim_{k \to +\infty} |u_k|_{p(x)} = 0 \ \Leftrightarrow \ \lim_{k \to +\infty} \rho(u_k) = 0$;
(*vi*) $\lim_{k \to +\infty} |u_k|_{p(x)} = +\infty \ \Leftrightarrow \ \lim_{k \to +\infty} \rho(u_k) = +\infty$.

The generalized Lebesgue - Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the norm

$$\|u\| := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Denoting $\rho_{1,p(x)} := \displaystyle\int_\Omega (|u|^{p(x)} + |\nabla u|^{p(x)}) dx \ \forall u \in W^{1,p(x)}(\Omega)$, we have the following proposition:

Proposition 3. *For all* $u, u_j \in W^{1,p(x)}(\Omega)$, *we have:*

(*i*) $\|u\| < 1 \ (= 1; > 1) \ \Leftrightarrow \ \rho_{1,p(x)}(u) < 1 \ (= 1; > 1)$;
(*ii*) *If* $\|u\| > 1$, *then* $\|u\|^{p^-} \leq \rho_{1,p(x)}(u) \leq \|u\|^{p^+}$;
(*iii*) *If* $\|u\| < 1$, *then* $\|u\|^{p^+} \leq \rho_{1,p(x)}(u) \leq \|u\|^{p^-}$;
(*iv*) $\lim_{j \to +\infty} \|u_j\| = 0 \ \Leftrightarrow \ \lim_{j \to +\infty} \rho_{1,p(x)}(u_j) = 0$;
(*v*) $\lim_{j \to +\infty} \|u_j\| = +\infty \ \Leftrightarrow \ \lim_{j \to +\infty} \rho_{1,p(x)}(u_j) = +\infty$.

Proposition 4. Suppose that Ω is a bounded smooth domain in $\mathbb{R}^N$ and $p \in C(\overline{\Omega})$ with $p(x) < N$ for all $x \in \overline{\Omega}$. If $p_1 \in C(\overline{\Omega})$ and $1 \leq p_1(x) \leq p^*(x)$ $(1 \leq p_1(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, where $p^*(x) = \dfrac{Np(x)}{N-p(x)}$.

The Lebesgue spaces on $\partial\Omega$ are defined as

$$L^{q(x)}(\partial\Omega) := \{u|\ u : \partial\Omega \to \mathbb{R} \ \text{ is measurable and } \int_{\partial\Omega} |u|^{q(x)} dS < \infty\},$$

and the corresponding (Luxemburg) norm is given by

$$\|u\|_{L^{q(x)}(\partial\Omega)} := \|u\|_{q(x),\partial\Omega} := \inf\left\{\lambda > 0 : \int_{\partial\Omega} \left|\frac{u(x)}{\lambda}\right|^{q(x)} dS \leq 1\right\}.$$

Proposition 5. Suppose that Ω is a bounded smooth domain in $\mathbb{R}^N$ and $p, q \in C(\overline{\Omega})$ with $p(x) < N$ for all $x \in \overline{\Omega}$. Then there is a continuous and compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$, where $q(x) < p_*(x) = \dfrac{(N-1)p(x)}{N-p(x)}$.

Proposition 6. (**Fan and Zhang** [17]) Let $L_{p(x)} : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))'$ be such that

$$L_{p(x)}(u)(v) = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \quad \forall\ u, v \in W^{1,p(x)}(\Omega),$$

then
(i) $L_{p(x)} : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))'$ is a continuous, bounded and strictly monotone operator;
(ii) $L_{p(x)}$ is a mapping of type S_+, i.e., if $u_n \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$ and $\limsup(L_{p(x)}(u_n) - L_{p(x)}(u), u_n - u) \leq 0$, then $u_n \to u$ in $W^{1,p(x)}(\Omega)$;
(iii) $L_{p(x)} : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))'$ is a homeomorphism.

Proposition 7. (**Bonder, Saintier and Silva** [5]) Let $(u_j)_{j\in\mathbb{N}} \subset W^{1,p(x)}(\Omega)$ be a sequence such that $u_j \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$. Then there exists countable set I of positive numbers $(\mu_i)_{i\in I}$ and $(\nu_i)_{i\in I}$ and points $\{x_i\} \subset \mathcal{A} := \{x \in \partial\Omega : q(x) = p_*(x)\}$ such that

$$|u_j|^{q(x)} dS \rightharpoonup \nu = |u|^{q(x)} dS + \sum_{i\in I} \nu_i \delta_{x_i}, \quad \text{weakly} - * \text{ in the sense of measures.} \tag{2.1}$$

$$|\nabla u_j|^{p(x)} dx \rightharpoonup \mu \geq |\nabla u|^{p(x)} dx + \sum_{i\in I} \mu_i \delta_{x_i}, \text{weakly} - * \text{ in the sense of measures.} \tag{2.2}$$

$$\overline{T}_{x_i} \nu_i^{\frac{1}{q(x_i)}} \leq \mu_i^{\frac{1}{p(x_i)}}, \tag{2.3}$$

where $\overline{T}_{x_i} = \sup_{\epsilon>0} T(p(.), q(.), \Omega_{\epsilon,i}, \Gamma_{\epsilon,i})$ is the localized Sobolev trace constant where

$$\Omega_{\epsilon,i} = \Omega \cap B_\epsilon(x_i) \ \text{ and } \ \Gamma_{\epsilon,i} = \partial B_\epsilon(x_i) \cap \Omega.$$

Definition 1. We say that a sequence $(u_j) \subset W^{1,p(x)}(\Omega)$ is a Palais-Smale sequence for the functional $J_{\lambda,\gamma}$ if

$$J_{\lambda,\gamma}(u_j) \to c_* \quad \text{and} \quad J'_{\lambda,\gamma}(u_j) \to 0 \quad \text{in} \quad (W^{1,p(x)}(\Omega))', \tag{2.4}$$

where

$$c_* = \inf_{h \in \mathcal{C}} \sup_{t \in [0,1]} J_{\lambda,\gamma}(h(t)) > 0 \tag{2.5}$$

and

$$\mathcal{C} = \{h : [0,1] \to W^{1,p(x)}(\Omega) : h \text{ continuous and } h(0) = 0, J_{\lambda,\gamma}(h(1)) < 0\}.$$

The number c_* is called the *energy level* c_*.

