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POSITIVE SOLUTIONS FOR A CLASS OF NONLOCAL PROBLEMS INVOLVING LEBESGUE GENERALIZED SPACES: SCALAR AND SYSTEM CASES

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Abstract. In this work we prove the existence of positive solutions for a class of scalar nonlocal problems and systems of such equations. We use sub-supersolution method combined with fixed point arguments and apply the results to some concrete problems.

1. Introduction

In this work we employ the sub-supersolution method in order to prove the existence of positive solutions for a class of nonlocal problems. More precisely, we study the class of nonlocal problems given by

$$(P)\qquad \begin{cases} -\mathcal{A}(x,|u|_{L^{r(x)}})\Delta u = f_1(x,u)|u|^{\alpha(x)}_{L^{q(x)}} + f_2(x,u)|u|^{\gamma(x)}_{L^{s(x)}} \ \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega \end{cases}$$

and

$$(S)\qquad \begin{cases} -\mathcal{A}(x,|v|_{L^{r_1(x)}})\Delta u = f_1(x,u,v)|v|^{\alpha_1(x)}_{L^{q_1(x)}} + f_2(x,u,v)|v|^{\gamma_1(x)}_{L^{s_1(x)}} & \text{in} \quad \Omega, \\ -\mathcal{A}(x,|u|_{L^{r_2(x)}})\Delta v = g_1(x,u,v)|u|^{\alpha_2(x)}_{L^{q_2(x)}} + g_2(x,u,v)|u|^{\gamma_2(x)}_{L^{s_2(x)}} & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, $|\cdot|_{L^{m(x)}}$ is the Luxemburg norm of Lebesgue generalized space $L^{m(x)}(\Omega)$, $\mathcal{A} : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ and $r, r_i, q, q_i, s, s_i, \alpha, \alpha_i, \gamma, \gamma_i : \overline{\Omega} \to \mathbb{R}$ are continuous functions. In the scalar case, we consider $f_1, f_2 : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ continuous functions and in the system case $f_1, f_2, g_1, g_2 : \overline{\Omega}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are also continuous functions. This class of problems is called nonlocal because of the presence of the terms $\mathcal{A}(.,|u|_{L^{r(x)}}), |u|^{\alpha_i(.)}_{L^{q_i(x)}}, |u|^{\gamma_i(.)}_{L^{s_i(x)}}$, which imply that equations in (P) and in (S) are no longer pontwise equalities. It is important to stress that this kind of problem comes from important applications of biology, physics and chemistry, as can be seen in [2], [3], [4], [5], [6], [7], [10], [11], [12], [13], [15], [16], [17], [22], [23], [25], [27], [28] and references therein.

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In the last decades, many authors have studied such classes of problems. For example, in [18], the author studied the following problem

$$\begin{cases} -a(\displaystyle\int_\Omega |u|^q)\Delta u = H(x)f(u) \text{ in } \Omega, \\ \qquad\qquad\qquad u = 0 \text{ on } \partial\Omega \end{cases}$$

using the Krasnoselskii and Schaefer Fixed Point Theorem in order to prove the existence of positive solutions. In [21], using the Galerkin method, the authors proved the existence of positive solutions to the problem

$$\begin{cases} -\Delta u = a(x,u)|u|^p_{L^q} \text{ in } \Omega, \\ \qquad u = 0 \text{ on } \partial\Omega. \end{cases}$$

The problem

$$\begin{cases} -\Delta_p u = |u|^{\alpha(x)}_{L^{q(x)}} \text{ in } \Omega, \\ \qquad u = 0 \text{ on } \partial\Omega \end{cases}$$

was studied in [19]. The authors used the sub-supersolution method with monotonic iteration and also proved the existence of a positive solution.

