JEPE Vol 2, 2016, p. 297-321

Orthogonal ⊥

SPREADING AND VANISHING FOR NONLINEAR STEFAN PROBLEMS IN HIGH SPACE DIMENSIONS

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Abstract. We classify the long-time behavior of solutions to nonlinear diffusive equations of the form $u_t - \Delta u = f(u)$ for $t > 0$ and x over a variable domain $\Omega(t) \subset \mathbb{R}^N$, with a Stefan condition for u over the free boundary $\Gamma(t) = \partial\Omega(t)$, and $u(0, x) > 0$ in $\Omega(0) = \Omega_0$. For monostable type of f and bistable type of f, we obtain a rather complete classification of the long-time dynamical behavior of the solution to this nonlinear Stefan problem, and examine how the behavior changes when $u(0, x)$ takes initial functions of the form $\sigma\phi(x)$ and $\sigma > 0$ is varied.

Dedicated to Professor David Kinderlehrer on the occassion of his 75th birthday

1. Introduction

We are interested in the long-time behavior of nonlinear Stefan problems of the form

$$(1.1) \qquad \begin{cases} u_t = \Delta u + f(u) & \text{for } x \in \Omega(t),\ t > 0, \\ u = 0 \text{ and } u_t = \mu|\nabla_x u|^2 & \text{for } x \in \Gamma(t),\ t > 0, \\ u(0, x) = u_0(x) & \text{for } x \in \Omega_0, \end{cases}$$

where $\Omega(t) \subset \mathbb{R}^N$ $(N \geq 2)$ is a varying domain with boundary $\Gamma(t)$ (commonly called the free boundary), $\mu > 0$ is a constant. We assume that $\Omega(0) = \Omega_0$ is a bounded domain which agrees with the interior of its closure $\overline{\Omega}_0$, $\partial\Omega_0$ satisfies the interior ball condition, and u_0 is taken from the set

$$\mathcal{I}(\Omega_0) := \{\phi \in C(\overline{\Omega}_0) \cap H^1(\Omega_0) \ :\ \phi(x) > 0 \text{ in } \Omega_0,\ \phi(x) = 0 \text{ on } \partial\Omega_0\}.$$

The basic assumptions on f are:

$$(F) \qquad \begin{cases} \text{(i) } f(0) = 0 \text{ and } f \in C^{1+\alpha}([0, \delta_0]) \text{ for some } \delta_0 > 0,\ \alpha \in (0, 1); \\ \text{(ii) } f(u) \text{ is locally Lipschitz in } [0, \infty),\ f(u) \leq 0 \text{ in } [M, \infty) \text{ for some } M > 0. \end{cases}$$

Under these assumptions, by [7], (1.1) has a unique weak solution and it is defined for all $t > 0$. Moreover, the following properties have been proved in [13]:

2010 *Mathematics Subject Classification.* Primary: 35B40, 35K55; Secondary: 35R35, 35K20.

Key words and phrases. Nonlinear diffusion equation, free boundary problem, asymptotic behavior, monostable nonlinearity, bistable nonlinearity, sharp threshold, spreading speed.

Received 16/11/2016, accepted 22/11/2016.

Research supported by the Australian Research Council and by the NSFC (No. 11671262).

Theorem A. (Theorem 1.1 in [13]) For each $t > 0$, $\tilde{\Gamma}(t) := \Gamma(t)\backslash\overline{\text{co}}(\Omega_0)$ is a $C^{2+\alpha}$ hypersurface in $\mathbb{R}^N$, and $\tilde{\Gamma} := \{(t,x) \ : \ x \in \tilde{\Gamma}(t), \ t > 0\}$ is a $C^{2+\alpha}$ hypersurface in $\mathbb{R}^{N+1}$.

Here $\overline{\text{co}}(\Omega_0)$ denotes the closed convex hull of Ω_0.

Theorem B. (Theorem 1.2 in [13]) $\Omega(t)$ is expanding in the sense that $\overline{\Omega}_0 \subset \Omega(t) \subset \Omega(s)$ if $0 < t < s$. Moreover, $\Omega_\infty := \cup_{t>0}\Omega(t)$ is either the entire space $\mathbb{R}^N$, or it is a bounded set. Furthermore,
(i) when $\Omega_\infty = \mathbb{R}^N$, for all large t, $\Gamma(t)$ is a $C^{2+\alpha}$ closed hypersurface in $\mathbb{R}^N$, and there exists a continuous function $M(t)$ such that

$$\Gamma(t) \subset \{x \ : \ M(t) - \pi d_0 \le |x| \le M(t)\}, \quad \text{where } d_0 \text{ is the diameter of } \Omega_0; \tag{1.2}$$

(ii) when Ω_∞ is bounded, $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty(\Omega(t))} = 0$.

Theorem C. (Theorem 1.3 in [13]) If $f(u) = u(a - bu)$ with a, b positive constants, then there exists $\mu^* \ge 0$ such that $\Omega_\infty = \mathbb{R}^N$ if and only if $\mu > \mu^*$. Moreover, when $\Omega_\infty = \mathbb{R}^N$, the following holds:

$$\lim_{t\to\infty} \frac{M(t)}{t} = k_0(\mu) > 0, \ \lim_{t\to\infty} \max_{|x|\le ct} \left| u(t,x) - \frac{a}{b} \right| = 0 \ \forall c \in (0, k_0(\mu)),$$

where $k_0(\mu)$ is increasing in μ and $\lim_{\mu\to\infty} k_0(\mu) = 2\sqrt{a}$.

The proof of the above results is built upon techniques in [2, 18, 19, 20] and a number of new ones. In particular, the following monotonicity lemma has played an important role.

Lemma D. (Lemma 4.2 in [13]) For any $t > 0$, $x \in \Omega(t)\backslash\overline{\text{co}}(\Omega_0)$ and $\nu \in \mathbb{S}^{N-1}$ satisfying $\nu \cdot (z - x) < 0 \ \forall z \in \overline{\text{co}}(\Omega_0)$, we have $\partial_\nu u(t,x) < 0$.

Problem (1.1) is often used to describe the spreading of an invasive species or a new species, whose population density at time t and space location x is given by $u(t,x)$, and the population range at time t is $\Omega(t)$, with $\Gamma(t) = \partial\Omega(t)$ representing the moving spreading front of the species. In this context, when Ω_∞ is a bounded set and $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty(\Omega(t))} = 0$, one often says that the species is **vanishing**. On the other hand, the case for $\Omega_\infty = \mathbb{R}^N$ described in Theorem C indicates a situation that the species can invade the entire available space and establish itself, which is widely called the **spreading** case. Thus Theorem C gives a spreading-vanishing dichotomy for the long-time dynamical behavior of $u(t,x)$. Such a dichotomy was first discovered in [8] for the one space dimension case, and the result has subsequently been extended in several directions (see, for example, [6, 7, 9, 10, 13]).

The main purpose of this paper is to obtain sharp long-time behavior for (1.1) in the spirit of Theorem C above, but for much more general nonlinear functions $f(u)$. Moreover, while μ is treated as a parameter in Theorem C to distinguish the spreading-vanishing behaviors, here we will fix $\mu > 0$ and take $u_0 = \sigma\phi$, with $\phi \in \mathcal{I}(\Omega_0)$ fixed and $\sigma > 0$ regarded as a varying parameter.

Such a problem in one space dimension has been treated in [9] and [10] (after the initial work [8]), where a rather complete theory has been established for f belonging to three types of nonlinearities: Monostable type, bistable type and combustion type. However, in higher space dimensions, the problem is much more difficult to treat, and the behavior of (1.1) may be more complicated and even different, especially in the combustion case (which is studied separately in [11]). Here we only consider the case that f is monostable and the case that f is bistable, as precisely described below.

In this paper, the nonlinear function f is said to be of monostable type if

$$(f_M) \qquad \begin{cases} \text{(i) } f(0)=0 \text{ and } f\in C^1([0,\infty)); \\ \text{(ii) } f(s)>0 \text{ for } s\in(0,1) \text{ and } f(s)<0 \text{ for } s\in(1,\infty); \\ \text{(iii) } f'(0)>0,\ f'(1)<0; \\ \text{(iv) } f\in C^{1+\alpha}([0,\delta_1]) \text{ for some } \delta_1>0 \text{ and } \alpha\in(0,1). \end{cases}$$

A typical example of $f(u)$ satisfying all the above conditions is $u(1-u)$.

The function f is said to be of bistable type if it satisfies

$$(f_B) \qquad \begin{cases} \text{(i) } f(0)=0 \text{ and } f\in C^1([0,\infty)); \\ \text{(ii) } f(s)<0 \text{ for } s\in(0,\theta)\cup(1,\infty) \text{ and } f(s)>0 \text{ for } s\in(\theta,1); \\ \text{(iii) } \int_0^1 f(s)ds>0; \\ \text{(iv) } f'(0)<0, f'(1)<0; \\ \text{(v) } f(u)/(u-\bar\theta) \text{ is non-increasing in } u\in(\bar\theta,1), \text{ where } \bar\theta\in(\theta,1) \\ \qquad\qquad \text{is uniquely determined by } \int_0^{\bar\theta} f(s)ds=0; \\ \text{(vi) } \lim_{s\searrow\theta} f(s)/(s-\theta)^\kappa\in(0,\infty], \\ \qquad \text{where } \kappa=\frac{N}{N-2} \text{ when } N>2, \text{ and } \kappa\in(0,\infty) \text{ when } N=2; \\ \text{(vii) } f\in C^{1+\alpha}([0,\delta_1]) \text{ for some } \delta_1>0,\ \alpha\in(0,1). \end{cases}$$

A typical example of $f(u)$ satisfying all the conditions in (f_B) except for (v) is $f(u) = u(u-\theta)(1-u)$ with $\theta\in(0,\frac{1}{2})$, and (v) is also satisfied if $\theta\in(\frac{5}{16},\frac{1}{2})$.

The definitions for monostable and bistable nonlinearities here are more restrictive than those in [9]. For monostable f, the extra requirement (iv) is due to the same restriction used in [13]; for bistable f, the extra restrictions (v) and (vi) in (f_B) are used to guarantee the uniqueness of the "ground state solution" v for (1.3) below (cf. [21, 22]), and (vii) is due to [13] as in the monostable case.

For monostable type of f our main results are contained in the following theorem.

Theorem 1. *Assume that f satisfies (f_M). Let $\phi\in\mathcal{I}(\Omega_0)$ and let $u_\sigma(t,x)$ be the solution of* (1.1) *with initial function $u_0(x)=\sigma\phi(x)$, $\sigma>0$. Then there exists $\sigma^*=\sigma^*(\phi)\in[0,\infty]$ such that*

(i) *for $0<\sigma\le\sigma^*$,* **vanishing** *happens, i.e., Ω_∞ is bounded and*

$$\lim_{t\to\infty}\max_{x\in\Omega(t)} u_\sigma(t,x)=0;$$

(ii) *for $\sigma>\sigma^*$,* **spreading** *happens, i.e., $\Omega_\infty=\mathbb{R}^N$ and*

$$\lim_{t\to\infty} u_\sigma(t,x)=1 \text{ locally uniformly in } x\in\mathbb{R}^N,$$

$$M(t)=c^*t-c_*(N-1)\log t+O(1) \text{ as } t\to\infty,$$

where c^* and c_* are positive constants independent of N and u_0 (they depend on f and μ only), and $M(t)$ is given in (1.2).

The constants c^* and c_* are given in [14]; see Theorems E and F in Section 2 below. The cases $\sigma^*=0$ and $\sigma^*=\infty$ can actually happen; see Remark 3 for details.

The next theorem is about bistable type of f.