When (2.4) implies the existence of a subsequence of (u_j), still denoted by (u_j), which converges in $W^{1,p(x)}(\Omega)$, we say that $J_{\lambda,\gamma}$ satisfies the Palais-Smale condition.

3. Preliminaries on Krasnoselskii's genus

In this section, we present some basic notions on the Krasnoselskii's genus, whose details may be found in Clark [8] and Krasnoselskii [20], that we will use in the proof of our main result. Let X be a real Banach space. Let us denote by $\mathcal{U}$ the class of all closed subsets $A \subset X \setminus \{0\}$ which are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Definition 2. Let $A \in \mathcal{U}$. The genus $\gamma(A)$ of A is defined as being the least positive integer k such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for all $x \in A$. If such a k does not exist we set $\gamma(A) = \infty$. Furthermore, by definition $\gamma(\emptyset) = 0$.

In the sequel, we will establish only the properties of genus that will be used through this work. More information on this subject may be found in the references Ambrosetti [2], Castro [7], Costa [13] and Krasnoselskii [20].

Theorem 2. Let $X = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.

Corollary 1. $\gamma(S^{N-1}) = N$.

As a consequence of this, if X is a separable infinite dimensional vector space and S is the unit sphere in X, then $\gamma(S) = \infty$.

We now establish a result due to Clark [8].

Theorem 3. Let $J \in C^1(X, \mathbb{R})$ be a functional satisfying the Palais-Smale condition. Furthermore, let us suppose that:

(i) J is bounded from below and even;

(ii) There is a compact set $K \in \mathcal{U}$ such that $\gamma(K) = k$ and $\sup_{x \in K} J(x) < J(0)$.

Then J possesses at least k pairs of distinct critical points and their corresponding critical values are less than $J(0)$.

4. Proof of Theorem 1

Proof. (i) The proof follows from several lemmas.

Lemma 1. If $(u_j) \subset W^{1,p(x)}(\Omega)$ is a Palais-Smale sequence, with energy level c, then (u_j) is bounded in $W^{1,p(x)}(\Omega)$.

Proof. Since (u_j) is a Palais-Smale sequence with energy level c, we have $J_{\lambda,\gamma}(u_j) \to c$ and $J'_{\lambda,\gamma}(u_j) \to 0$. Then, taking θ such that

$$h^+ < \theta < q^-, \tag{4.1}$$

we obtain

$$C + \|u_j\| \geq \left(J_{\lambda,\gamma}(u_j) - \frac{1}{\theta} J'_{\lambda,\gamma}(u_j)u_j\right) = a\psi(u_j) + \frac{b}{\eta+1}\psi^{\eta+1}(u_j) - \gamma \int_{\partial\Omega} \frac{1}{q(x)}|u_j|^{q(x)} dS$$
$$-\frac{\lambda}{r+1}\left[\int_\Omega F(x,u_j)dx\right]^{r+1} - \frac{a}{\theta}\int_\Omega |\nabla u_j|^{p(x)-2}\nabla u_j \nabla u_j dx - \frac{a}{\theta}\int_\Omega |u_j|^{p(x)-2}u_j u_j dx$$
$$+\frac{\lambda}{\theta}\left[\int_\Omega F(x,u_j)dx\right]^r \int_\Omega f(x,u_j)u_j dx + \frac{\gamma}{\theta}\int_{\partial\Omega} |u_j|^{q(x)-2}u_j u_j dS - \frac{b}{\theta}\psi^\eta(u_j)\rho_{1,p(x)}(u_j)$$
$$+\int_\Omega \frac{1}{h(x)}|u_j|^{h(x)}dx - \frac{1}{\theta}\int_\Omega |u_j|^{h(x)-2}u_j u_j dx.$$

Thus,

$$C + \|u_j\| \geq \left(\frac{a}{p^+} - \frac{a}{\theta}\right)\rho_{1,p(x)}(u_j) + \left(\frac{b}{(\eta+1)(p^+)^{\eta+1}} - \frac{b}{\theta(p^-)^\eta}\right)\rho^{\eta+1}_{1,p(x)}(u_j)$$
$$+\left(\frac{\lambda A_1^{r+1}}{\theta(\beta^+)^r} - \frac{\lambda A_2^{r+1}}{(r+1)(\beta^-)^{r+1}}\right)\left[\int_\Omega |u|^{\beta(x)}dx\right]^{r+1} + \left(\frac{\gamma}{\theta} - \frac{\gamma}{q^-}\right)\int_{\partial\Omega}|u_j|^{q(x)}dS$$
$$+\left(\frac{1}{h^+} - \frac{1}{\theta}\right)\int_\Omega |u_j|^{h(x)}dx.$$

Now, let us suppose that (u_j) is unbounded in $W^{1,p(x)}(\Omega)$. Passing to a subsequence if necessary, we get $\|u_j\| > 1$ and we obtain

$$C + \|u_j\| \geq \left(\frac{a}{p^+} - \frac{a}{\theta}\right)\|u\|^{p^-} + \left(\frac{b}{(\eta+1)(p^+)^{\eta+1}} - \frac{b}{\theta(p^-)^\eta}\right)\|u\|^{(\eta+1)p^-}$$
$$+\lambda\left(\frac{A_1^{r+1}}{\theta}\frac{1}{(\beta^+)^r} - \frac{A_2^{r+1}}{r+1}\frac{1}{(\beta^-)^{r+1}}\right)\|u\|^{\beta^\pm(r+1)},$$

which is a contradiction because $p^- > \beta^\pm(r+1) > 1$. Hence (u_j) is bounded in $W^{1,p(x)}(\Omega)$. $\square$

Lemma 2. Let $(u_j) \subset W^{1,p(x)}(\Omega)$ be a Palais-Smale sequence with energy level c and $t_0 = \lim_{j\to\infty}\psi(u_j)$. If

$$c < \left(\frac{1}{\theta} - \frac{1}{q^-_{\mathcal{A}}}\right)\inf_{i\in I}\left(\gamma^{1-1/p(x_i)}\overline{a}^{1/p(x_i)}\overline{T}_{x_i}\right)^{\frac{p(x_i)p_*(x_i)}{p_*(x_i)-p(x_i)}}$$
$$+K\min\{\lambda^{\frac{(h/\beta)^-}{(h/\beta)^- - (r+1)}}, \lambda^{\frac{(h/\beta)^+}{(h/\beta)^+ - (r+1)}}\},$$

where $\overline{a} = t_1$ with $0 < t_1 < bt_0^\eta$ and K independent on λ. Then, there exists $\lambda_0 > 0$ such that, for all $0 < \lambda < \lambda_0$ the index set I given in the Proposition 7 is empty and $u_j \to u$ strongly in $L^{q(x)}(\partial\Omega)$ for some $u \in W^{1,p(x)}(\Omega)$.