In [1], using the sub-supersolution method and a version of Minty-Browder's Theorem for pseudomonotonic operators, the authors proved the existence of a positive solutions of the problem

$$\begin{cases} -a(\int_\Omega |u|^q)\Delta u = h_1(x,u)f(\int_\Omega |u|^p) + h_2(x,u)g(\int_\Omega |u|^r) \text{ in } \Omega, \\ \qquad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $h_i : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ are continuous functions, $q, p, r \in [1, \infty)$ and the functions $a, f, g : [0, \infty) \to \mathbb{R}^+$ satisfy $f, g \in L^\infty([0, \infty))$ and

$$a(t), f(t), g(t) \geq a_0 > 0, \ \forall\, t \in [0, \infty).$$

Another problem that we would like to comment on is

$$(P)_\lambda \qquad \begin{cases} -\mathfrak{A}(x,u)\Delta u = \lambda f(u) \text{ in } \Omega, \\ \qquad u = 0 \text{ on } \partial\Omega, \end{cases}$$

that was studied in [9] with $f \in C^1([0,\theta], \mathbb{R})$, $f(0) = 0 = f(\theta)$, $f'(0) > 0$, $f(t) > 0$ in $(0, \theta)$. The function $\mathfrak{A} : \Omega \times L^p(\Omega) \to \mathbb{R}$ is such that $x \mapsto \mathfrak{A}(x,u)$ is measurable, $u \mapsto \mathfrak{A}(x,u)$ is continuous and there are constants $a_0, a_\infty > 0$, such that

$$a_0 \leq \mathfrak{A}(x,u) \leq a_\infty \text{ a.e in } \Omega, \ \forall\, u \in L^p(\Omega).$$

In that work, the authors used Schauder's Fixed Point Theorem to prove the existence of a positive solution. The multiplicity of solutions to this problem was studied in [14], where the authors proved that if f has n different roots, then $(P)_\lambda$ has n different positive solutions. They used Schauder's Fixed Point Theorem and Variational Methods. Moreover, in both works, the authors have studied the asymptotic behavior of solutions when λ approaches to ∞.

Motivated by the results found in [1], [9], [14] and [19], in this work we study problem (P) and we prove the following result:

Theorem 1. *Suppose that* $r(x), q(x), s(x) \in C_+(\overline{\Omega})$, $0 \le \alpha(x), \gamma(x) \in C^0(\overline{\Omega})$, $(\underline{u}, \overline{u})$ *is a pair of sub-supersolution for* (P) *with* $\underline{u} > 0$ *a.e in* Ω *and* $f_1(x,t), f_2(x,t) \ge 0$ *in* $\overline{\Omega} \times [0, |\overline{u}|_{L^\infty}]$. *Suppose also that* $\mathcal{A} : \overline{\Omega} \times (0, \infty) \to \mathbb{R}$ *is a continuous function and*

$$\mathcal{A}(x,t) > 0 \ \ in \ \ \overline{\Omega} \times \left[|\underline{u}|_{L^{r(x)}},\ |\overline{u}|_{L^{r(x)}}\right].$$

Then, problem (P) *has a weak positive solution* u *with*

$$\underline{u} \le u \le \overline{u}.$$

In the next sections, we define sub-supersolutions and weak solutions for (P) and the space $C_+(\overline{\Omega})$.

With respect to system (S), in [20] the authors studied the system

$$\begin{cases} -\Delta u^m = a|v|^{\alpha}_{L^p} & \text{in} \quad \Omega, \\ -\Delta v^n = b|u|^{\beta}_{L^q} & \text{in} \quad \Omega, \\ \qquad u = v = 0 & \text{on} \quad \partial\Omega \end{cases}$$

and proved the existence of a solution using Rabinowitz's result [26] concerning connected components of solutions. In [8], the authors studied

$$\begin{cases} -\Delta u = f_1(x,u)|v|^{p_1}_{L^{\alpha_1}} & \text{in} \quad \Omega, \\ -\Delta v = f_2(x,v)|u|^{p_2}_{L^{\alpha_2}} & \text{in} \quad \Omega, \\ \qquad u = v = 0 & \text{on} \quad \partial\Omega \end{cases}$$

using the Galerkin method combined with the sub-supersolution method and monotonic iteration. The system

$$\begin{cases} -\Delta_{p_1} u = |v|^{\alpha_1(x)}_{L^{q_1(x)}} & \text{in} \quad \Omega, \\ -\Delta_{p_2} v = |u|^{\alpha_2(x)}_{L^{q_2(x)}} & \text{in} \quad \Omega, \\ \qquad u = v = 0 & \text{on} \quad \partial\Omega \end{cases}$$

was studied in [19], where the authors have also used Rabinowitz's result [26] concerning connected components of solutions. In this article, motivated by results that can be found in [8], [19] and [20], we prove the following result:

Theorem 2. *Suppose* $r_i(x), q_i(x), s_i(x) \in C_+(\overline{\Omega})$, $0 \le \alpha_i(x), \gamma_i(x) \in C^0(\overline{\Omega})$. *Assume that the pairs* $(\underline{u}, \underline{v}), (\overline{u}, \overline{v})$ *are sub-supersolution to* (S) *with* $\underline{u}, \underline{v} > 0$ *a.e in* Ω, $f_i(x,t,s), g_i(x,t,s) \ge 0$ *in* $\overline{\Omega} \times [0, |\overline{u}|_{L^\infty}] \times [0, |\overline{v}|_{L^\infty}]$. *Suppose also that* $\mathcal{A} : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ *is a continuous function and*

$$\mathcal{A}(x,t) > 0 \ in \, \overline{\Omega} \times \left[\underline{\sigma}, \overline{\sigma}\right],$$

where $\underline{\sigma} := \min\left\{|\underline{w}|_{L^{r_1(x)}}, |\underline{w}|_{L^{r_2(x)}}\right\}$ *and* $\overline{\sigma} := \max\left\{|\overline{w}|_{L^{r_1(x)}}, |\overline{w}|_{L^{r_2(x)}}\right\}$. *Then, system* (S) *has a weak positive solution* (u,v) *with*

$$\underline{u} \le u \le \overline{u} \quad e \quad \underline{v} \le v \le \overline{v}.$$

In the next sections, we define subsolutions, supersolutions and weak solutions for (S). Now we make some comparisons concerning our results and some previously published results. For example:

i) We study a more general problem than the problems studied in [9], [14] *and* [19].

ii) In contrast with [9], [14] *and* [19], *we do not need that* $\mathcal{A}$ *be bounded in all its domain.*

iii) In contrast with [1]*, we discarded the hypotheses* $f, g \in L^\infty([0,\infty))$ *and* $f(t), g(t) \geq a_0 > 0$*. In this paper we deal with the functions* $f(t) = t^{\alpha(x)}$ *and* $g(t) = t^{\beta(x)}$ *that, even in the case where* $\alpha(x) = \alpha$ *and* $\beta(x) = \beta$*, do not satisfy the hypotheses of the main theorem in* [1]*. Moreover, in* [1] *the authors consider* $\mathcal{A}(x,t) = a(t) \geq a_0 > 0$*. Our result includes this case and other cases like* $\mathcal{A}(x,0) = 0$ *and* $\lim\limits_{t\to 0} \mathcal{A}(x,t) = \pm\infty$*.*
iv) We also provide three applications of Theorem 1. More precisely, we make applications of this theorem to a problem of sublinear type, to a problem of concave and convex type and to a problem that is a generalization of the classic logistic equation.
v) We study a more general system than the systems studied in [8], [19] *and* [20]*.*
vi) We also provide three applications of Theorem 2. More precisely, we apply this theorem to a system of sublinear type, to a concave and a convex system and to a system that is a generalization of the classic logistic system.

2. Basic results involving Generalized Lebesgue space

In this section we state some basic properties of the space $L^{p(x)}(\Omega)$. Firstly, we recall that thoughout this work we are always considering $\Omega \subset \mathbb{R}^N$ a bounded smooth domain. Let

$$C_+(\overline{\Omega}) = \left\{h; h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\right\}$$

and

$$h^+ = \max_{\overline{\Omega}} h(x),\ h^- = \min_{\overline{\Omega}} h(x).$$

For $p \in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega) = \left\{u; u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)} < \infty\right\}.$$

We introduce the norm on $L^{p(x)}(\Omega)$ defined by

$$|u|_{p(x)} = \inf\left\{\lambda > 0; \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \leq 1\right\}$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space. We call it a generalized Lebesgue space.