Theorem 2. *Assume that f satisfies (f_B). Let $\phi \in \mathcal{I}(\Omega_0)$ and let $u_\sigma(t,x)$ be the solution of (1.1) with initial function $u_0(x) = \sigma\phi(x)$, $\sigma > 0$. Then either vanishing happens for u_σ for all $\sigma > 0$, or there exists $\sigma^* = \sigma^*(\phi) \in (0,\infty)$ such that*

(i) *for $0 < \sigma < \sigma^*$,* **vanishing** *happens, i.e., Ω_∞ is bounded and*

$$\lim_{t\to\infty} \max_{x\in\Omega(t)} u_\sigma(t,x) = 0;$$

(ii) *for $\sigma > \sigma^*$,* **spreading** *happens, i.e., $\Omega_\infty = \mathbb{R}^N$ and*

$$\lim_{t\to\infty} u_\sigma(t,x) = 1 \text{ locally uniformly in } x \in \mathbb{R}^N,$$

$$M(t) = c^*t - c_*(N-1)\log t + O(1) \text{ as } t \to \infty,$$

where c^* and c_* are positive constants independent of N and u_0 (they depend on f and μ only), and $M(t)$ is given in (1.2);

(iii) *for $\sigma = \sigma^*$, a* **transition** *case happens, where we have $\Omega_\infty = \mathbb{R}^N$ and for any sequence $\{t_k\}$ increasing to ∞ as $k \to \infty$, there is a subsequence $\{t_{k_j}\}$ and a point $x_0 \in \overline{\text{co}}(\Omega_0)$ such that*

$$\lim_{j\to\infty} u_\sigma(t_{k_j}, x) = v(|x - x_0|) \text{ locally uniformly for } x \in \mathbb{R}^N,$$

where $v(r)$ is the unique positive solution to

$$v_{rr} + \frac{N-1}{r}v_r + f(v) = 0 \text{ for } r > 0, \quad v_r(0) = 0,\ v(\infty) = 0. \tag{1.3}$$

The rest of the paper is organized as follows. In Section 2, we study the radially symmetric case of (1.1). In Section 3, we give the proofs for Theorems 1 and 2, where the results in Section 2 are used extensively.

2. The Radially Symmetric Case

In this section, we study the radially symmetric case of (1.1), which forms an important first step towards the proof of Theorems 1 and 2. In such a case, $u = u(t,r)$ and $\Omega(t) = \{r < h(t)\}$, with $r = |x|$. So the free boundary is given by $r = h(t)$. To avoid confusion, we will use $w(t,r)$ to replace $u(t,r)$ and rewrite (1.1) in the form

$$\begin{cases} w_t = \Delta w + f(w), & 0 < r < h(t),\ t > 0, \\ w_r(t,0) = 0,\ w(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu w_r(t,h(t)), & t > 0, \\ h(0) = h_0, \quad w(0,r) = w_0(r), & r \le h_0, \end{cases} \tag{2.1}$$

where $\Delta w = w_{rr} + \frac{N-1}{r}w_r$ and h_0 is a positive constant. For simplicity, we will choose the initial function w_0 from

$$\mathscr{K}(h_0) := \Big\{ \psi \in C^2([0,h_0]) : \psi'(0) = \psi(h_0) = 0,\ \psi(r) > 0 \text{ in } [0,h_0) \Big\}. \tag{2.2}$$

For (2.1), apart from results needed for the proof of Theorems 1 and 2, we also consider cases which are of independent interest. For that purpose, we will consider general f satisfying

$$f : [0,\infty) \to \mathbb{R} \text{ is } C^1 \text{ and } f(0) = 0, \tag{2.3}$$

and

$$f(w) \le Kw \text{ for all } w \ge 0 \text{ and some } K > 0. \tag{2.4}$$

With f satisfying (2.3) and (2.4), by [6], for any given $h_0 > 0$ and $w_0 \in \mathscr{K}(h_0)$, (2.1) has a classical solution $(w(t,r), h(t))$ belonging to $C^{1,2}(D) \times C^1([0,\infty))$, such that all the identities in (2.1) are satisfied pointwisely, where

$$D := \big\{(t,r) : t \in (0,\infty),\ r \in [0, h(t)]\big\}.$$

In the rest of the paper, the solution of (2.1) may also be denoted by $(w(t,r;w_0), h(t;w_0))$, or simply (w,h), depending on the context.

By [6] we have $h'(t) > 0$ and $w(t,r) > 0$ for $r \in [0,h(t))$. So $h_\infty := \lim_{t\to\infty} h(t) \in (h_0, \infty]$ always exists.

2.1. **Main Results.** Our first result on (2.1) is a general convergence theorem, which is an analogue of Theorem 1.1 in [12] and Theorem 1.1 in [9].

Theorem 3. *Suppose that f satisfies* (2.3) *and (w,h) is a solution of* (2.1) *defined for all $t > 0$. If $w(t,r)$ is bounded, namely*

$$w(t,r) \le C \text{ for all } t > 0,\ \ r \in [0,h(t)] \text{ and some } C > 0,$$

then either $h_\infty < \infty$ and $\lim_{t\to\infty} \|w(t,r)\|_{L^\infty([0,h(t)])} = 0$; or $h_\infty = \infty$ and $\lim_{t\to\infty} w(t,r) = v(r)$ locally uniformly for $r \in [0,\infty)$, where $v(r)$ satisfies

$$v'' + \frac{N-1}{r} v' + f(v) = 0 \ \text{ for } r > 0, \quad v'(0) = 0, \tag{2.5}$$

and either $v \equiv$ constant or $v'(r) < 0$ for $r \ge h_0$; in the former case, the constant is necessarily a nonnegative zero of f.

The proof of this theorem is given in subsection 2.4. Using this theorem we can prove the following results, which give a rather complete description for the asymptotic behavior of (2.1) with monostable and bistable types of f. We remark that the condition (iv) in (f_M) and (vii) in (f_B) is not required for the radial case here.

Theorem 4. *Assume that f satisfies* (i)-(iii) *in (f_M). If $h_0 > 0$, $\psi \in \mathscr{K}(h_0)$ and if $w_\sigma(t,r)$ is the solution of* (2.1) *with initial function $w_0 = \sigma\psi$, then there exists $\sigma^* = \sigma^*(h_0,\psi) \in [0,\infty]$ such that*

(i) *for $0 < \sigma \le \sigma^*$, vanishing happens, i.e., $h_\infty < \infty$ and*

$$\lim_{t\to\infty} \max_{0\le r\le h(t)} w(t,r) = 0;$$

(ii) *for $\sigma > \sigma^*$, spreading happens, i.e., $h_\infty = \infty$ and*

$$\lim_{t\to\infty} w(t,r) = 1 \text{ locally uniformly in } r \in [0,\infty),$$

For bistable f we have a trichotomy result.

Theorem 5. *Assume that f satisfies* (i)-(vi) *in (f_B). If $h_0 > 0$, $\psi \in \mathscr{K}(h_0)$ and if $w_\sigma(t,r)$ is the solution of* (2.1) *with initial function $w_0 = \sigma\psi$, then either w_σ vanishes for every $\sigma > 0$, or there exists $\sigma^* = \sigma^*(h_0,\psi) \in (0,\infty)$ such that*

(i) *for $0 < \sigma < \sigma^*$, vanishing happens, i.e., $h_\infty < \infty$ and*

$$\lim_{t\to\infty} \max_{0\le r\le h(t)} w(t,r) = 0;$$

(ii) *for $\sigma > \sigma^*$, spreading happens, i.e., $h_\infty = \infty$ and*

$$\lim_{t\to\infty} w(t,r) = 1 \text{ locally uniformly in } r \in [0,\infty)$$

(iii) *for $\sigma = \sigma^*$, $h_\infty = \infty$ and*

$$\lim_{t\to\infty} |w(t,r) - v(r)| = 0 \text{ locally uniformly in } r \in [0,\infty),$$

where v is the unique positive solution to

$$v'' + \frac{N-1}{r}v' + f(v) = 0 \text{ for } r > 0, \quad v'(0) = 0,\ v(\infty) = 0. \tag{2.6}$$

When spreading happens in Theorems 4 and 5, much better descriptions of the long-time behavior of $(w(t,r), h(t))$ have been obtained in [14]. For convenience of reference and later use, we recall them below.

We need the following result for the semi-wave from [9].

Theorem E (Proposition 1.8 and Theorem 6.2 of [9]) Suppose that f satisfies (i)-(iii) in (f_M), or it satisfies (i)-(iv) in (f_B). Then for any $\mu > 0$ there exists a unique $c^* = c^*(\mu) > 0$ and a unique solution q_{c^*} to

$$q'' - cq' + f(q) = 0,\ q > 0 \text{ in } (0,\infty),\ \ q(0) = 0,\ q(\infty) = 1 \tag{2.7}$$

with $c = c^*$ such that $q'_{c^*}(0) = \frac{c^*}{\mu}$.

We remark that this function q_{c^*} is shown in [9] to satisfy $q'_{c^*}(z) > 0$ for $z \geq 0$, and it is called a *semi-wave with speed c^**.

When spreading happens for (2.1), namely

$$h_\infty = \infty \text{ and } \lim_{t\to\infty} w(t,r) = 1 \text{ locally uniformly for } r \in [0,\infty),$$

it is shown in [14] that the following holds.

Theorem F (Theorem 4.1 of [14]) Let f be as in Theorem E, and suppose spreading happens for (2.1). Then there exists $c_* > 0$ independent of N, and $\hat{h} \in \mathbb{R}^1$ such that

$$\lim_{t\to\infty} \big[h(t) - c^*t + (N-1)c_* \log t\big] = \hat{h}, \tag{2.8}$$

$$\lim_{t\to\infty} \sup_{r\in[0,\,h(t)]} \Big| w(t,r) - q_{c^*}(c^*t - (N-1)c_*\log t + \hat{h} - r)\Big| = 0.$$

Moreover, the constant c_* is given by

$$c_* = \frac{1}{\zeta\, c^*},\ \ \zeta = 1 + \frac{c^*}{\mu^2 \int_0^\infty q'_{c^*}(z)^2 e^{-c^* z} dz}.$$

2.2. **Comparison results.** We give some basic comparison results which will be used later in the paper. In these results, we always assume that f satisfies (2.3).

Lemma 1. *Suppose that $T \in (0,\infty)$, $\overline{h} \in C^1([0,T])$, $\overline{w} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ with $D_T = \{(t,r) \in \mathbb{R}^2 : 0 < t \leq T, 0 < r < \overline{h}(t)\}$, and*

$$\begin{cases} \overline{w}_t \geq \Delta \overline{w} + f(\overline{w}) & \text{for } 0 < t \leq T,\ 0 < r < \overline{h}(t), \\ \overline{w}(t,\overline{h}(t)) = 0, \quad \overline{h}'(t) \geq -\mu \overline{w}_r(t,\overline{h}(t)) & \text{for } 0 < t \leq T, \\ \overline{w}_r(t,0) \leq 0 & \text{for } 0 < t \leq T. \end{cases}$$

If

$$h_0 \le \overline{h}(0) \quad \text{and} \quad w_0(r) \le \overline{w}(0,r) \text{ in } [0,h_0],$$

then the solution (w,h) *of the free boundary problem* (2.1) *satisfies*

$$h(t) \le \overline{h}(t) \text{ in } (0,T], \quad w(t,r) \le \overline{w}(t,r) \text{ for } t \in (0,T] \text{ and } r \in [0,h(t)).$$

Lemma 2. *Suppose that* $T \in (0,\infty)$, $\xi, \overline{h} \in C^1([0,T])$, $\overline{w} \in C(\overline{D_T^*}) \cap C^{1,2}(D_T^*)$ *with* $D_T^* = \{(t,r) \in \mathbb{R}^2 : 0 < t \le T,\ \xi(t) < r < \overline{h}(t)\}$, $\xi(t) \ge 0$ *in* $[0,T]$ *and*

$$\begin{cases} \overline{w}_t \ge \Delta \overline{w} + f(\overline{w}) & \text{for } 0 < t \le T,\ \xi(t) < r < \overline{h}(t), \\ \overline{w}(t,\xi(t)) \ge w(t,\xi(t)) & \text{for } 0 < t \le T, \\ \overline{w}(t,\overline{h}(t)) = 0, \quad \overline{h}'(t) \ge -\mu \overline{w}_r(t,\overline{h}(t)) & \text{for } 0 < t \le T, \end{cases}$$

with

$$\xi(0) \le h_0 \le \overline{h}(0), \quad w_0(r) \le \overline{w}(0,r) \text{ in } [\xi(0),h_0],$$

where (w,h) *solves* (2.1). *Then*

$$h(t) \le \overline{h}(t) \text{ in } (0,T], \quad w(t,r) \le \overline{w}(t,r) \text{ for } t \in (0,T] \text{ and } \xi(t) \le r \le h(t).$$

The proof of Lemma 1 is identical to that of Lemma 5.7 in [8], and a minor modification of this proof yields Lemma 2 (see also [9]).