Proof. Assume that $u_j \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$. By Proposition 7 and Lemma 1, we have

$$|u_j|^{q(x)}dS \rightharpoonup \nu = |u|^{q(x)}dS + \sum_{i\in I}\nu_i\delta_{x_i}, \quad \text{weakly} - * \text{ in the sense of measures.}$$

$$|\nabla u_j|^{p(x)}dx \rightharpoonup \mu \geq |\nabla u|^{p(x)}dx + \sum_{i\in I}\mu_i\delta_{x_i}, \text{weakly} - * \text{ in the sense of measures.}$$

$$\overline{T}_{x_i}\nu_i^{\frac{1}{q(x_i)}} \leq \mu_i^{\frac{1}{p(x_i)}}, \ \forall i \in I.$$

If $I = \emptyset$ then $u_j \to u$ strongly in $L^{q(x)}(\partial\Omega)$. Suppose that $I \neq \emptyset$. Let x_i be a singular point of the measures μ and ν. We consider $\phi \in C_0^\infty(\mathbb{R}^N)$, such that $0 \leq \phi(x) \leq 1$, $\phi(0) = 0$ and support in the unit ball of $\mathbb{R}^N$. Consider the functions $\phi_{i,\varepsilon}(x) = \phi\left(\dfrac{x - x_i}{\varepsilon}\right)$ for all $x \in \mathbb{R}^N$ and $\varepsilon > 0$.

As $J'_{\lambda,\gamma}(u_j) \to 0$ in $\left(W^{1,p(x)}\Omega\right)'$ we obtain

$$\lim J'_{\lambda,\gamma}(u_j)(\phi_{i,\varepsilon}u_j) = 0.$$

$$\begin{aligned} J'_{\lambda,\gamma}(u)\phi_{i,\varepsilon}u_j \ &= \ (a + b\psi^\eta(u_j))\int_\Omega \Big(|\nabla u_j|^{p(x)-2}\nabla u_j\nabla\phi_{i,\varepsilon}u_j + |u_j|^{p(x)-2}u_j\phi_{i,\varepsilon}u_j\Big)\,dx \\ &-\gamma\int_{\partial\Omega}|u_j|^{q(x)-2}u_j\phi_{i,\varepsilon}u_j dS - \lambda\left[\int_\Omega F(x,u_j)dx\right]^r\int_\Omega f(x,u_j)(\phi_{i,\varepsilon}u_j)dx \\ &+\int_\Omega |u_j|^{h(x)-2}u_j\phi_{i,\varepsilon}u_j dx \to 0, \end{aligned}$$

i.e.,

$$\begin{aligned} J'_{\lambda,\gamma}(u)\phi_{i,\varepsilon}u_j \ &= \ (a + b\psi^\eta(u_j))\int_\Omega |\nabla u_j|^{p(x)-2}\nabla u_j\nabla\phi_{i,\varepsilon}u_j dx - \gamma\int_{\partial\Omega}|u_j|^{q(x)}\phi_{i,\varepsilon}dS \\ &+(a + b\psi^\eta(u_j))\int_\Omega |\nabla u_j|^{p(x)}\nabla\phi_{i,\varepsilon}dx + (a + b\psi^\eta(u_j))\int_\Omega |u_j|^{p(x)}\phi_{i,\varepsilon}dx \\ &-\lambda\left[\int_\Omega F(x,u_j)dx\right]^r\int_\Omega f(x,u_j)(\phi_{i,\varepsilon}u_j)dx + \int_\Omega |u_j|^{h(x)}\phi_{i,\varepsilon}dx \to 0. \end{aligned}$$

When $j \to \infty$ we get

$$0 = \lim\left[(a + b\psi^\eta(u_j))\int_\Omega(|\nabla u_j|^{p(x)-2}\nabla u_j\nabla\phi_{i,\varepsilon}u_j)dx + (a + b\psi^\eta(u_j))\int_\Omega |u|^{p(x)}\phi_{i,\varepsilon}dx\right]$$

$$(a + bt_0^\eta)\int_\Omega \phi_{i,\varepsilon}d\mu - \gamma\int_{\partial\Omega}\phi_{i,\varepsilon}d\nu - \lambda\left[\int_\Omega F(x,u)dx\right]^r\int_\Omega f(x,u)(\phi_{i,\varepsilon}u)dx.$$

We may show that,

$\lim\limits_{\varepsilon\to 0}\ (a + bt_0^\eta)\displaystyle\int_\Omega(|\nabla u|^{p(x)-2}\nabla u\nabla\phi_{i,\varepsilon}u)dx \to 0$, see Shang and Wang [25].

On the other hand,

$$\lim_{\varepsilon\to 0}(a + bt_0^\eta)\int_\Omega \phi_{i,\varepsilon}d\mu = (a + bt_0^\eta)\mu\phi(0), \quad \gamma\lim_{\epsilon\to 0}\int_{\partial\Omega}\phi_{i,\varepsilon}d\nu = \gamma\nu\phi(0)$$

and

$$\lambda\left[\int_\Omega F(x,u)dx\right]^r\int_\Omega f(x,u)(\phi_{i,\varepsilon}u)dx \to 0, \quad (a + bt_0)\int_\Omega |u|^{p(x)}\phi_{i,\varepsilon}dx \to 0,$$

$$\int_\Omega |u_j|^{h(x)}\phi_{i,\varepsilon}dx \to 0 \ \ as \ \ \varepsilon \to 0.$$