Proposition 1. **(i)** *The space* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ *is a separable, uniformly convex Banach space, and its conjugate space is* $(L^{q(x)}(\Omega), |\cdot|_{q(x)})$*, where* $q \in C_+(\overline{\Omega})$ *and* $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$*. For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$*, we have*

$$\left|\int_\Omega uv\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)} |v|_{q(x)}.$$

(ii) *If* $p_1, p_2 \in C_+(\overline{\Omega}), p_1(x) \leq p_2(x)$*, for any* $x \in \overline{\Omega}$*, then*

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$$

and the embedding is continuous.

Proposition 2. *For* $\rho(u) = \int_\Omega |u|^{p(x)}$ *and for all* $u, u_n \in L^{p(x)}(\Omega)$*, we have*
(i) *If* $u \neq 0$*, the* $|u|_{L^{p(x)}} = \lambda$ *is equivalent to* $\rho(\frac{u}{\lambda}) = 1$*.*
(ii) *If* $|u|_{L^{p(x)}} < 1$ $(= 1;\ > 1)$*, then* $\rho(u) < 1$ $(= 1;\ > 1)$*.*
(iii) *If* $|u|_{L^{p(x)}} > 1$*, then* $|u|^{p^-}_{L^{p(x)}} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}}$*.*

(iv) *If* $|u|_{L^{p(x)}} < 1$, *then* $|u|^{p^+}_{L^{p(x)}} \le \rho(u) \le |u|^{p^-}_{L^{p(x)}}$.
(v) $|u_n|_{L^{p(x)}} \to 0$ *is equivalent to* $\rho(u_n) \to 0$.
(vi) $|u_n|_{L^{p(x)}} \to \infty$ *is equivalent to* $\rho(u_n) \to \infty$.

More information on these spaces may be found in Fan-Zhang [24] and in its references.

3. The scalar case

From now on we use in $H^1_0(\Omega)$ the usual norm $\|u\|^2 = \int_\Omega |\nabla u|^2$. We start by defining a weak solution and a pair of sub-supersolution to problem (P).

Definition 1. The function $u \in H^1_0(\Omega) \bigcap L^\infty(\Omega)$ is a weak solution of problem (P) if $u > 0$ a.e in Ω and

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega \left(\frac{f_1(x,u)|u|^{\alpha(x)}_{L^{q(x)}}}{\mathcal{A}(x,|u|_{L^{r(x)}})} + \frac{f_2(x,u)|u|^{\gamma(x)}_{L^{s(x)}}}{\mathcal{A}(x,|u|_{L^{r(x)}})} \right) \varphi, \ \ \forall\, \varphi\ H^1_0(\Omega).$$

Definition 2. Given $z, w \in L^\infty(\Omega)$, with $z \le w$ a.e in Ω, we define

$$[z,w] := \big\{ u \in L^\infty(\Omega) : z(x) \le u(x) \le w(x) \text{ a.e in } \Omega \big\}.$$

Definition 3. The pair $(\underline{u}, \overline{u})$ is a sub-supersolution to problem (P) if $\underline{u} \in H^1_0(\Omega) \bigcap L^\infty(\Omega)$, $\overline{u} \in H^1(\Omega) \bigcap L^\infty(\Omega)$ with $\underline{u} \le \overline{u}$ a.e in Ω, $\underline{u} = 0 \le \overline{u}$ a.e in $\partial\Omega$ and for each $\varphi \in H^1_0(\Omega)$ with $\varphi \ge 0$, we have

$$\int_\Omega \nabla \underline{u} \nabla \varphi \le \int_\Omega \left(\frac{f_1(x,\underline{u})|\underline{u}|^{\alpha(x)}_{L^{q(x)}}}{\mathcal{A}(x,|w|_{L^{r(x)}})} + \frac{f_2(x,\underline{u})|\underline{u}|^{\gamma(x)}_{L^{s(x)}}}{\mathcal{A}(x,|w|_{L^{r(x)}})} \right) \varphi, \ \ \forall\ w \in [\underline{u}, \overline{u}] \tag{3.1}$$

and

$$\int_\Omega \nabla \overline{u} \nabla \varphi \ge \int_\Omega \left(\frac{f_1(x,\overline{u})|\overline{u}|^{\alpha(x)}_{L^{q(x)}}}{\mathcal{A}(x,|w|_{L^{r(x)}})} + \frac{f_2(x,\overline{u})|\overline{u}|^{\gamma(x)}_{L^{s(x)}}}{\mathcal{A}(x,|w|_{L^{r(x)}})} \right) \varphi, \ \ \forall\ w \in [\underline{u}, \overline{u}]. \tag{3.2}$$