Remark 1. The function $\overline{w}$, or the pair $(\overline{w},\overline{h})$ in Lemmas 1 and 2 is often called an upper solution to (2.1). A lower solution can be defined analogously by reversing all the inequalities. We also have corresponding comparison results for lower solutions in each case.

The following result shows that an upper solution in one space dimension can often be used to construct an upper solution in high space dimensions.

Lemma 3. *Suppose that* $t_1 < t_2$ *and* $\xi, \overline{h} \in C^1([t_1,t_2])$, $\overline{w}(t,y) \in C(\overline{D}_*) \cap C^{1,2}(D_*)$ *with* $D_* = \{(t,y) \in \mathbb{R}^2 : t_1 < t < t_2, 0 \le \xi(t) < y < \overline{h}(t)\}$, *and*

$$\overline{w}_t \ge \overline{w}_{yy} + f(\overline{w}) \text{ for } y \in [\xi(t),\overline{h}(t)],\ t \in [t_1,t_2].$$

Assume that

$$\overline{w}_y(t,y) \le 0 \quad \text{for } \ y \in [\xi(t),\overline{h}(t)],\ t \in [t_1,t_2].$$

Then $\tilde{w}(t,r) := \overline{w}(t,r)$ *satisfies*

$$\tilde{w}_t \ge \Delta \tilde{w} + f(\tilde{w}) \text{ for } r \in [\xi(t),\overline{h}(t)],\ t \in [t_1,t_2].$$

Proof. Since $\overline{w}_y(t,y) \le 0$, a direct calculation gives

$$\tilde{w}_t - \Delta \tilde{w} - f(\tilde{w}) = \overline{w}_t - \overline{w}_{rr} - \frac{N-1}{r}\overline{w}_r - f(\overline{w}) \ge \overline{w}_t - \overline{w}_{rr} - f(\overline{w}) \ge 0.$$

□

2.3. **Local and global existence.** The following local existence result can be proved in the same way as in [6] (see Theorems 2.1 and 4.1 there).

Theorem 6. *Suppose that* (2.3) *holds. For any given* $w_0 \in \mathscr{K}(h_0)$ *and any* $\alpha \in (0,1)$*, there is a* $T > 0$ *such that problem* (2.1) *admits a unique solution*

$$(w,h) \in C^{(1+\alpha)/2,1+\alpha}(\overline{G}_T) \times C^{1+\alpha/2}([0,T]);$$

moreover,

$$\|w\|_{C^{(1+\alpha)/2,1+\alpha}(\overline{G}_T)} + \|h\|_{C^{1+\alpha/2}([0,T])} \leq C, \tag{2.9}$$

where $G_T = \{(t,r) \in \mathbb{R}^2 : t \in (0,T],\ r \in [0,h(t)]\}$*,* C *and* T *depend on* h_0*,* α *and* $\|w_0\|_{C^2([0,h_0])}$*.*

Remark 2. As in [6, 8], problem (2.1) can be converted to a problem in a fixed domain. Using the Schauder estimates to the latter problem, one can obtain additional regularity for the solution. Therefore, we indeed have $w \in C^{1+\alpha/2,2+\alpha}(G_T)$.

Lemma 4. *Suppose that* (2.3) *holds,* (w,h) *is a solution to* (2.1) *defined for* $t \in [0,T_0)$ *for some* $T_0 \in (0,\infty)$*, and there exists* $C_1 > 0$ *such that*

$$w(t,r) \leq C_1 \text{ for } t \in [0,T_0) \text{ and } r \in [0,h(t)].$$

Then there exists C_2 *depending on* C_1 *but independent of* T_0 *such that*

$$0 < h'(t) \leq C_2 \quad \text{for } t \in (0,T_0).$$

Moreover, the solution can be extended to some interval $(0,T)$ *with* $T > T_0$*.*

The proof of this lemma is identical to that of [6, Lemma 4.2] (see also [8, Lemma 2.2] and [9, Lemma 2.6]). This lemma implies that the solution of (2.1) can be extended as long as w remains bounded. In particular, if f satisfies (2.3) and (2.4), then the solution of (2.1) exists for all $t > 0$.

2.4. **Proof of Theorem 3.** To prove Theorem 3, we need the following lemma.

Lemma 5. *Suppose that* $(w(t,r),h(t))$ *is a solution of* (2.1) *as given in Theorem 3. Then*

$$w_r(t_0,r) < 0 \text{ for all } t_0 > 0,\ r \in (h_0,h(t_0)]. \tag{2.10}$$

Proof. First $w_r(t_0,h(t_0)) < 0$ follows from the Hopf lemma directly. For any $r_1 \in (h_0,h(t_0))$, there exists a unique $t_1 \in (0,t_0)$ such that $h(t_1) = r_1$ and $h(t) > r_1$ for $t > t_1$. By the Hopf lemma again, we have $w_r(t_1,r_1) = w_r(t_1,h(t_1)) < 0$. To prove (2.10) we need to show that $w_r(t_0,r_1) < 0$.

We actually show that $w_r(t,r_1) < 0$ for $t \in (t_1,t_0]$. Note that $u(t,x) := w(t,|x|) = w(t,r)$ is a radially symmetric solution of (1.1) for $t \in [0,\infty)$ and $x \in B_{h(t)}$. For $t > t_1$ denote

$$G_t := \{x = (x_1,x') \in B_{h(t)} : x_1 > r_1\}$$

and define

$$z(t,x) := u(t,x_1,x') - u(t,2r_1 - x_1,x') \quad \text{for } x \in G_t.$$

Then we have

$$z_t = \Delta z + c(t,x)z \quad \text{for } t > t_1,\ x \in G_t \text{ and some } c \in L^\infty,$$

and

$$z(t,x)=0 \text{ for } x\in\partial G_t\cap\{x:x_1=r_1\},\quad z(t,x)<0 \text{ for } x\in\partial G_t\cap\{x:x_1>r_1\}.$$

Hence we can apply the strong maximum principle and the Hopf lemma to deduce

$$z(t,x)<0 \text{ in } G_t,\ \ z_{x_1}(t,x)<0 \quad \text{for } x\in\partial G_t\cap\{x:x_1=r_1\},\ t>t_1.$$

We thus have $0>z_{x_1}(t,r_1,0)=2u_{x_1}(t,r_1,0)=2w_r(t,r_1)$ for $t>t_1$. □

Proof of Theorem 3: We will make use of Lemma 5 and follow the ideas of [9, 12] with considerable variations.

Let (w,h) be a solution of (2.1) as given in the theorem. Then h_∞ is either a finite positive number or $h_\infty=\infty$. If $h_\infty<\infty$, then the same reasoning as in [8, 9] shows that $\lim_{t\to\infty}\|w(t,\cdot)\|_{L^\infty([0,h(t)])}=0$.

Next we consider the case that $h_\infty=\infty$. Denote by $\omega(w)$ the ω-limit set of $w(t,\cdot)$ in the topology of $L^\infty_{loc}([0,\infty))$, namely, a function $\eta\in\omega(w)$ if and only if there exists a sequence $0<t_1<t_2<t_3<\cdots\to\infty$ such that

$$\lim_{n\to\infty} w(t_n,r)=\eta(r) \qquad \text{locally uniformly in } [0,\infty). \tag{2.11}$$

By standard parabolic estimates, we see that the convergence (2.11) implies convergence in the $C^2_{loc}([0,\infty))$ topology. Thus the definition of $\omega(w)$ remains unchanged if the topology of $L^\infty_{loc}((0,\infty))$ is replaced by that of $C^2_{loc}([0,\infty))$.

It is well-known that $\omega(w)$ is compact and connected, and for any $\eta\in\omega(w)$ there exists an entire orbit passing through it, namely a solution of $\tilde\eta_t=\tilde\eta_{rr}+f(\tilde\eta)$ defined for all $t\in\mathbb{R}$ and $r\in[0,\infty)$, that satisfies $\tilde\eta(0,r)=\eta(r)$. We can find such an entire solution $\tilde\eta(t,r)$ by choosing a suitable sequence $0<t_1<t_2<t_3<\cdots\to\infty$ such that

$$w(t+t_n,r)\to\tilde\eta(t,r) \quad \text{as } n\to\infty. \tag{2.12}$$

Here the convergence is understood in the L^∞_{loc} sense in $(t,r)\in\mathbb{R}\times[0,\infty)$, but, by parabolic regularity, it takes place in the $C^{1,2}_{loc}(\mathbb{R}\times[0,\infty))$ sense as well.

For clarity we divide the arguments below into two parts, each proving a specific claim.

Claim 1: $\omega(w)$ consists of solutions of (2.5), namely functions $v(r)$ satisfying

$$v_{rr}+\frac{N-1}{r}v_r+f(v)=0 \quad \text{for } r>0, \qquad v_r(0)=0. \tag{2.13}$$

Let $\eta(r)$ be an arbitrary element of $\omega(w)$ and $\tilde\eta(t,r)$ be the entire orbit satisfying $\tilde\eta(0,r)=\eta(r)$. Since $\tilde\eta$ is a nonnegative solution of

$$\tilde\eta_t=\Delta\tilde\eta+f(\tilde\eta),\quad t\in\mathbb{R},\ r\in[0,\infty),$$

and $f(0)=0$, by the strong maximum principle we have either $\tilde\eta(t,r)>0$ for all $t\in\mathbb{R}$ and $r\in[0,\infty)$, or $\tilde\eta\equiv0$. In the latter case we have $\eta\equiv0$, which is a solution to (2.13). In what follows we assume the former, namely $\eta(r)>0$ for $r\in[0,\infty)$.

By Lemma 5, $\eta'(r)\le0$ for $r\in[h_0,\infty)$, and $\eta'(0)=0$. Let $v(r)$ be the solution of the following initial value problem:

$$v_{rr}+\frac{N-1}{r}v_r+f(v)=0 \quad \text{for } r>0, \qquad v(0)=\eta(0)>0,\quad v_r(0)=0. \tag{2.14}$$

This problem has a unique positive solution defined for all small $r > 0$ (see [22]). By unique continuation, $v(r)$ can be extended to a positive solution of (2.14) in $[0, \infty)$, or a solution of (2.14) with compact positive support, namely there exists $R > 0$ such that

$$v(r) > 0 \text{ in } [0, R) \text{ and } v(R) = 0 \text{ or } v(R) = \infty.$$

In case v has compact positive support, denote

$$h_1(t) := \min\{R, h(t)\} \quad \text{for} \quad t > 0.$$

Since $h(t)$ is strictly increasing with $h_\infty = \infty$, we can find some large $t_1 > 0$ such that $h(t) > h_1(t) = R$ for $t \geq t_1$.