Then,

$(a+bt_0^\eta)\mu_i\phi(0)=\gamma\nu_i\phi(0)$ implies that $\gamma^{-1}\overline{a}\mu_i\le\nu_i$. By $\overline{T}_{x_i}\nu_i^{1/p_*(x_i)}\le\mu_i^{1/p(x_i)}$ we obtain $\gamma^{-1}\overline{a}\left(\overline{T}_{x_i}\right)^{p(x_i)}\nu_i^{p(x_i)/p_*(x_i)}\le\gamma^{-1}\overline{a}\mu_i\le\nu_i$. Thus $\gamma^{-1}\overline{a}\left(\overline{T}_{x_i}\right)^{p(x_i)}\le\nu_i^{1-p(x_i)/p_*(x_i)}=\nu_i^{(p_*(x_i)-p(x_i))/p_*(x_i)}$ and $\gamma^{-1/p(x_i)}\overline{a}^{1/p(x_i)}\overline{T}_{x_i}\le\nu_i^{(p_*(x_i)-p(x_i))/p(x_i)p_*(x_i)}$. Therefore

$$\nu_i\ge\left(\gamma^{-1/p(x_i)}\overline{a}^{1/p(x_i)}\overline{T}_{x_i}\right)^{\frac{p(x_i)p_*(x_i)}{p_*(x_i)-p(x_i)}}.$$

On the other hand, using θ satisfying (4.1)

$$c=\lim J_{\lambda,\gamma}(u_j)=\lim\left(J_{\lambda,\gamma}(u_j)-\frac{1}{\theta}J'_{\lambda,\gamma}(u_j)u_j\right).$$

Thus,

$$\begin{aligned}c &\ge \lim\Bigg(\lambda\left(\frac{A_1^{r+1}}{\theta}\frac{1}{(\beta^+)^r}-\frac{A_2^{r+1}}{r+1}\frac{1}{(\beta^-)^{r+1}}\right)\left[\int_\Omega|u_j|^{\beta(x)}\right]^{r+1}+\\ &+\left(\frac{1}{h^+}-\frac{1}{\theta}\right)\int_\Omega|u_j|^{h(x)}dx+\gamma\int_{\partial\Omega}\left(\frac{1}{\theta}-\frac{1}{q(x)}\right)|u_j|^{q(x)}\Bigg).\end{aligned}$$

Now, setting $\mathcal{A}_\delta=\bigcup_{x\in\mathcal{A}}(B_\delta(x)\cap\Omega)=\{x\in\Omega:\operatorname{dist}(x,\mathcal{A})<\delta\}$, we obtain

$$\begin{aligned}c &\ge \lambda\left(\frac{A_1}{\theta}\frac{1}{(\beta^+)^r}-\frac{A_2^{r+1}}{r+1}\frac{1}{(\beta^-)^{r+1}}\right)\left[\int_\Omega|u|^{\beta(x)}\right]^{r+1}+\left(\frac{1}{h^+}-\frac{1}{\theta}\right)\int_\Omega|u|^{h(x)}dx\\ &\quad+\gamma\left(\frac{1}{\theta}-\frac{1}{q^-}\right)\int_{\partial\Omega}|u|^{q(x)}dS+\gamma\left(\frac{1}{\theta}-\frac{1}{q^-_{\mathcal{A}_\delta}}\right)\sum_{i\in I}\nu_i.\end{aligned}$$

Since $\delta>0$ is arbitrary and q is continuous, we get

$$\begin{aligned}c &\ge \lambda\left(\frac{A_1^{r+1}}{\theta}\frac{1}{(\beta^+)^r}-\frac{A_2^{r+1}}{r+1}\frac{1}{(\beta^-)^{r+1}}\right)\left[\int_\Omega|u|^{\beta(x)}\right]^{r+1}+\left(\frac{1}{h^+}-\frac{1}{\theta}\right)\int_\Omega|u|^{h(x)}dx\\ &\quad+\left(\frac{1}{\theta}-\frac{1}{q^-_{\mathcal{A}}}\right)\inf_{i\in I}\left(\gamma^{1-1/p(x_i)}\overline{a}^{1/p(x_i)}\overline{T}_{x_i}\right)^{\frac{p(x_i)p_*(x_i)}{p_*(x_i)-p(x_i)}}.\end{aligned}$$

Applying now Hölder inequality, we find

$$\begin{aligned}c &\ge \lambda\left(\frac{A_1^{r+1}}{\theta}\frac{1}{(\beta^+)^r}-\frac{A_2^{r+1}}{r+1}\frac{1}{(\beta^-)^{r+1}}\right)\left[||u|^{\beta(x)}|_{h(x)/\beta(x)}|\Omega|^{\frac{h^+-\beta^-}{h^-}}\right]^{r+1}\\ &\quad\left(\frac{1}{h^+}-\frac{1}{\theta}\right)\int_\Omega|u|^{h(x)}dx+\left(\frac{1}{\theta}-\frac{1}{q^-_{\mathcal{A}}}\right)\inf_{i\in I}\left(\gamma^{1-1/p(x_i)}\overline{a}^{1/p(x_i)}\overline{T}_{x_i}\right)^{\frac{p(x_i)p_*(x_i)}{p_*(x_i)-p(x_i)}}.\end{aligned}$$

If $||u|^{\beta(x)}|_{h(x)/\beta(x)}\ge1$, we have

$$c\ge c_1\left||u|^{\beta(x)}\right|_{h(x)/\beta(x)}^{(h/\beta)^-}-\lambda c_2\left||u|^{\beta(x)}\right|_{h(x)/\beta(x)}^{r+1}+c_3,$$

where $0<c_2=\left(\frac{A_2^{r+1}}{r+1}\frac{1}{(\beta^-)^{r+1}}-\frac{A_1^{r+1}}{\theta}\frac{1}{(\beta^+)^r}\right)|\Omega|^{\frac{(h^+-\beta^-)(r+1)}{h^-}}$ and

$c_3=\left(\frac{1}{\theta}-\frac{1}{q^-_{\mathcal{A}}}\right)\inf_{i\in I}\left(\gamma^{1-1/p(x_i)}\overline{a}^{1/p(x_i)}\overline{T}_{x_i}\right)^{\frac{p(x_i)p_*(x_i)}{p_*(x_i)-p(x_i)}}$.