3.1. **Proof of Theorem 1.**

Proof. Consider the truncation operator associated with $\underline{u}$ and $\overline{u}$, that is,

$$T : L^2(\Omega) \to L^\infty(\Omega)$$

$$Tu(x) = \begin{cases} \underline{u}(x) & \text{if } \quad u(x) \le \underline{u}(x) \\ u(x) & \text{if } \ \underline{u}(x) \le u(x) \le \overline{u}(x) \\ \overline{u}(x) & \text{if } \quad u(x) \ge \overline{u}(x). \end{cases}$$

Since $\underline{u}, \overline{u} \in L^\infty(\Omega)$ and $\underline{u} \le Tu \le \overline{u}$, T is well-defined. Now consider the operator

$$H : [\underline{u}, \overline{u}] \to L^2(\Omega)$$

given by

$$H(v)(x) = \frac{f_1(x,v)|v|^{\alpha(x)}_{L^{q(x)}}}{\mathcal{A}(x,|v|_{L^{r(x)}})} + \frac{f_2(x,v)|v|^{\gamma(x)}_{L^{s(x)}}}{\mathcal{A}(x,|v|_{L^{r(x)}})}.$$

We are going to prove that H is well-defined because, since $Tu \in [\underline{u}, \overline{u}] \subset L^\infty(\Omega)$ and $\underline{u} > 0$ a.e in Ω, we have

$$|\underline{u}|_{L^\infty} \le |Tu|_{L^\infty} \le |\overline{u}|_{L^\infty}, \ \ \forall\, u \in L^2(\Omega).$$

Moreover,

$$|\underline{u}|_{L^{m(x)}} \leq |Tu|_{L^{m(x)}} \leq |\overline{u}|_{L^{m(x)}},\ \forall\ u \in L^2(\Omega),\ m(x) \in C_+(\overline{\Omega}).$$

Therefore, since $\mathcal{A}(x,t)$ is positive and continuous in the compact set $\overline{\Omega} \times \left[|\underline{u}|_{L^{r(x)}},\ |\overline{u}|_{L^{r(x)}}\right]$, there are constants $k, K > 0$ such that

$$0 < k \leq \mathcal{A}(x, |Tu|_{L^{r(x)}}) \leq K \text{ in } \Omega,\ \forall\, u \in L^2(\Omega).$$

By the continuity of $f_1(x,t)$ and $f_2(x,t)$ in $\overline{\Omega} \times \left[0,\ |\overline{u}|_{L^\infty}\right]$, there are constants $c_1, c_2 > 0$ such that

$$|H(v)| \leq \frac{c_1(|\overline{u}|^{\alpha^-}_{L^{q(x)}} + |\overline{u}|^{\alpha^+}_{L^{q(x)}}) + c_2(|\overline{u}|^{\gamma^-}_{L^{s(x)}} + |\overline{u}|^{\gamma^+}_{L^{s(x)}})}{k} \text{ in } \overline{\Omega},\ \forall\, v \in [\underline{u}, \overline{u}].$$

Then, H is well-defined. Now, we prove that $u \mapsto H(Tu)$ of $L^2(\Omega)$ in $L^2(\Omega)$ is continuous. Let $(u_n) \subset L^2(\Omega)$ be a sequence such that $u_n \to u$ in $L^2(\Omega)$, for some $u \in L^2(\Omega)$. Then, up to subsequences,

$$u_n(x) \to u(x) \text{ a.e in } \Omega,$$

$$Tu_n(x) \to Tu(x) \text{ a.e in } \Omega,$$

$$|Tu_n(x) - Tu(x)|^{m(x)} \to 0 \text{ a.e in } \Omega$$

and

$$|Tu_n(x) - Tu(x)|^{m(x)} \leq 2^{m(x)}|\overline{u}|^{m(x)}_{L^\infty} \leq 2^{m+}(|\overline{u}|^{m^-}_{L^\infty} + |\overline{u}|^{m^+}_{L^\infty}) \text{ a.e } \Omega,$$