In case $v > 0$ in $[0, \infty)$, we simply take $h_1(t) := h(t)$ and $t_1 = 1$. Then

$$z(t, r) := w(t, r) - v(r), \quad r \in [0, h_1(t)),\ t > t_1$$

satisfies

$$\begin{cases} z_t = \Delta z + c(t, r)z, & r \in [0, h_1(t)),\ t > t_1, \\ z_r(t, 0) = 0,\ z(t, h_1(t)) \neq 0, & t > t_1, \end{cases} \tag{2.15}$$

for some $c \in L^\infty_{loc}(D)$, with $D := \{(t, r) : t \geq t_1,\ 0 \leq r < h_1(t)\}$. Denote by $\mathcal{Z}(z(t, \cdot))$ the number of zeros of $z(t, \cdot)$ on $[0, h_1(t))$.

By Theorem 2.1 of [4][1], one sees that

(i) $\mathcal{Z}(z(t, \cdot)) < \infty$ for any $t > t_1$;
(ii) $\mathcal{Z}(z(t, \cdot))$ is monotone nonincreasing in t;
(iii) if $z(t_0, r_0) = z_r(t_0, r_0) = 0$ for some $t_0 > t_1$ and $0 \leq r_0 < h_1(t_0)$, then $\mathcal{Z}(z(t, \cdot)) > \mathcal{Z}(z(s, \cdot))$ for any $t_1 < t < t_0 < s$.

Consequently, for sufficiently large t, the function z has fixed number of simple zeros on $[0, h_1(t))$. In view of this and the fact that

$$\lim_{n\to\infty} z(t + t_n, r) = \tilde{\eta}(t, r) - v(r) \quad \text{in} \quad C^{1,2}_{loc}(\mathbb{R} \times [0, h_1(\infty))),$$

we see (cf. [12, Lemma 2.6]) that for each $t \in \mathbb{R}$, either $\tilde{\eta}(t, r) \equiv v(r)$ on $[0, h_1(\infty))$, or $\tilde{\eta}(t, r) - v(r)$ has only simple zeros on $[0, h_1(\infty))$. The latter, however, is impossible for $t = 0$ since $r = 0$ is a degenerate zero of $\tilde{\eta}(0, r) - v(r) \equiv \eta(r) - v(r)$. Consequently, $\tilde{\eta}(0, r) = \eta(r) \equiv v(r)$. This proves Claim 1.

Claim 2: $\omega(w)$ consists of a single function which is a solution of (2.5), and it is either a nonnegative constant or a positive radially symmetric function strictly decreasing for $r > h_0$.

By Claim 1, $\omega(w)$ consists of solutions of (2.5). First, we show that $\omega(w)$ is a singleton. Otherwise, $\omega(w)$ contains infinitely many elements since it is connected. Let v_1, v_2 and v_3 be three distinct functions in $\omega(w)$. Then $v_1(0)$, $v_2(0)$ and $v_3(0)$ are different from each other since $v_1(r)$, $v_2(r)$ and $v_3(r)$ are distinct solutions of (2.14) with initial values $(v(0), v'(0)) = (v_i(0), 0)$ for $i = 1, 2, 3$, respectively. Without loss of generality we can assume $v_1(0) < v_2(0) < v_3(0)$. Using Lemma 6 below, we see that $w(t, 0) - v_2(0)$ never changes sign for sufficiently large t. This clearly contradicts the fact that $v_1, v_3 \in \omega(w)$.

[1]Theorem 2.1 of [4] treats the case where the linear parabolic equation is satisfied over $(t, r) \in [T_1, T_2] \times [0, 1]$. However, the proof there easily carries over to the case of (2.15) where the boundary is $r = h_1(t)$ instead of $r = 1$.

So we have proved that $\omega(w)$ is a singleton. Suppose $\omega(w) = \{\eta\}$. By Claim 1, $\eta(r)$ is a solution of (2.5). Then $\zeta := \eta_r$ satisfies

$$\zeta'' + \frac{N-1}{r}\zeta' + \Big[f'(\eta) - \frac{N-1}{r^2}\Big]\zeta = 0 \quad \text{for} \quad r > 0.$$

It follows from Lemma 5 that $\zeta(r) \le 0$ for $r \ge h_0$. Hence by Harnack's inequality either $\zeta(r) \equiv 0$ for all $r > h_0$, or $\zeta(r) < 0$ for all $r > h_0$. In the former case, by the unique continuation property of the solutions to the above linear ODE, we deduce $\eta(r) = 0$ for all $r > 0$. This proves Claim 2.

The proof of Theorem 3 is now complete under the assumption that Lemma 6 bow is valid. □

Lemma 6. *Let (w, h) be given in Theorem 3, with $h_\infty = \infty$. Suppose that $v \in \omega(w)$. Then there exists $t^* > 0$ large such that $w(t, 0) - v(0)$ does not change sign for $t \ge t^*$.*

Proof. The function $z(t,r) := w(t,r) - v(r)$ satisfies an equation of the form (2.15), with $h_1(t)$ replaced by $h(t)$, and t_1 replaced by 0. Hence the number of zeros of $z(t,\cdot)$ in $[0, h(t)]$, denoted by $\mathcal{Z}(z(t,\cdot))$, has the properties (i), (ii) and (iii) stated just below (2.15).

We now use the idea of [17]. Denote

$$A^+ := \{(t,r) : t > 0,\ r \in [0, h(t)], z(t,r) > 0\},$$

$$A^- := \{(t,r) : t > 0,\ r \in [0, h(t)], z(t,r) < 0\},$$

and for each $s > 0$ set

$$\Sigma_s := \{(t,r) : t \ge s,\ r \in [0, h(t)]\},\ l_s := \{s\} \times [0, h(s)].$$

The proof of Lemma 2 in [17] yields the following conclusion: *Fix $t_0 > 0$ and let C be any connected component of $A^+ \cap \Sigma_{t_0}$. Then $C \cap l_{t_0} \ne \emptyset$. The same assertion holds for any connected component of $A^- \cap \Sigma_{t_0}$.*

If there is a sequence $t_0 < t_1 < t_2 < ... < t_k < ...$ satisfying

$$\lim_{k\to\infty} t_k = \infty,\ (-1)^k z(t_k, 0) > 0,$$

then let C_j^+ be the connected component of $A^+ \cap \Sigma_{t_0}$ that contains $(t_{2j}, 0)$, and C_j^- be the connected component of $A^- \cap \Sigma_{t_0}$ that contains $(t_{2j-1}, 0)$, $j = 1, 2, ...$ Clearly these are disjoint connected sets, and the above stated conclusion indicates that

$$C_j^+ \cap l_{t_0} \ne \emptyset,\ C_j^- \cap l_{t_0} \ne \emptyset \text{ for all } j \ge 1.$$

So we may choose x_j^+ and x_j^- such that

$$(t_0, x_j^+) \in C_j^+ \cap l_{t_0},\ (t_0, x_j^-) \in C_j^- \cap l_{t_0},\ j = 1, 2, ...$$

Since t_k is strictly increasing in k, it is easily seen that $x_j^- < x_j^+ < x_{j+1}^-$ for $j = 1, 2, ...$ This implies that $z(t_0, r)$ changes sign infinitely many times for $r \in [0, h(t_0)]$, a contradiction to property (i) for $\mathcal{Z}(z(t,\cdot))$.

Hence there exists some large $t^* > 0$ such that $z(t, 0)$ does not change sign for $r \ge t^*$. □

2.5. **Stationary solutions.** In this subsection we assume that f satisfies (i)-(iii) in (f_M) or (i)-(vi) in (f_B), and (w,h) solves (2.1) with $h_\infty = \infty$. We want to better understand the nonnegative solutions of (2.5) which belongs to $\omega(w)$. By Lemma 5, $w_r(t,r) \le 0$ for $r \in (h_0, h(t)]$, so for any $v \in \omega(w)$, we easily see that

$$0 \le v(r) \le 1 \ \text{ for } r > 0, \quad v'(r) \le 0 \ \text{ for } r > h_0. \tag{2.16}$$

Therefore, we only focus on solutions of (2.5) which satisfy (2.16).

For monostable f, by Theorem 1.1 of [5], we immediately obtain

Lemma 7. *Assume that f satisfies* (i)-(iii) *in (f_M), and v is a nonnegative solution of* (2.5) *satisfying $0 \le v \le 1$. Then either $v \equiv 0$ or $v \equiv 1$.*

For bistable f, we first prove

Lemma 8. *Assume that f satisfies* (i)-(vi) *in (f_B). Then* (2.5) *has a unique ground state solution, namely a positive solution $V_0(r)$ satisfying $V_0'(r) < 0$ for $r > 0$ and $V_0(\infty) = 0$.*

Proof. The following initial value problem

$$v'' + \frac{N-1}{r} v' + f(v) = 0 \ \text{ for } r > 0, \qquad v(0) = \beta > 0, \quad v_r(0) = 0 \tag{2.17}$$

has a unique solution defined on some maximal interval $[0, r^*)$ with $0 < r^* \le \infty$, which we denote by $v(r;\beta)$.

Clearly $v(r;\theta) \equiv \theta$ and $v(r;1) \equiv 1$. For $\beta \in I := (\theta, 1)$, since $f(v) > 0$ for $v \in (\theta, 1)$, we easily see from (2.17) that $v'(r;\beta) < 0$ for small $r > 0$. Set

$$I_- := \{\beta \in I \ : \ \exists\, r_0 > 0 \text{ such that } v(r_0;\beta) = 0 \text{ and } v'(r;\beta) < 0 \text{ for } r \in (0, r_0]\}$$

and

$$I_+ := \{\beta \in I \ : \ \exists\, r_0 > 0 \text{ such that } v'(r_0;\beta) = 0 \text{ and } v(r;\beta) > 0 \text{ for } r \in [0, r_0]\}.$$

By [1, Lemma I.1] and its proof, I_+ and I_- are nonempty, disjoint, open and $(\theta, \bar\theta] \subset I_+$. Therefore, the set $J := I \backslash (I_+ \cup I_-) \subset (\bar\theta, 1)$ is not empty, and for any $\beta_1 \in J$, the unique solution $v(r;\beta_1)$ of (2.17) satisfies

$$v'(r;\beta_1) < 0 \text{ for all } r > 0, \quad \lim_{r\to\infty} v(r;\beta_1) = v_* \ge 0.$$

By [1, Lemma I.2], $v_* = \theta$ or 0. However, due to (vi) in (f_B), we can apply Theorem 1.1 of [5] to exclude the possibility of $v_* = \theta$. Therefore, $v(r;\beta_1) \searrow 0$ as $r \to \infty$, and so $v(r;\beta_1)$ is a solution of (2.17) with $\beta = \beta_1$ and

$$v(r) > 0, \quad v'(r) < 0 \ \text{ for } r > 0, \ v(\infty) = 0. \tag{2.18}$$

Such a solution is generally called a *ground state* solution.

On the other hand, by [21, 22] the ground state solution is unique if (i), (ii) (iii) and (v) in (f_B) are satisfied. Hence, when f satisfies (i)-(vi) of (f_B), the problem (2.5) has a unique ground state solution $V_0(r) := v(r;\beta_1)$. □

The above proof indicates that

$$J = \{V_0(0)\} = \{\beta_1\}, \ I_+ = (\theta, V_0(0)) \text{ and } I_- = (V_0(0), 1). \tag{2.19}$$

Lemma 9. *Assume that f satisfies* (i)-(vi) *in* (f_B)*, and v is a solution of* (2.5) *satisfying* (2.16)*. Then $v \in \{0, 1, \theta, V_0\}$ or it satisfies*

$$v(0) \in (0, V_0(0)) \setminus \{\theta\},\ v(\infty) = \theta.$$

Proof. Let $v(r)$ be a solution of (2.5) satisfying (2.16) and $v \notin \{0, 1, \theta, V_0\}$. By the definition of I_- we see that $v(0) \notin I_- = (V_0(0), 1)$. Clearly we also have $v(0) \notin \{0, 1, \theta, V_0(0)\}$. Hence necessarily $v(0) \in (0, \theta) \cup (\theta, V_0(0))$.