So, if $g_1(t)=c_1t^{(h/\beta)^-}-\lambda c_2t^{r+1}$, this function attains its absolute minimum, for $t>0$, at the point

$$\bar{t}=\left(\frac{(r+1)\lambda c_2}{(h/\beta)^-c_1}\right)^{\dfrac{1}{(h/\beta)^--(r+1)}}.$$

Note that,

$g_1(\bar{t}) = c_1 \left(\frac{(r+1)\lambda c_2}{(h/\beta)^- c_1}\right)^{\frac{(h/\beta)^-}{(h/\beta)^- - (r+1)}} - \lambda c_2 \left(\frac{(r+1)\lambda c_2}{(h/\beta)^- c_1}\right)^{\frac{r+1}{(h/\beta)^- - (r+1)}}$, which implies

$$g_1(\bar{t}) = c_1 \left(\frac{(r+1)\lambda c_2}{c_1 (h/\beta)^-}\right)^{\frac{(h/\beta)^-}{(h/\beta)^- - (r+1)}}$$

$$- \frac{(r+1)c_1(h/\beta)^-}{(r+1)c_1(h/\beta)^-}\lambda c_2 \left(\frac{(r+1)\lambda c_2}{c_1 (h/\beta)^-}\right)^{\frac{r+1}{(h/\beta)^- - (r+1)}}.$$

Thus,

$$g_1(\bar{t}) = c_1 \left(\frac{(r+1)\lambda c_2}{c_1 (h/\beta)^-}\right)^{\frac{(h/\beta)^-}{(h/\beta)^- - (r+1)}}$$

$$- \frac{c_1(h/\beta)^-}{r+1} \left(\frac{(r+1)\lambda c_2}{c_1 (h/\beta)^-}\right)^{\frac{r+1}{(h/\beta)^- - (r+1)} + 1}.$$

So,

$$g_1(\bar{t}) = c_1 \left(\frac{(r+1)\lambda c_2}{c_1 (h/\beta)^-}\right)^{\frac{(h/\beta)^-}{(h/\beta)^- - (r+1)}} \left(1 - \frac{(h/\beta)^-}{r+1}\right).$$

Using the fact that $\beta^+(r+1) < h^-$, we can write

$g_1(\bar{t}) = \lambda^{\frac{(h/\beta)^-}{(h/\beta)^- - (r+1)}} K$, where K is a negative constant depending only on A_1, A_2, r, h, β and Ω.

If $||u|^{\beta(x)}|_{h(x)/\beta(x)} < 1$, we have

$$c \geq c_1 \left||u|^{\beta(x)}\right|_{h(x)/\beta(x)}^{(h/\beta)^+} - \lambda c_2 \left||u|^{\beta(x)}\right|_{h(x)/\beta(x)}^{r+1} + c_3,$$

where $0 < c_2 = \left(\frac{A_2^{r+1}}{r+1}\frac{1}{(\beta^-)^{r+1}} - \frac{A_1^{r+1}}{\theta}\frac{1}{(\beta^+)^r}\right)|\Omega|^{\frac{(h^+-\beta^-)(r+1)}{h^-}}$ and

$c_3 = \left(\frac{1}{\theta} - \frac{1}{q_{\mathcal{A}}^-}\right)\inf_{i\in I}\left(\gamma^{1-1/p(x_i)}\overline{a}^{1/p(x_i)}\overline{T}_{x_i}\right)^{\frac{p(x_i)p_*(x_i)}{p_*(x_i)-p(x_i)}}.$

So, if $g_2(t) = c_1 t^{(h/\beta)^+} - \lambda c_2 t^{r+1}$, this function attains its absolute minimum, for $t > 0$, at the point

$$\underline{t} = \left(\frac{(r+1)\lambda c_2}{(h/\beta)^+ c_1}\right)^{\frac{1}{(h/\beta)^+ - (r+1)}}.$$

Thus, we obtain

$g_2(\underline{t}) = \lambda^{\frac{(h/\beta)^+}{(h/\beta)^+ - (r+1)}} K$, where K is a negative constant depending only on A_1, A_2, r, h, β and Ω. Then

$$\begin{aligned} c \quad \geq \quad & \left(\frac{1}{\theta} - \frac{1}{q_{\mathcal{A}}^-}\right) \inf_{i \in I} \left(\gamma^{1-1/p(x_i)} \overline{a}^{1/p(x_i)} \overline{T}_{x_i}\right)^{\frac{p(x_i)p_*(x_i)}{p_*(x_i)-p(x_i)}} \\ & + K \min\{\lambda^{\frac{(h/\beta)^-}{(h/\beta)^- - (r+1)}}, \lambda^{\frac{(h/\beta)^+}{(h/\beta)^+ - (r+1)}}\}. \end{aligned}$$

Therefore $I = \emptyset$.

□

Lemma 3. Let $(u_j) \subset W^{1,p(x)}(\Omega)$ be a Palais-Smale sequence with energy level c. If

$$\begin{aligned} c \quad < \quad & \left(\frac{1}{\theta} - \frac{1}{q_{\mathcal{A}}^-}\right) \inf_{i \in I} \left(\gamma^{1-1/p(x_i)} \overline{a}^{1/p(x_i)} \overline{T}_{x_i}\right)^{\frac{p(x_i)p_*(x_i)}{p_*(x_i)-p(x_i)}} \\ & + K \min\{\lambda^{\frac{(h/\beta)^-}{(h/\beta)^- - (r+1)}}, \lambda^{\frac{(h/\beta)^+}{(h/\beta)^+ - (r+1)}}\}, \end{aligned}$$

there exist $u \in W^{1,p(x)}(\Omega)$ and a subsequence, still denoted by (u_j), such that $u_j \to u$ in $W^{1,p(x)}(\Omega)$.

Proof. From

$$J'_{\lambda,\gamma}(u_j) \to 0,$$

we have

$$\begin{aligned} J'_{\lambda,\gamma}(u_j)(u_j - u) \quad = \quad & (a + b\psi^\eta(u_j)) \int_\Omega \Big(|\nabla u_j|^{p(x)-2} \nabla u_j \nabla (u_j - u) dx \\ & + (a + b\psi^\eta(u_j)) \int_\Omega |u_j|^{p(x)-2} u_j (u_j - u) \Big) dx + \int_\Omega |u_j|^{h(x)-2} u_j (u_j - u) dx \\ & - \gamma \int_{\partial\Omega} |u_j|^{q(x)-2} u_j (u_j - u) dS - \lambda \left[\int_\Omega F(x, u_j) dx\right]^r \int_\Omega f(x, u_j)(u_j - u) dx \to 0, \end{aligned}$$