for $m(x) \in C_+(\overline{\Omega})$. From Lebesgue's Dominated Convergence Theorem we obtain

$$\int_\Omega |Tu_n - Tu|^{m(x)} \to 0.$$

By Proposition 2, item v), we derive

$$Tu_n \to Tu \text{ in } L^{m(x)}(\Omega),\ m(x) \in C_+(\overline{\Omega}).$$

Using the continuity of $f_1(x,t)$, $f_2(x,t)$, $\mathcal{A}(x,t)$ and Lebesgue's Dominated Convergence Theorem, we get

$$H(Tu_n) \to H(Tu) \text{ in } L^2(\Omega).$$

Given $v \in L^2(\Omega)$, let $u = S(v)$ be the unique solution of the problem

$$(P_L) \qquad \begin{cases} -\Delta u = H(Tv) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where S is the solution operator. We are going to prove the existence of a fixed point of S using Schaefer's Fixed Point Theorem. Note that S is compact because, for $(v_n) \subset L^2(\Omega)$ bounded and $u_n = S(v_n)$, by the definition of S we get

$$\int_\Omega \nabla u_n \nabla \varphi = \int_\Omega H(Tv_n)\varphi,\ \forall\, \varphi \in H^1_0(\Omega).$$

Taking $\varphi = u_n$ and using the definition of T and H, we have

$$\|u_n\|^2 \leq K_0|u_n|_{L^1},\ \forall\, n \in \mathbb{N}, \text{ and for some } K_0 > 0$$

and, hence, $(u_n) \subset H^1_0(\Omega)$ is bounded. Now, up to a subsequence,

$$u_n \rightharpoonup u \text{ in } H^1_0(\Omega).$$

From Sobolev's embedding, we derive

$$S(v_n) = u_n \to u \text{ in } L^2(\Omega).$$

Now we prove that S is continuous. In fact, for $v_n \to v$ in $L^2(\Omega)$, $u_n = S(v_n)$ and $u = S(v)$, by the definition of S we obtain

$$\int_\Omega \nabla u_n \nabla \varphi = \int_\Omega H(Tv_n)\varphi, \ \forall \, \varphi \in H_0^1(\Omega)$$

and

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega H(Tv)\varphi, \ \forall \, \varphi \in H_0^1(\Omega).$$

Taking $\varphi = u_n$, by Holder's inequality, we get

$$\begin{aligned} \Big| \int_\Omega \nabla u_n \nabla (u_n - u) \Big| &\leq \int_\Omega |H(Tv_n) - H(Tv)||u_n| \\ &\leq |H(Tv_n) - H(Tv)|_{L^2} |u_n|_{L^2}. \end{aligned}$$

Since, (u_n) is bounded in $H_0^1(\Omega)$ and by the continuity of $u \mapsto H(Tu)$, we conclude

$$\int_\Omega \nabla u_n \nabla (u_n - u) \to 0,$$

which implies

$$\|u_n - u\|^2 \to 0.$$

Hence,

$$S(v_n) \to S(v) \text{ in } L^2(\Omega)$$

and S is continuous. Now, we are going to prove that there is $R > 0$ such that if $u = \theta S(u)$ with $\theta \in [0,1]$, then $|u|_{L^2} < R$. Indeed, if $\theta = 0$, we have $u = 0$. If $\theta \neq 0$, we get

$$S(u) = \frac{u}{\theta}.$$

From the definition of S, we obtain

$$\int_\Omega \nabla \Big(\frac{u}{\theta}\Big) \nabla \varphi = \int_\Omega H(Tv)\varphi, \ \forall \, \varphi \in H_0^1(\Omega).$$

Taking $\varphi = u$ and using the definition of T and H once again, for some $K_0 > 0$ we derive

$$\|u\|^2 \leq \theta K_0 |u|_{L^1},$$

which implies

$$|u|_{L^2} < R.$$

Hence, from Schaefer's Fixed Point Theorem, there is $u \in L^2(\Omega)$ with $u = S(u)$. Then,