By (2.16), $v(r) \searrow \gamma$ as $r \to \infty$ for some $\gamma \geq 0$. Therefore, for some sequence $r_n \to \infty$ satisfying $v'(r_n) \to 0$, by replacing $r = r_n$ in the equation of (2.17) and letting $n \to \infty$ we deduce $\lim_{n\to\infty} v''(r_n) = f(\gamma)$. This, together with $v(\infty) = \gamma$, implies that $f(\gamma) = 0$, and so $\gamma = \theta$ or 0.

We claim that $\gamma = 0$ is impossible. In fact, when $v(0) = \beta \in I_+ = (\theta, V_0(0))$, there exists $r_* > 0$ such that $v'(r) < 0$ for $r \in (0, r_*)$ and $v'(r_*) = 0$. Hence $v''(r_*) = -f(v(r_*)) \geq 0$. Clearly, $f(v(r_*)) \neq 0$, for otherwise, $v \equiv v(r_*)$ is the unique solution of the equation in (2.17), a contradiction. Therefore, $f(v(r_*)) < 0$, which implies

$$v(r_*) \in (0, \theta), \quad v'(r_*) = 0, \quad v''(r_*) > 0. \tag{2.20}$$

If $\gamma = 0$, that is, $v(r) \searrow 0$ as $r \to \infty$, then there exists a sequence $r_n \to \infty$ such that $v(r_n) \to 0$, $v'(r_n) \to 0$ as $n \to \infty$. Multiplying the equation in (2.17) by $2v'$ and integrating it on $[r_*, r_n]$ we have

$$0 = [v'(r_n)]^2 + \int_{r_*}^{r_n} \frac{2(N-1)}{r}[v'(r)]^2 dr + 2\int_{v(r_*)}^{v(r_n)} f(s)ds > [v'(r_n)]^2 + 2\int_{v(r_*)}^{v(r_n)} f(s)ds.$$

Letting $n \to \infty$ we deduce

$$\int_0^{v(r_*)} f(s)ds \geq 0.$$

Since $v(r_*) \in (0, \theta)$ and $f(s) < 0$ for $s \in (0, v(r_*))$, the above inequality is impossible.

If $v(0) = \beta \in (0, \theta)$ and $\gamma = 0$, we obtain a similar contradiction by taking $r_* = 0$ in the above argument. Hence $\gamma = 0$ is impossible, and $v(r)$ satisfies $v(0) \in (0, V_0(0)) \setminus \{\theta\}$, $v(\infty) = \theta$. □

Next we consider solutions of (2.5) which have compact positive support, namely, functions $v(r)$ that satisfies (2.5) over some finite interval $[0, R)$, with $v(R) = 0$, $v(r) > 0$ for $r \in [0, R)$. These solutions will be used to find sufficient conditions for spreading of (2.1).

Lemma 10. *Suppose that f satisfies* (i)-(iii) *in* (f_M)*. Then there exists $R_M > 0$ such that for any $R \geq R_M$,* (2.5) *has a solution $v_R(r)$ satisfying*

$$v_R(0) \in (0, 1), \ \ v_R'(r) < 0 \ \ \text{for } r \in (0, R], \quad v_R(R) = 0. \tag{2.21}$$

Proof. We use a standard upper and lower solution argument, the details are included here for completeness. Let λ_1^R be the first eigenvalue of the problem

$$-\Delta u = \lambda u \ \text{ in } B_R, \quad u = 0 \text{ on } \partial B_R,$$

and $\phi > 0$ the corresponding eigenfunction with $\|\phi\|_\infty = 1$. It is well known that $0 < \lambda_1^R \to 0$ as $R \to \infty$. Choose $R_0 > 0$ large enough such that $\lambda_1^R < f'(0)$ for all $R \geq R_0$. Then for all small $\epsilon > 0$, $f(\epsilon\phi(x)) \geq \lambda_1^R \epsilon\phi(x)$ in B_R. It follows that

$$-\Delta(\epsilon\phi) \leq f(\epsilon\phi) \text{ in } B_R, \quad \epsilon\phi = 0 \text{ on } \partial B_R.$$

Thus $\epsilon\phi$ is a lower solution to the problem

$$-\Delta u = f(u) \text{ in } B_R, \quad u = 0 \text{ on } \partial B_R. \tag{2.22}$$

Since $f(1) = 0$, clearly the constant function 1 is an upper solution of (2.22). Therefore by the standard upper and lower solution argument we know that (2.22) has a solution u satisfying $0 < u < 1$ in B_R, for every $R \geq R_0$. The well-known Gidas-Ni-Nirenberg symmetry result infers that such a solution is radially symmetric, and $u'(r) < 0$ for $r \in (0, R]$. □

Lemma 11. *Assume that f satisfies* (i)-(vi) *in* (f_B). *Then for every $\beta \in (V_0(0), 1)$, there exists $R_\beta > 0$ such that* (2.5) *has a solution $v_\beta(r)$ satisfying*

$$v_\beta(0) = \beta, \ v'_\beta(r) < 0 \textit{ for } r \in (0, R_\beta], \ v_\beta(R_\beta) = 0.$$

Proof. By (2.19), we have $I_- = (V_0(0), 1)$, and the existence of v_β follows directly from the definition of I_-. □

2.6. **Sufficient conditions for vanishing.** For f satisfying (i)-(iii) in (f_M) or (i)-(vi) in (f_B), we give some sufficient conditions ensuring that vanishing happens for (2.1). The following upper bound for w is an easy consequence of the standard comparison principle.

Lemma 12. *Assume that f satisfies* (2.3) *and* (2.4). *Then, for any $h_0 > 0$ and any $w_0 \in \mathscr{K}(h_0)$, the solution $w(t, r)$ of* (2.1) *satisfies*

$$w(t, r) \leq \frac{e^{Kt}}{(4\pi t)^{N/2}} \int_{B_{h_0}} w_0(|\xi|) d\xi \quad \textit{for } \ 0 \leq r \leq h(t), \ t > 0. \tag{2.23}$$

Proof. Consider the Cauchy problem

$$\begin{cases} \tilde{w}_t = \Delta \tilde{w} + K\tilde{w}, & x \in \mathbb{R}^N, \ t > 0, \\ \tilde{w}(0, x) = \Phi(|x|), & x \in \mathbb{R}^N, \end{cases} \tag{2.24}$$

where

$$\Phi(r) = \begin{cases} w_0(r), & r \in [0, h_0], \\ 0, & r \in [h_0, \infty). \end{cases}$$

Using the fundamental solution we have

$$\tilde{w}(t, |x|) = \frac{e^{Kt}}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x-\xi|^2}{4t}} \Phi(|\xi|) d\xi \leq \frac{e^{Kt}}{(4\pi t)^{N/2}} \int_{B_{h_0}} w_0(|\xi|) d\xi.$$

By the standard comparison theorem, we have $w(t, r) \leq \tilde{w}(t, r)$ for $t > 0$ and $r \in [0, h(t)]$, and the required inequality follows. □

Proposition 1. *Let $h_0 > 0$ and $\phi \in \mathscr{K}(h_0)$. Then vanishing happens to* (2.1) *with $u_0 = \phi$, namely*

$$h_\infty < \infty \textit{ and } \lim_{t\to\infty} \|w(t, r; \phi)\|_{L^\infty([0, h(t)])} = 0,$$

if one of the following conditions holds:

(i) *f satisfies* (i)-(iii) *in* (f_M), $h_0 < \sqrt{\lambda_1 / f'(0)}$ *and $\|\phi\|_{L^\infty}$ is sufficiently small, where $\lambda_1 > 0$ is the first eigenvalue of*

$$-\Delta \varphi = \lambda \varphi \textit{ in } \ B_1(0) \subset \mathbb{R}^N, \qquad \varphi = 0 \textit{ on } \ \partial B_1(0); \tag{2.25}$$

(ii) *f satisfies* (i)-(vii) *in* (f_B), *and $\|\phi\|_{L^\infty} \leq \theta$;*

(iii) f *satisfies* (i)-(vii) *in* (f_B), *and*

$$\int_{B_{h_0}} \phi(x)dx \leq \theta \cdot \Big(\frac{2\pi N}{eK}\Big)^{N/2}. \tag{2.26}$$

Proof. (i) Since $h_0^2 f'(0) < \lambda_1$, there exists a small $\delta > 0$ such that

$$\frac{\lambda_1}{(1+\delta)^2 h_0^2} - f'(0) \geq 2\delta. \tag{2.27}$$

Moreover, there exists an $s > 0$ small such that

$$-2\mu\varphi_1'(1)s \leq \delta^2 h_0^2, \qquad \text{and} \quad f(u) \leq (f'(0)+\delta)u \ \text{ for } u \in [0,s],$$

where $\varphi_1 = \varphi_1(|x|)$ is the first eigenfunction of (2.25) corresponding to λ_1 which satisfies

$$\varphi_1(r) > 0 \ \text{ for } r \in [0,1), \quad \varphi_1'(0) = 0 \quad \text{and} \quad \varphi_1'(r) < 0 \ \text{ for } r \in (0,1].$$

Set

$$k(t) := h_0\Big(1 + \delta - \frac{\delta}{2}e^{-\delta t}\Big) \quad \text{and} \quad \hat{w}(t,r) := se^{-\delta t}\varphi_1\Big(\frac{r}{k(t)}\Big).$$

Clearly $\hat{w}(t,k(t)) = \hat{w}_r(t,0) = 0$. A direct calculation shows that, for $t > 0$ and $r \in [0,k(t)]$,

$$\hat{w}_t - \Delta\hat{w} - f(\hat{w}) \geq \left(\frac{\lambda_1}{h_0^2(1+\delta)^2} - f'(0) - 2\delta\right)\hat{w} \geq 0.$$

On the other hand, by the choice of s we have

$$\mu\hat{w}_r(t,k(t)) = \mu s e^{-\delta t}\frac{\varphi_1'(1)}{k(t)} \geq \frac{-\delta^2 h_0}{2}e^{-\delta t} = -k'(t).$$

Choose $\varepsilon_1 := s\varphi_1(\frac{2}{2+\delta})$. When $\|\phi\|_{L^\infty} \leq \varepsilon_1$, we have

$$\phi(r) \leq \varepsilon_1 \leq s\varphi_1\Big(\frac{2r}{h_0(2+\delta)}\Big) = \hat{w}(0,r) \quad \text{for } r \in [0,h_0].$$

Therefore, $(\hat{w}(t,r), k(t))$ is an upper solution to (2.1). By Lemma 1 we have

$$h(t) \leq k(t) \leq h_0(1+\delta), \quad h_\infty < \infty.$$

By Theorem 3, $\lim_{t\to\infty}\|w(t,\cdot)\|_{L^\infty([0,h(t)])} = 0$. This proves (i).

By Lemma 3 we can use the same upper solutions as in the proof of Theorem 3.2 of [9] to prove (ii). Finally, (iii) follows from (ii) by making use of (2.23) with $t = \frac{N}{2K}$, and regard this t value as the initial time. □

From this proposition, we immediately obtain

Corollary 1. *If* f *satisfies* (i)-(vi) *in* (f_B) , *then* $\lim_{t\to\infty}\|w(t,\cdot)\|_{L^\infty([0,h(t)])} = 0$ *implies that* $h_\infty < \infty$.

2.7. **Sufficient conditions for spreading.** We will use Lemmas 10, 11 and the notations v_R, R_M, v_β and R_β there.