Note that there exists nonnegative constants c_1, c_2, c_3 and c_4 such that

$$c_1 \leq a + b\psi^\eta(u_j) \leq c_2$$

and

$$c_3 \leq \left[\int_\Omega F(x, u_j) dx\right]^r \leq c_4.$$

But using Hölder inequality, we have

$$\left|\int_\Omega |u_j|^{p(x)-2} u_j (u_j - u) dx\right| \leq \int_\Omega |u_j|^{p(x)-1} |u_j - u| dx \leq C_1 |u|_{p(x)/p(x)-1} |u_j - u|_{p(x)},$$

$$\left|\int_\Omega |u_j|^{h(x)-2} u_j (u_j - u) dx\right| \leq \int_\Omega |u_j|^{h(x)-1} |u_j - u| dx \leq C_2 |u|_{h(x)/h(x)-1} |u_j - u|_{h(x)},$$

and

$$\left|\int_\Omega f(x, u_j)(u_j - u) dx\right| \leq A_2 \int_\Omega |u_j|^{\beta(x)-1} |u_j - u| dx \leq C_3 |u|_{\beta(x)/\beta(x)-1} |u_j - u|_{\beta(x)},$$

where C_1, C_2 and C_2 are positive constants. Thus

$$\left|\int_\Omega |u_j|^{p(x)-2}u_j(u_j-u)dx\right| \to 0,$$

$$\left|\int_\Omega |u_j|^{h(x)-2}u_j(u_j-u)dx\right| \to 0$$

and

$$\left|\int_\Omega f(x,u_j)(u_j-u)dx\right| \to 0.$$

By Lemma 2, $u_j \to u$ in $L^{q(x)}(\partial\Omega)$ and using Hölder inequality we obtain

$$\left|\int_{\partial\Omega} |u_j|^{q(x)-2}u_j(u_j-u)\right| dx \to 0.$$

Taking

$$L_{p(x)}(u_j)(u_j-u) = \int_\Omega |\nabla u_j|^{p(x)-2}\nabla u_j \nabla(u_j-u)dx,$$

we obtain $L_{p(x)}(u_j)(u_j-u) \to 0$. We also have $L_{p(x)}(u)(u_j-u) \to 0$. So

$$(L_{p(x)}(u_j) - L_{p(x)}(u), u_j-u) \to 0.$$

From Proposition 6 we have $u_j \to u$ in $W^{1,p(x)}(\Omega)$.

□

Lemma 4. The energy functional $J_{\lambda,\gamma}$ associated to (1.1) is unbounded below.

Proof. Recall that

$$\begin{aligned} J_{\lambda,\gamma}(u) &= a\psi(u) + \frac{b}{\eta+1}\psi^{\eta+1}(u) - \frac{\lambda}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1} + \int_\Omega \frac{1}{h(x)}|u|^{h(x)}dx \\ &\quad -\gamma\int_{\partial\Omega}\frac{1}{q(x)}|u|^{q(x)}dS. \end{aligned}$$

Take $0 < w \in W^{1,p(x)}(\Omega)$. For $t > 1$,

$$\begin{aligned} J_{\lambda,\gamma}(tw) &\leq \frac{a}{p^-}t^{p^+}\rho_{1,p(x)}(w) + \frac{bt^{p^+(\eta+1)}}{(\eta+1)(p^-)^{\eta+1}}\rho_{1,p(x)}^{\eta+1}(w) + \frac{1}{h^-}t^{h^+}\int_\Omega |w|^{h(x)}dx \\ &\quad + \frac{\lambda}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1} t^{\beta^-(r+1)}\left(\int_\Omega |w|^{\beta(x)}dx\right)^{r+1} - \frac{\gamma}{q^+}t^{q^-}\int_{\partial\Omega}|w|^{q(x)}dS. \end{aligned}$$

Then we have

$$\lim_{t\to\infty} J_{\lambda,\gamma}(tw) = -\infty,$$

and the proof is over.

□

In what follows we will use a truncation, like in Azorero and Alonso [3], on the functional $J_{\lambda,\gamma}$, to obtain a special bounded from below functional, as follows.

Assuming $\|u\| \leq 1$ sufficiently small, by Proposition 2 and (4) we have

$$J_{\lambda,\gamma}(u) \geq \frac{a}{p^+}\|u\|^{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}}\|u\|^{(\eta+1)p^+} + \frac{1}{h^+}\frac{1}{S_h^{h^+}}\|u\|^{h^+} - \frac{\lambda}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}\frac{1}{S_\beta^{\beta^-(r+1)}}\|u\|^{\beta^-(r+1)} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^-}}\|u\|^{q^-} \geq J_{1,\lambda,\gamma}(\|u\|),$$

where

$$J_{1,\lambda,\gamma}(t) = \frac{a}{p^+}t^{h^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}}t^{h^+} + \frac{1}{h^+}\frac{1}{S_h^{h^+}}t^{h^+} - \frac{\lambda}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}\frac{1}{S_\beta^{\beta^-(r+1)}}t^{\beta^-(r+1)} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^-}}t^{q^-}.$$

First note that $J_{1,\lambda,\gamma}(t) < 0$ for $t \approx 0$, because $\beta^-(r+1) < h^+$.
Furthermore,

$$\left(\frac{a}{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}} + \frac{1}{h^+}\frac{1}{S_h^{h^+}}\right)t^{h+} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^-}}t^{q^-} = t^{h+}\left(\frac{a}{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}} + \frac{1}{h^+}\frac{1}{S_h^{h^+}} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^-}}t^{q^- - h^+}\right).$$

In view of $q^- > h^+$, we can take R_1 small enough, such that

$$\left(\frac{a}{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}} + \frac{1}{h^+}\frac{1}{S_h^{h^+}}\right)R_1^{h^+} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^-}}R_1^{q^-} > 0.$$

Let us define

$$\lambda_1 = \frac{(r+1)}{2}\left(\frac{\beta^-}{A_2}\right)^{r+1}\frac{S_\beta^{\beta^-(r+1)}}{R_1^{\beta^-(r+1)}}\left(\left(\frac{a}{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}} + \frac{1}{h^+}\frac{1}{S_h^{h^+}}\right)R_1^{h^+} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^-}}R_1^{q^-}\right)$$

and $R_0 = \max\{0 < t \leq R_1; J_{1,\lambda_1} \leq 0\}$. Thus, there exist $0 < \lambda_1, R_0$ and R_1 with $R_0 < R_1$, such that

$$J_{\lambda,\gamma}(u) \geq J_{1,\lambda,\gamma}(\|u\|) \geq J_{1,\lambda_1}(\|u\|) = \left(\frac{a}{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}} + \frac{1}{h^+}\frac{1}{S_h^{h^+}}\right)\|u\|^{h^+} - \frac{\lambda_1}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}\frac{1}{S_\beta^{\beta^-(r+1)}}\|u\|^{\beta^-(r+1)} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^-}}\|u\|^{q^-}.$$

for all $\|u\| < R_1$ and $0 < \lambda < \lambda_1$, with $J_{1,\lambda_1}(R_1) > 0$ and $J_{1,\lambda_1}(R_0) = 0$.