Proposition 2. *Let (w,h) be the solution of (2.1). Then spreading happens, i.e.,*

$$h_\infty = \infty \quad \text{and} \quad \lim_{t\to\infty} w(t,\cdot) = 1 \text{ locally uniformly in } [0,\infty),$$

provided that one of the following conditions is satisfied:

(i) *f satisfies (i)-(iii) in (f_M) and $h_0 \geq R$, $w_0 \geq v_R$ on $[0,R]$ for some $R > R_M$;*

(ii) *f satisfies (i)-(vi) in (f_B) and $h_0 \geq R_\beta$, $w_0 \geq v_\beta$ on $[0,R_\beta]$ for some $\beta \in (V_0(0),1)$.*

Proof. (i) In this case, by the comparison principle we have $w(t,r) \geq v_R(r)$ for all $t>0$ and $r \in [0,R]$. The convergence result (Theorem 3) implies that w converges to a solution $\bar{v}$ of $\Delta v + f(v) = 0$ on $[0,\infty)$. We thus obtain $\bar{v}(r) \geq v_R(r)$ for $r \in [0,R]$. In view of Lemma 7, we must have $\bar{v} \equiv 1$.

(ii) Similarly by the comparison principle we have $w(t,r) \geq v_\beta(r)$ for all $t>0$ and $r \in [0,R_\beta]$. The convergence result (Theorem 3) indicates that w converges to a solution $\bar{v}$ of $\Delta v + f(v) = 0$ on $[0,\infty)$. Hence $\bar{v}(r) \geq v_\beta(r)$ for $r \in [0,R_\beta]$. It follows that $\bar{v}(0) \geq \beta > V_0(0)$. By Lemma 9 we necessarily have $\bar{v} \equiv 1$. □

2.8. **Dichotomy and sharp threshold for monostable f.** In this subsection, based on the results in the previous subsections, we give a complete description of the long-time dynamical behavior of the solutions of (2.1) for monostable f.

Theorem 7 (Dichotomy)**.** *Suppose f satisfies (i)-(iii) in (f_M), and $h_0 > 0$, $u_0 \in \mathscr{K}(h_0)$ and (w,h) is the solution of (2.1). Then either spreading happens, namely, $h_\infty = \infty$ and*

$$\lim_{t\to\infty} w(t,r) = 1 \text{ locally uniformly in } [0,\infty),$$

or vanishing happens, namely, $h_\infty < \infty$ and

$$\lim_{t\to\infty} \max_{r\in[0,h(t)]} w(t,r) = 0.$$

Moreover, when vanishing happens, $h_\infty \leq \sqrt{\frac{\lambda_1}{f'(0)}}$, where λ_1 is the first eigenvalue of (2.25).

Proof. This is a simple variation of that of [9, Theorem 5.1]; the details are omitted. □

Theorem 8 (Sharp threshold)**.** *Let f satisfy (i)-(iii) in (f_M). Suppose that $h_0 > 0$, $\psi \in \mathscr{K}(h_0)$, and (w_σ, h_σ) is the solution of (2.1) with $w_0 = \sigma\psi$, $\sigma > 0$. Then we have the following conclusions:*

(i) *If $h_0 \geq \sqrt{\frac{\lambda_1}{f'(0)}}$ then spreading happens for w_σ for every $\sigma > 0$.*

(ii) *If $h_0 < \sqrt{\frac{\lambda_1}{f'(0)}}$, then there exists $\sigma^* = \sigma^*(h_0,\phi) \in (0,\infty]$ such that spreading happens when $\sigma > \sigma^*$, and vanishing happens when $0 < \sigma \leq \sigma^*$.*

Proof. The proof is a simple variation of that of [9, Theorem 5.2]. We again omit the details. □

2.9. **Trichotomy and sharp thresholds for bistable f.**

Theorem 9 (Trichotomy). *Suppose that f satisfies* (i)-(vi) *in* (f_B), $h_0 > 0$, $w_0 \in \mathscr{K}(h_0)$ *and* (w, h) *is the solution of* (2.1)*. Then exactly one of the following three cases happens:*

(i) Spreading: $h_\infty = \infty$ *and*

$$\lim_{t\to\infty} w(t,r) = 1 \text{ locally uniformly in } [0,\infty),$$

(ii) Vanishing: $h_\infty < \infty$ *and*

$$\lim_{t\to\infty} \max_{0\le x\le h(t)} w(t,r) = 0,$$

(iii) Transition: $h_\infty = \infty$ *and*

$$\lim_{t\to\infty} w(t,r) = V_0(r) \text{ locally uniformly in } [0,\infty),$$

where V_0 is the unique ground state solution of (2.5)*.*

Proof. By Theorem 3, we have either $h_\infty < \infty$ or $h_\infty = \infty$, and in the former case, $\lim_{t\to\infty} \max_{r\in[0,h(t)]} w(t,r) = 0$.

Suppose now $h_\infty = \infty$. By Lemma 5, Theorem 3 and Lemma 9, $v(r) := \lim_{t\to\infty} w(t,r)$ is a function in $\{0, \theta, 1, V_0\}$, or it satisfies $v(0) \in (0, V_0(0)) \setminus \{\theta\}$, $v(\infty) = \theta$.

We now show that $v \in \{1, V_0\}$. Since $h_\infty = \infty$, by Corollary 1, $v \equiv 0$ is impossible. It remains to exclude the case $v \equiv \theta$, and the case $v(0) \in (0, V_0(0)) \setminus \{\theta\}$, $v(\infty) = \theta$. Clearly in both cases we have $v(\infty) = \theta$.

For the solution w of (2.1), we set

$$E(t) := \int_{B_{h(t)}(0)} \left[\frac{1}{2}|\nabla w|^2 - F(w)\right] dx = \int_0^{h(t)} \omega_N \left[\frac{1}{2} w_r^2 - F(w)\right] r^{N-1} dr, \tag{2.28}$$

with ω_N denoting the surface area of the unit sphere in $\mathbb{R}^N$, and

$$F(w) = \int_0^w f(s)ds.$$

It is easily checked that $E(t)$ is well-defined and $E'(t) \le 0$.

Next we make use of $E(t)$ to show that $v(\infty) = \theta$ leads to a contradiction. Choose $R_0 > 0$ large such that $v(r) \le \frac{1}{2}(\theta + \bar\theta)$ for $r \ge R_0$. For any fixed $R \ge R_0$, we can find $T = T_R > 0$ large so that

$$h(t) > R,\ w(t,R) \le \frac{1}{3}(\theta + 2\bar\theta) \text{ for } t \ge T.$$

Due to the monotonicity of $w(t,r)$ in r for $r \ge h_0$, it follows that $w(t,r) \le w(t,R) < \bar\theta$ for $t \ge T$ and $r \ge R$. Hence $F(w(t,r)) < 0$ for such (t,r), and we have, for $t \ge T$,

$$E(0) \ge E(t) > \int_0^R \omega_N \left[\frac{1}{2} w_r^2 - F(w)\right] r^{N-1} dr \ge -\int_0^R \omega_N F(w) r^{N-1} dr.$$

Letting $t \to \infty$ we obtain

$$-\int_0^R \omega_N F(v(r)) r^{N-1} dr \le E(0) \text{ for all } R \ge R_0.$$

Since $F(v(r)) \to F(\theta) < 0$ as $r \to \infty$, the above inequality cannot hold for all large R. This contradiction shows that $v(\infty) = \theta$ is impossible. Hence $v \in \{1, V_0\}$. ◻

Theorem 10 (Sharp threshold). *Suppose that f satisfies* (i)-(vi) *in* (f_B)*, and* $h_0 > 0$*,* $\phi \in \mathscr{K}(h_0)$*, and* (w, h) *is a solution of* (2.1) *with* $w_0 = \sigma\phi$*,* $\sigma > 0$*. Then there exists* $\sigma^* = \sigma^*(h_0, \phi) \in (0, \infty]$ *such that spreading happens when* $\sigma > \sigma^*$*, vanishing happens when* $0 < \sigma < \sigma^*$*, and transition happens when* $\sigma = \sigma^*$*.*

Proof. By Proposition 1 (ii) we find that vanishing happens if $\sigma < \theta/\|\phi\|_\infty$. Hence

$$\sigma^* = \sigma^*(h_0, \phi) := \sup\left\{\sigma_0 : \text{ vanishing happens for } \sigma \in (0, \sigma_0]\right\} \in (0, +\infty].$$

If $\sigma^* = +\infty$, then there is nothing left to prove. So we assume that σ^* is a finite positive number.

By definition, vanishing happens for all $\sigma \in (0, \sigma^*)$. We now consider the case $\sigma = \sigma^*$. In this case, we cannot have vanishing, for otherwise we have, for some large $t_0 > 0$, $w(t_0, r) < \theta$ in $[0, h(t_0)]$, and due to the continuous dependence of the solution on the initial values, we can find $\epsilon > 0$ sufficiently small such that the solution (w_ϵ, h_ϵ) of (2.1) with $w_0 = (\sigma^* + \epsilon)\phi$ satisfies

$$w_\epsilon(t_0, r) < \theta \text{ in } [0, h_\epsilon(t_0)].$$

Hence we can apply Proposition 1 (ii) to conclude that vanishing happens to w_ϵ, a contradiction to the definition of σ^*. Thus at $\sigma = \sigma^*$ either spreading or transition happens.

We show next that spreading cannot happen at $\sigma = \sigma^*$. Suppose this happens. Let v_β and R_β with $\beta \in (V_0(0), 1)$ be given by Lemma 11. Then we can find $t_0 > 0$ large (depending on β) such that

$$h(t_0) > R_\beta,\ w(t_0, r) > v_\beta(r) \text{ in } [0, R_\beta]. \tag{2.29}$$

By the continuous dependence of the solution on initial values, we can find a small $\epsilon > 0$ such that the solution (w^ϵ, h^ϵ) of (2.1) with $w_0 = (\sigma^* - \epsilon)\phi$ satisfies (2.29), and by Proposition 2, spreading happens for (w^ϵ, h^ϵ). But this is a contradiction to the definition of σ^*.

Hence transition must happen when $\sigma = \sigma^*$. We show next that spreading happens when $\sigma > \sigma^*$. Let (w, h) be the solution of (2.1) with $w_0 = \sigma\phi$ for some $\sigma > \sigma^*$, and denote the solution of (2.1) with $w_0 = \sigma^*\phi$ by (w^*, h^*). By the comparison results we know that

$$h^*(1) < h(1),\ w^*(1, r) < w(1, r) \text{ on } [0, h^*(1)].$$

Hence we can find $\epsilon_0 > 0$ small such that for any given $e \in \mathbb{S}^{N-1}$ and any $\epsilon \in [0, \epsilon_0]$ we have

$$B_{h^*(1)}(\epsilon e) \subset B_{h(1)}(0)$$

and

$$w^*(1, |x - \epsilon e|) < w(1, |x|) \quad \text{ for } x \in B_{h^*(1)}(\epsilon e).$$

Now define

$$w_\epsilon(t, x) = w^*(t + 1, |x - \epsilon e|),\ h_\epsilon(t) = h^*(t + 1).$$

Clearly $(w_\epsilon, B_{h_\epsilon(t)}(\epsilon e))$ is a solution of (1.1) with initial function $w_0(x) = w^*(1, |x - \epsilon e|)$. By the comparison principle for the solutions of (1.1) (see [7, 13]) we have, for all $t > 0$ and $\epsilon \in (0, \epsilon_0]$,

$$B_{h_\epsilon(t)}(\epsilon e) \subset B_{h(t+1)}(0), \quad w_\epsilon(t, x) \le w(t + 1, |x|) \text{ in } B_{h_\epsilon(t)}(\epsilon e).$$

If $\omega(w) \ne \{1\}$, then by Theorem 9 necessarily $w(t, x) \to V_0(|x|)$ as $t \to \infty$, and by letting $t \to \infty$ in the above inequality we obtain

$$V_0(|x - \epsilon e|) \le V_0(|x|) \quad \text{ for all } x \in \mathbb{R}^N.$$

Taking $x = \epsilon e$ we obtain $V_0(0) \le V_0(\epsilon)$, a contradiction to the fact that $V'(r) < 0$ for $r > 0$. Thus we must have $\omega(w) = \{1\}$. This proves that spreading happens for $\sigma > \sigma^*$. □

3. The General Case

In this section we prove Theorems 1 and 2 for the general problem (1.1).

3.1. **Proof of Theorem 1.** We first prove that $\Omega_\infty = \mathbb{R}^N$ implies spreading, i.e., $\lim_{t\to\infty} u(t,x) = 1$ locally uniformly in $x \in \mathbb{R}^N$.

In fact, when $\Omega_\infty = \mathbb{R}^N$, by Theorem B (i), there exists $T > 0$ such that $\Omega(T) \supset \overline{B}_{R_0}(0)$, where $R_0 := \sqrt{\frac{\lambda_1}{f'(0)}}$ and $\lambda_1 > 0$ is the first eigenvalue of (2.25). So there exists $R_1 > R_0$ such that $\Omega(T) \supset B_{R_1}(0)$. We can then select a radially symmetric function $\phi_0(r)$ such that

$$\phi_0 \in \mathscr{K}(R_1), \quad 0 < \phi_0(|x|) < u(T,x) \text{ for } x \in B_{R_1}(0). \tag{3.1}$$

Then Theorem 8 and the comparison principle imply that, as $t \to \infty$,

$$u(t+T,x) \ge w(t,|x|;\phi_0) \to 1 \ \text{ locally uniformly in } \ |x| \in [0,\infty). \tag{3.2}$$

On the other hand, $\lim_{t\to\infty} \max_{x\in\Omega(t)} u(t,x) \le 1$ follows easily from the assumption (f_M). Therefore, $\lim_{t\to\infty} u(t,x) = 1$ locally uniformly in $x \in \mathbb{R}^N$. So we have proved that spreading happens when $\Omega_\infty = \mathbb{R}^N$.

Denote

$$\Sigma := \{\sigma > 0 : \lim_{t\to\infty} u_\sigma(t,\cdot) = 1 \text{ locally uniformly in } \mathbb{R}^N\}$$

and define

$$\sigma^* := \inf \Sigma \text{ if } \Sigma \ne \emptyset, \ \sigma^* = \infty \text{ if } \Sigma = \emptyset.$$

Clearly there are only three possible cases:

$$\text{(i) } \sigma^* = 0; \qquad \text{(ii) } \sigma^* \in (0,\infty); \qquad \text{(iii) } \sigma^* = \infty.$$

In cases (i), by the comparison principle we have $\Sigma = (0,\infty)$ and nothing further is left to prove. In case (iii), by Theorem B and what we have proved above, Ω_∞ must be a bounded set and so vanishing happens for every $\sigma > 0$.

It remains to consider case (ii). By the comparison principle we easily see that $\Sigma \supset (\sigma^*,\infty)$. We now show that $\sigma^* \notin \Sigma$. If $\sigma^* \in \Sigma$ then as above we can find $\phi_0(r)$ satisfying (3.1) with $u = u_{\sigma^*}$. By continuous dependence on initial data, for sufficiently small $\epsilon > 0$, (3.1) also holds for $u = u_{\sigma^*-\epsilon}$. Hence (3.2) holds for $u = u_{\sigma^*-\epsilon}$ and so spreading happens for $u_{\sigma^*-\epsilon}$, contradicting the definition of σ^*. Thus $\Sigma = (\sigma^*,\infty)$, and for $\sigma \le \sigma^*$, Ω_∞ must be a bounded set, and so vanishing happens.

Finally we prove the estimate for $M(t)$ when spreading happens. Let ϕ_0 and $w(t,r;\phi_0)$ satisfy (3.1) and (3.2). Clearly

$$M(t+T) \ge h(t;\phi_0) \text{ for } t > 0.$$

Fix $R_2 > R_1$ such that $\overline{\Omega}(T) \subset B_{R_2}(0)$, and then choose a radially symmetric function $\phi^0(r)$ such that

$$\phi^0 \in \mathscr{K}(R_2), \ \phi^0(|x|) > u(T,x) \text{ for } x \in \Omega(T).$$

Then the comparison principle implies that

$$w(t,|x|;\phi^0) \ge u(t+T,x) \ \text{ for } t > 0, \ x \in \Omega(t).$$

It follows that

$$M(t+T) \le h(t;\phi^0) \text{ for } t > 0.$$

By Theorem 7 and Theorem F, we have

$$h(t;\phi^0) - [c^*t - c_*(N-1)\log t] \to \hat{h}^0 \in \mathbb{R}^1 \text{ as } t \to \infty,$$

$$h(t;\phi_0) - [c^*t - c_*(N-1)\log t] \to \hat{h}_0 \in \mathbb{R}^1 \text{ as } t \to \infty.$$

It thus follows from $h(t;\phi_0) \le M(t+T) \le h(t;\phi^0)$ that

$$M(t) = c^*t - c_*(N-1)\log t + O(1) \text{ as } t \to \infty.$$

The proof of Theorem 1.1 is now complete. □

Remark 3. (i) The above proof shows that $\sigma^* = 0$ if there exists $x_0 \in \mathbb{R}^N$ such that $\Omega_0 \supset \overline{B}_{R_0}(x_0)$, $R_0 := \sqrt{\frac{\lambda_1}{f'(0)}}$.

(ii) The case $\sigma^* = \infty$ is also possible. This happens, for example, when Ω_0 is contained in a small ball and $f(u)$ converges to $-\infty$ sufficiently fast as $u \to \infty$. In [9] we proved that, when

$$\liminf_{u\to\infty} \frac{-f(u)}{u^8} > 0, \tag{3.3}$$

there exists $h^* > 0$ such that, for any initial function u_0 with support in $[-h^*, h^*]$, the solution u of (1.1) (with $N = 1$) vanishes. By Lemma 3, we can use such a solution for the one dimensional problem to construct an upper solution for the problem (1.1). Therefore, when $\Omega_0 \subset B_{h^*}(0)$ and $f(u)$ satisfies (3.3), any solution u_σ of (1.1) with initial function $\sigma\phi \in \mathcal{I}(\Omega_0)$ vanishes no matter how large σ is.

3.2. **Proof of Theorem 2.** By Proposition 1 and by the comparison principle, it is easily seen that

$$\Sigma_0 := \{\sigma > 0 : \lim_{t\to\infty} \|u_\sigma(t,\cdot)\|_{L^\infty(\Omega(t))} = 0\}$$

is a non-empty open interval of the form $(0, \sigma_*)$. In case $\Sigma_0 = (0,\infty)$, there is nothing left to prove. So, in what follows, we suppose $\Sigma_0 = (0,\sigma_*)$ with $\sigma_* < \infty$.

Define

$$\Sigma_1 := \{\sigma > 0 : \lim_{t\to\infty} u_\sigma(t,\cdot) = 1 \text{ in } L^\infty_{loc}(\mathbb{R}^N)\}.$$

By Proposition 2 and by the comparison principle, one sees that the set Σ_1 is an open interval of the form (σ^*, ∞) if it is not empty. Set

$$\Sigma^* := (0,\infty)\backslash(\Sigma_0 \cup \Sigma_1).$$

To complete the proof of Theorem 2 we only need to show that

$$\sigma_* = \sigma^* \text{ (and so } \Sigma^* = \{\sigma^*\})$$

and

$$\omega(u_{\sigma^*}) \subset \{V_0(x_0 + \cdot) : x_0 \in \overline{\text{co}}(\Omega_0)\}.$$

We prove these conclusions by several lemmas.

Lemma 13. *Suppose that u is a solution of (1.1) with $\Omega_\infty = \mathbb{R}^N$, and there exist a constant $c > 0$ and a sequence $\{(t_n, x_n)\}$ satisfying*

$$u(t_n, x_n) \geq c, \ \lim_{n\to\infty} |x_n| = \infty.$$

Then for any given $R_0,\ R > 0$, there exist n and some $y_n \in \mathbb{R}^N$, such that

$$u(t_n, x) \geq c \text{ for } x \in B_R(y_n),\ B_R(y_n) \cap B_{R_0}(0) = \emptyset.$$

Proof. Since $|x_n| \to \infty$ as $n \to \infty$, necessarily $t_n \to \infty$ as $n \to \infty$. By Theorem B, for all large n, $\Gamma(t_n)$ is a $C^{2+\alpha}$ closed hypersurface in $\mathbb{R}^N$, and

$$\Gamma(t_n) \subset B_{M(t_n)}(0) \setminus B_{M(t_n)-\pi d_0}(0).$$

Let $r_0 > 0$ be chosen such that $B_0 := B_{r_0}(0) \supset \Omega_0$. For any $x \in \mathbb{R}^N \setminus \overline{B}_0$, denote

$$S_x^0 := \{\nu \in \mathbb{S}^{N-1} : \nu \cdot (z - x) < 0\ \forall z \in \overline{B}_0\}.$$

Then

$$\{\nu \in \mathbb{S}^{N-1} : \nu \cdot (z - x) < 0\ \forall z \in \overline{\text{co}}(\Omega_0)\} \supset S_x^0, \tag{3.4}$$

and it follows from Lemma D and (3.4) that

$$\partial_\nu u(t, x) < 0 \text{ for } x \in \Omega(t) \setminus \overline{B}_0 \text{ and } \nu \in S_x^0. \tag{3.5}$$

It is easily seen that the cone

$$K_x := \{y \in \mathbb{R}^N : y - x = s\nu,\ \nu \in S_x^0, s > 0\}$$

is circular, with vertex at x, and axis passing through 0 and x. Let us denote its openning angle by θ_x. Clearly θ_x depends only on $|x|$, and is a strictly increasing function of $|x|$, with

$$\lim_{|x|\to r_0} \theta_x = 0,\ \lim_{|x|\to\infty} \theta_x = \pi.$$

For any given $R > 0$, $r > r_0 + R$ and $x \in \mathbb{R}^N$ with $|x| > r + R$, we consider another circular cone

$$K_r^x := \{y \in \mathbb{R}^N : y - x = s(z - x),\ z \in B_R(rx/|x|), s > 0\}.$$

This cone has vertex at x, axis passing through 0 and x, and contains the ball $B_R(rx/|x|)$. If we denote its opening angle by θ^x, then it depends only on $|x| - r$, is a strictly decreasing function of $|x| - r$, and

$$\theta^x \to 0 \text{ as } |x| - r \to \infty,\ \theta^x \to \pi \text{ as } |x| - r \to R.$$

We now consider the set

$$\tilde{K}_r^x := \{x \in K_r^x : |x| > r\} \cup B_R(rx/|x|).$$

By the regularity of $\Gamma(t_n)$ for large n, and the above properties of θ^x, it is easily seen that

$$\tilde{K}_r^{x_n} \subset \Omega(t_n) \text{ for all large } n.$$

Moreover, in view of the properties of θ_x, a simple geometrical consideration shows that, if r is chosen large enough, then for all large $|x|$ and $y \in \tilde{K}_r^x$, we have $\frac{x-y}{|x-y|} \in S_y^0$, and $\tilde{K}_r^x \cap B_{R_0}(0) = \emptyset$. Hence, by (3.5), for such r and all large n, $u(t_n, \cdot)$ is decreasing from $y \in \tilde{K}_r^{x_n}$ to x_n along the line segment $\overline{yx_n}$. We thus obtain

$$u(t_n, y) \geq u(t_n, x_n) \geq c \text{ for } y \in \tilde{K}_r^{x_n}.$$

In particular, $u(t_n,x)\ge c$ for $x\in B_R(y_n)$, $y_n=rx_n/|x_n|$, and $B_R(y_n)\cap B_{R_0}(0)=\emptyset$. □