We can choose a nonincreasing function $\tau_1 : [0,\infty) \to [0,1]$, $\tau_1 \in C^\infty([0,\infty))$ such that

$$\tau_1(x) = 1, \quad \text{se} \quad x \leq R_0$$

and

$$\tau_1(x) = 0, \quad \text{se} \quad x \geq R_1.$$

If $\|u\| > 1$, we obtain

$$J_{\lambda,\gamma}(u) \geq \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{(\eta+1)(p^+)^{\eta+1}}\|u\|^{(\eta+1)p^-} + \frac{1}{h^+}\frac{1}{S_h^{h^\pm}}\|u\|^{h^\pm} - \frac{\lambda}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}\frac{1}{S_\beta^{\beta^\pm(r+1)}}\|u\|^{\beta^\pm(r+1)} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^\pm}}\|u\|^{q^\pm}.$$

So,

$$J_{\lambda,\gamma}(u) \geq \left(\frac{a}{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}} + \frac{1}{h^+}\frac{1}{S_h^{h^\pm}}\right)\|u\|^{p^-} - \frac{\lambda}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}\frac{1}{S_\beta^{\beta^\pm(r+1)}}\|u\|^{\beta^+(r+1)} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^\pm}}\|u\|^{q^+} = J_{2,\lambda,\gamma}(\|u\|),$$

where

$$J_{2,\lambda,\gamma}(t) = \left(\frac{a}{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}} + \frac{1}{h^+}\frac{1}{S_h^{h^\pm}}\right)t^{p^-} - \frac{\lambda}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}\frac{1}{S_\beta^{\beta^\pm(r+1)}}t^{\beta^+(r+1)} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^\pm}}t^{q^+}.$$

First note that $J_{2,\lambda,\gamma}(t) < 0$ for $t \approx 0^+$, because $\beta^-(r+1) < p^-$.

Furthermore,

$$\left(\frac{a}{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}} + \frac{1}{h^+}\frac{1}{S_h^{h^\pm}}\right)t^{p^-} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^\pm}}t^{q^+} = t^{p^-}\left(\left(\frac{a}{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}} + \frac{1}{h^+}\frac{1}{S_h^{h^\pm}}\right) - \frac{\gamma}{q^-}\frac{1}{S_q^{q^\pm}}t^{q^+-p^-}\right).$$

In view of the $q^+ > p^-$, we can take R_1, sufficiently small, such that

$$\left(\frac{a}{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}} + \frac{1}{h^+}\frac{1}{S_h^{h^\pm}}\right)R_1^{p^-} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^\pm}}R_1^{q^+} > 0.$$

Let us define

$$\lambda_2 = \frac{(r+1)}{2}\left(\frac{\beta^-}{A_2}\right)^{r+1}\frac{S_\beta^{\beta^\pm(r+1)}}{R_1^{\beta^+(r+1)}}\left(\left(\frac{a}{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}} + \frac{1}{h^+}\frac{1}{S_h^{h^\pm}}\right)R_1^{p-} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^-}}R_1^{q^+}\right)$$

and $R_0 = \max\{0 < t \leq R_1; J_{2,\lambda_1} \leq 0\}$. Thus, there are $0 < \lambda_2, R_0$ and R_1 with $R_0 < R_1$, such that

$$J_{\lambda,\gamma}(u) \geq J_{2,\lambda,\gamma}(\|u\|) \geq J_{2,\lambda_2}(\|u\|) = \left(\frac{a}{p^+} + \frac{b}{(\eta+1)(p^+)^{\eta+1}} + \frac{1}{h^+}\frac{1}{S_h^{h^\pm}}\right)\|u\|^{p^-} - \frac{\lambda_2}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}\frac{1}{S_\beta^{\beta^\pm(r+1)}}\|u\|^{\beta^+(r+1)} - \frac{\gamma}{q^-}\frac{1}{S_q^{q^\pm}}\|u\|^{q^+}.$$

for all $\|u\| < R_1$ and $0 < \lambda < \lambda_2$, with $J_{2,\lambda_2}(R_1) > 0$ and $J_{2,\lambda_2}(R_0) = 0$.

We can choose a nonincreasing function $\tau_2 : [0,\infty) \to [0,1]$, $\tau_2 \in C^\infty([0,\infty))$, such that

$$\tau_2(x) = 1, \quad \text{se} \quad x \leq R_0$$

and

$$\tau_2(x) = 0, \quad \text{se} \quad x \geq R_1.$$

Finally, we define

$$\tau(t) = \begin{cases} \tau_1(t) & , \quad t \leq 1 \\ \tau_2(t) & , \quad t > 1 \end{cases}.$$

Now, we consider the truncated functional, with $0 < \lambda < \lambda' = \min\{\lambda_1, \lambda_2\}$,

$$\begin{aligned} I_\lambda(u) &= a\psi(u) + \frac{b}{\eta+1}\psi^{\eta+1}(u) - \frac{\lambda}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1} + \int_\Omega \frac{1}{h(x)}|u|^{h(x)}dx \\ &\quad -\gamma\int_{\partial\Omega}\frac{1}{q(x)}|u|^{q(x)}\tau(\|u\|)dS. \end{aligned}$$

Note that, if $\|u\| \le R_0$ then $J_{\lambda,\gamma}(u) = I_\lambda(u)$ and if $\|u\| \ge R_1$, then

$$I_\lambda(u) = a\psi(u) + \frac{b}{\eta+1}\psi^{\eta+1}(u) + \int_\Omega \frac{1}{h(x)}|u|^{h(x)}dx - \frac{\lambda}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1}.$$

We can see that I_λ is coercive, and, hence I_λ is bounded from below.

Lemma 5. I_λ is $C^1(W^{1,p(x)}(\Omega),\mathbb{R})$, if $I_\lambda(u) \le 0$ then $\|u\| < R_0$ and $I_\lambda(v) = J_\lambda(v)$ for all v in a small enough neighborhood of u. Moreover, I_λ satisfies a local Palais-Smale condition for $c \le 0$.

Proof. It is immediate that $I_\lambda \in C^1(W^{1,p(x)}(\Omega),\mathbb{R})$. If $I_\lambda(u) \le 0$ then $\|u\| < R_0$ by construction of truncated functional. Now for all $u \in B_{R_0}(0)$ there exists $\varepsilon > 0$ such that $B_\varepsilon(u) \subset B_{R_0}(0)$ and $I_\lambda(v) = J_\lambda(v)$ for all $v \in B_\varepsilon(u)$, since $\|v\| < R_0$. To prove a local Palais-Smale condition for $c \le 0$, observe que that all Palais-Smale sequences for I_λ with $c \le 0$ must be bounded because the functional is coercive. By Lemma 2, there exists λ_0 such that for $0 < \lambda < \lambda_0$

$$\begin{aligned} c &\le 0 < \left(\frac{1}{\theta} - \frac{1}{q_{\mathcal{A}}^-}\right)\inf_{i\in I}\left(\gamma^{1-1/p(x_i)}\overline{a}^{1/p(x_i)}\overline{T}_{x_i}\right)^{\frac{p(x_i)p_*(x_i)}{p_*(x_i)-p(x_i)}} \\ &\quad + K\min\{\lambda^{\frac{(h/\beta)^-}{(h/\beta)^- - (r+1)}}, \lambda^{\frac{(h/\beta)^+}{(h/\beta)^+ - (r+1)}}\}. \end{aligned}$$

Hence by Lemma 3, we conclude that I_λ enjoys a local Palais-Smale condition.

□

Lemma 6. For every $n \in \mathbb{N}$ there exists $\varepsilon > 0$ such that

$$\gamma(I_\lambda^{-\varepsilon}) \ge n,$$

where $I_\lambda^{-\varepsilon} = \{u \in W_0^{1,p(x)}(\Omega); I_\lambda^{-\varepsilon}(u) \le -\varepsilon\}$ and γ is the Krasnoselskii's genus.

Proof. Let $E_n \subset W^{1,p(x)}(\Omega)$ be an n-dimensional subspace. Hence we have for $u \in E_n$ such that $\|u\| = 1$ and $0 < t < R_0$, we get

$$\begin{aligned} I_\lambda(tu) &= a\psi(tu) + \frac{b}{\eta+1}\psi^{\eta+1}(tu) - \frac{\lambda}{r+1}\left[\int_\Omega F(x,tu)dx\right]^{r+1} + \int_\Omega \frac{1}{h(x)}|tu|^{h(x)}dx \\ &\quad -\gamma\int_{\partial\Omega}\frac{1}{q(x)}|tu|^{q(x)}\tau(\|u\|)dS. \\ I_\lambda(tu) &\le \frac{at^{p^-}}{p^-}\rho_{1,p(x)}(u) + \frac{bt^{(\eta+1)p^-}}{(\eta+1)(p^-)^{\eta+1}}\rho_{1,p(x)}^{\eta+1}(u) + \frac{t^{h^-}}{h^-}\int_\Omega |u|^{h(x)}dx \\ &\quad -\frac{\lambda A_1^{r+1}t^{\beta^+(r+1)}}{(r+1)(\beta^+)^{r+1}}\left(\int_\Omega |u|^{\beta(x)}dx\right)^{r+1} - \frac{\gamma t^{q^+}}{q^+}\int_{\partial\Omega}|u|^{q(x)}dS. \end{aligned}$$

$$I_\lambda(tu) \le \frac{at^{p^-}}{p^-} + \frac{bt^{(\eta+1)p^-}}{(\eta+1)(p^-)^{\eta+1}} + \frac{t^{h^-}}{h^-}a_n - \frac{\lambda A_1^{r+1}t^{\beta^+(r+1)}}{(r+1)(\beta^+)^{r+1}}b_n - \frac{\gamma t^{q^+}}{q^+}c_n,$$

where

$$a_n = \sup\left\{\left(\int_\Omega |u|^{h(x)}\right); u \in E_n, \|u\| = 1\right\},$$

,

$$b_n = \inf\left\{\left(\int_\Omega |u|^{\beta(x)}\right)^{r+1}; u \in E_n, \|u\| = 1\right\},$$

and

$$c_n = \inf\left\{\int_{\partial\Omega} |u|^{q(x)} dS; u \in E_n, \|u\| = 1\right\}.$$

Then,

$$I_\lambda(tu) \leq \frac{at^{p^-}}{p^-} + \frac{bt^{(\eta+1)p^-}}{(\eta+1)(p^-)^{\eta+1}} + \frac{t^{h^-}}{h^-}a_n - \frac{\lambda A_1^{r+1} t^{\beta^+(r+1)}}{(r+1)(\beta^+)^{r+1}} b_n$$

Note that $a_n > 0$, $b_n > 0$ and $c_n > 0$, because E_n is finite dimensional and the norm $W^{1,p(x)}(\Omega)$ and $L^{\beta(x)}(\Omega)$ are equivalent on E_n. As $\beta^+(r+1) < p^-$ and $0 < t < R_0$ we obtain that there exists positive constants ρ and ϵ such that

$$I_\lambda(\rho u) < -\epsilon \quad \text{for} \quad u \in E_n, \|u\| = 1.$$

Therefore, if we set $S_{\rho,n} = \{u \in E_n : \|u\| = \rho\}$, we have that $S_{\rho,n} \subset I_\lambda^{-\epsilon}$. Hence, by monotonicity of genus,

$$\gamma(I_\lambda^{-\epsilon}) \geq \gamma(S_{\rho,n}) = n$$

as we wanted to show.

□

Lemma 7. Let $\Sigma = \{A \subset W^{1,p(x)}(\Omega) - 0 : A \text{ is closed}, A = -A\}$, $\Sigma_k = \{A \subset \Sigma : \gamma(A) \geq k\}$ where γ stands for the Krasnoselskii's genus. Then

$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} J_{\lambda,\gamma}(u)$$

is a negative critical value of $J_{\lambda,\gamma}$ and moreover, if $c = c_k = \cdots = c_{k+r}$, then $\gamma(K_c) \geq r+1$ where $K_c = \{u \in W^{1,p(x)}(\Omega) : J_{\lambda,\gamma}(u) = c, J'_{\lambda,\gamma}(u) = 0\}$.

Proof. The proof follows exactly the steps in Azorero and Alonso [3], using Lemma 6. □

Remark 1. The item (*ii*) of Theorem 1, is shown following the same steps as item (*i*).

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