Lemma 14. *Assume $\sigma\in\Sigma^*$ and that $u_\sigma(t,x)$ is extended to $[0,\infty)\times\mathbb{R}^N$ by letting $u_\sigma(t,x)=0$ for $x\in\mathbb{R}^N\setminus\Omega(t)$, $t\ge 0$. Then there exists some large $R_0>0$ such that $u_\sigma(t,x)\le\tilde V_0(|x|-R_0)$ for $t>0$ and $|x|\ge R_0$, where $\tilde V_0(s)$ is the unique solution of the problem*

$$v''+f(v)=0 \ \ in\ \mathbb{R}^1,\quad v'(0)=0,\quad v(\pm\infty)=0.$$

Proof. It is well known that $\tilde V_0(0)=\bar\theta\in(\theta,1)$. Fix $\hat\theta\in(\theta,\tilde V_0(0))$. First we prove that, for any given large R_0,

$$u_\sigma(t,x)\le\hat\theta\quad\text{for all } t>0 \text{ and } |x|\ge R_0. \tag{3.6}$$

Otherwise, for any positive integer n, there exists $t_n>0$ and $x_n\in\Omega(t_n)$ such that $u_\sigma(t_n,x_n)>\hat\theta$ and $|x_n|\ge n$. By Lemma 13, for any $R>0$, we can find some large n and a ball $B_R(y_n)$ such that

$$u_\sigma(t_n,x)>\hat\theta \text{ for } x\in B_R(y_n). \tag{3.7}$$

We show below that this would imply spreading happens for u_σ, contradicting our assumption that $\sigma\in\Sigma^*$.

We start with an argument similar to the proof of [3, Lemma 3.2] (see also [15]). Let $\eta(t)$ be the solution of

$$\eta'=f(\eta),\ \eta(0)=\hat\theta.$$

Clearly $\eta(t)$ is an increasing function with $\eta(\infty)=1$.

Let $w(t,x)$ be the unique solution to

$$\begin{cases} w_t-\Delta w=f(w) & \text{for } x\in B_R(0),\ t>0,\\ w=0 & \text{for } x\in\partial B_R(0),\ t>0,\\ w(0,x)=\hat\theta & \text{for } x\in B_R(0).\end{cases}$$

By the comparison principle clearly $w(t,x)\le\eta(t)$ for $t>0, x\in B_R(0)$.

Denote

$$\rho(x)=(1+|x|^2)^{-1},\ \zeta(t,x)=\rho(x)[w(t,x)-\eta(t)].$$

A simple calculation gives

$$\zeta_t-\Delta\zeta-\frac{4x}{1+|x|^2}\cdot\nabla\zeta=\left[\frac{2N}{1+|x|^2}+f'(\theta(t,x)\right]\zeta$$

with $\theta(t,x)$ lying between $w(t,x)$ and $\eta(t)$, and hence $\theta(t,x)\in[0,1)$. It follows that

$$\frac{2N}{1+|x|^2}+f'(\theta(t,x))\le M:=2N+\max_{u\in[0,1]}f'(u) \text{ for } t>0,\ x\in B_R(0).$$

This implies that, for any $T>0$, the constant function $v:=-\eta(T)/(1+R^2)$ is a lower solution of the equation satisfied by $e^{-Mt}\zeta(t,x)$ over $[0,T]\times B_R(0)$. It follows that

$$\zeta(t,x)\ge -e^{Mt}\eta(T)/(1+R^2) \text{ for } t\in[0,T],\ x\in B_R(0).$$

In particular

$$\zeta(T,x)\ge -e^{MT}\eta(T)/(1+R^2) \text{ for } x\in B_R(0),$$

which gives

$$w(T,x)\ge\eta(T)\left[1-\frac{(1+|x|^2)e^{MT}}{1+R^2}\right] \text{ for } x\in B_R(0).$$

For $\beta \in (V_0(0), 1)$, let v_β and R_β be given by Lemma 11. Then we can find $T > 0$ large such that $\eta(T) > \beta$. From the above estimate for $w(T, x)$, we can subsequently find $R > 0$ large so that

$$w(T, x) > \beta \text{ for } x \in B_{R_\beta}(0).$$

For $R > 0$ chosen this way, we now have a ball $B_R(y_n)$ and $t_n > 0$ such that (3.7) holds. The comparison principle then yields

$$u_\sigma(t_n + t, x) \geq w(t, x - y_n) \text{ for } x \in B_R(y_n).$$

It follows that

$$u_\sigma(t_n + T, x) > \beta \geq v_\beta(|x - y_n|) \text{ for } x \in B_{R_\beta}(y_n).$$

Then by the spreading condition in Proposition 2 and by the comparison principle we see that spreading happens for u_σ, contradicting the assumption that $\sigma \in \Sigma^*$. This proves (3.6).

Next, we define $v_1(x) := \tilde{V}_0(|x| - R_0)$ for $x \in \mathbb{R}^N \setminus B_{R_0}$. Then, it is easily seen that

$$-\Delta v_1 \geq f(v_1) \text{ for } x \in \mathbb{R}^N \setminus B_{R_0}(0).$$

We may now use (3.6) and a simple comparison argument to show that

$$u_\sigma(t, x) \leq v_1(x) = \tilde{V}_0(|x| - R_0) \text{ for } t > 0,\ |x| > R_0.$$

This proves the lemma. □

Lemma 15. *Assume $\sigma \in \Sigma^*$. Then $\omega(\sigma\phi)$ is non-empty, and for each $\overline{U} \in \omega(\sigma\phi)$,*

$$\sup_{x \in \overline{\text{co}}(\Omega_0)} \overline{U}(x) = \sup_{x \in \mathbb{R}^N} \overline{U}(x) \in [\theta, 1].$$

Proof. By Lemma D, for every $t > 0$,

$$\sup_{x \in \overline{\text{co}}(\Omega_0)} u_\sigma(t, x) = \sup_{x \in \mathbb{R}^N} u_\sigma(t, x).$$

By Proposition 1 and by the comparison principle we see that vanishing happens if $\sup_{x \in \mathbb{R}^N} u_\sigma(t, x) \leq \theta$ for some $t \geq 0$. Since $\sigma \in \Sigma^*$, we must have

$$\sup_{x \in \overline{\text{co}}(\Omega_0)} u_\sigma(t, x) = \sup_{x \in \mathbb{R}^N} u_\sigma(t, x) > \theta \text{ for every } t \geq 0.$$

It follows that

$$\sup_{x \in \overline{\text{co}}(\Omega_0)} \overline{U}(x) = \sup_{x \in \mathbb{R}^N} \overline{U}(x) \geq \theta.$$

Since $\limsup_{t \to \infty} u_\sigma(t, x) \leq 1$ always holds, we must also have $\overline{U} \leq 1$. □

Lemma 16. *Assume $\sigma \in \Sigma^*$ and $\overline{U} \in \omega(\sigma\phi)$. Then $\overline{U}(x) = V(|x - x_0|)$ for some $x_0 \in \overline{\text{co}}(\Omega_0)$.*

Proof. For all large $t > 0$, say $t \geq T$, such that $\Gamma(t)$ is a smooth closed hypersurface in $\mathbb{R}^N$, define the energy of $u_\sigma(t, x)$ by

$$E[u_\sigma](t) := \int_{\Omega(t)} \Big[\frac{1}{2}|\nabla u_\sigma|^2 - F(u_\sigma)\Big] dx$$

where $F(u) = \int_0^u f(s) ds$. A direct calculation gives

$$\frac{d}{dt} E[u_\sigma](t) = -\int_{\Omega(t)} (u_\sigma)_t^2 dx - \frac{\mu}{2} \int_{\Gamma(t)} |\nabla u_\sigma|^3 dS \leq 0.$$

By Lemma 14, for all large $|x|$, say $|x| \geq R_1$, $u_\sigma(t,x) < \theta$. Hence $F(u_\sigma(t,x)) \leq 0$ for $x \geq R_1$. It follows that $E[u_\sigma](t)$ is bounded from below.

So, by a simple variant of the standard argument involving a Lyapunov functional (see, e.g. [16, Theorem 4.3.4]) we see that each element $\overline{U} \in \omega(\sigma\phi)$ is a stationary solution, that is, $\Delta\overline{U} + f(\overline{U}) = 0$ for $x \in \mathbb{R}^N$. By Lemma 15 we see that $\sup \overline{U} \geq \theta$. By Lemma 14 we deduce $\overline{U}(x) \to 0$ as $|x| \to \infty$. By the moving plane method, one further concludes that $\overline{U}$ is radially symmestric about some $x_0 \in \mathbb{R}^N$, and is strictly decreasing in the radial directions away from x_0. We may now apply Lemma 9 to conclude that $\overline{U}(x) \equiv V(|x-x_0|)$. By Lemma 15, the maximum of $\overline{U}(x)$ is achieved at some point in $\overline{\text{co}}(\Omega_0)$. Hence $x_0 \in \overline{\text{co}}(\Omega_0)$. □

Lemma 17. *Σ^* consists of exactly one element, that is, $\sigma_* = \sigma^*$.*

Proof. We argue by contradiction. Assume $\sigma_1, \sigma_2 \in \Sigma^*$ and $\sigma_1 < \sigma_2$. Then there exist a time sequence $\{t_n\}$ tending to infinity such that $u_{\sigma_i}(t_n, x) \to V_0(|x - \bar{x}_i|)$ $(i = 1, 2)$ as $n \to \infty$ in the topology of $C^2_{loc}(\mathbb{R}^N)$, where $\bar{x}_1, \bar{x}_2 \in \overline{\text{co}}(\Omega_0)$.

By comparison we have

$$u_{\sigma_1}(T, x) < u_{\sigma_2}(T, x) \text{ for } x \in \overline{\Omega_1(T)} \subset \Omega_2(T),$$

where $\Omega_i(t)$ $(i = 1, 2)$ denote the supporting domain of $u_{\sigma_i}(t, \cdot)$, and $T > 0$ is chosen large enough such that $\partial\Omega_i(T)$ $(i = 1, 2)$ are closed smooth hypersurfaces in $\mathbb{R}^N$. Hence there exists $\varepsilon_0 > 0$ such that

$$u_{\sigma_1}(T, x) < u_{\sigma_2}(T, x + \varepsilon e) \quad \text{for all } x \in \Omega_1(T),\ e \in \mathbb{S}^{N-1},\ \varepsilon \in (0, \varepsilon_0).$$

By the comparison principle again we have

$$u_{\sigma_1}(t, x) < u_{\sigma_2}(t, x + \varepsilon e) \quad \text{for all } t > T,\ x \in \Omega_1(t),\ e \in \mathbb{S}^{N-1},\ \varepsilon \in (0, \varepsilon_0).$$

Taking $t = t_n$ and letting $n \to \infty$ we thus deduce

$$V_0(|x - \bar{x}_1|) \leq V_0(|x + \varepsilon e - \bar{x}_2|) \quad \text{for all } x \in \mathbb{R}^N,\ e \in \mathbb{S}^{N-1},\ \varepsilon \in (0, \varepsilon_0).$$

In particular,

$$V_0(0) \leq V_0(|\varepsilon e + \bar{x}_1 - \bar{x}_2|) \quad \text{for all } \ e \in \mathbb{S}^{N-1},\ \varepsilon \in (0, \varepsilon_0).$$

This contradicts the fact that $V_0(0) > V_0(r)$ for $r > 0$.

This completes the proof of the lemma and hence the proof of Theorem 1.2. □

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