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega H(Tu)\varphi, \ \forall \ \ \varphi \in H_0^1(\Omega),$$

that is,

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega \left(\frac{f_1(x, Tu)|Tu|_{L^{q(x)}}^{\alpha(x)}}{\mathcal{A}(x, |Tu|_{L^{r(x)}})} + \frac{f_2(x, Tu)|Tu|_{L^{s(x)}}^{\gamma(x)}}{\mathcal{A}(x, |Tu|_{L^{r(x)}})} \right) \varphi, \ \forall \ \ \varphi \in H_0^1(\Omega). \tag{3.3}$$

Now we are going to prove that

$$\underline{u} \leq u \leq \overline{u} \ \text{ a.e in } \ \Omega.$$

In fact, since $Tu \in [\underline{u}, \overline{u}]$, using $w = Tu$ in (3.1) and considering (3.3), we have, for each $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$ that

$$\begin{aligned}\int_\Omega \nabla(\underline{u} - u)\nabla\varphi &\leq \int_\Omega \Big(\frac{f_1(x,\underline{u})|\underline{u}|^{\alpha(x)}_{L^{q(x)}} - f_1(x,Tu(x))|Tu|^{\alpha(x)}_{L^{q(x)}}}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\Big)\varphi \\ &+ \int_\Omega \Big(\frac{f_2(x,\underline{u})|\underline{u}|^{\gamma(x)}_{L^{s(x)}} - f_2(x,Tu(x))|Tu|^{\gamma(x)}_{L^{s(x)}}}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\Big)\varphi.\end{aligned}$$

Taking $\varphi = (\underline{u} - u)_+ = \max\{(\underline{u} - u), 0\}$ and recalling that $f_i(x,t) \geq 0$ in $[0, |\overline{u}|_{L^\infty}]$, $Tu = \underline{u}$ in $\big\{x \in \Omega : \underline{u}(x) \geq u(x)\big\}$ and $Tu \in [\underline{u}, \overline{u}]$, we get

$$\begin{aligned}\int_\Omega \nabla(\underline{u} - u)\nabla(\underline{u} - u)_+ &\leq \int_\Omega \Big(\frac{f_1(x,\underline{u})|\underline{u}|^{\alpha(x)}_{L^{q(x)}} - f_1(x,Tu(x))|Tu|^{\alpha(x)}_{L^{q(x)}}}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\Big)(\underline{u} - u)_+ \\ &+ \int_\Omega \Big(\frac{f_2(x,\underline{u})|\underline{u}|^{\gamma(x)}_{L^{s(x)}} - f_2(x,Tu(x))|Tu|^{\gamma(x)}_{L^{s(x)}}}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\Big)(\underline{u} - u)_+ \\ &= \int_{\{x\in\Omega\,:\,\underline{u}(x)\geq u(x)\}} f_1(x,\underline{u})\frac{(|\underline{u}|^{\alpha(x)}_{L^{q(x)}} - |Tu|^{\alpha(x)}_{L^{q(x)}})}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}(\underline{u} - u) \\ &+ \int_{\{x\in\Omega\,:\,\underline{u}(x)\geq u(x)\}} f_2(x,\underline{u})\frac{(|\underline{u}|^{\gamma(x)}_{L^{s(x)}} - |Tu|^{\gamma(x)}_{L^{s(x)}})}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}(\underline{u} - u) \\ &\leq 0,\end{aligned}$$

which implies

$$\|(\underline{u} - u)_+\|^2 \leq 0,$$

and conclude,

$$\underline{u} \leq u, \quad \text{a.e in } \Omega.$$

Now, using $w = Tu$ in (3.2) and considering (3.3), we have, for each $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$,

$$\begin{aligned}\int_\Omega \nabla(u - \overline{u})\nabla\varphi &\leq \int_\Omega \Big(\frac{f_1(x,Tu(x))|Tu|^{\alpha(x)}_{L^{q(x)}} - f_1(x,\overline{u}(x))|\overline{u}|^{\alpha(x)}_{L^{q(x)}}}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\Big)\varphi \\ &+ \int_\Omega \Big(\frac{f_2(x,Tu(x))|Tu|^{\gamma(x)}_{L^{s(x)}} - f_2(x,\overline{u}(x))|\overline{u}|^{\gamma(x)}_{L^{s(x)}}}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\Big)\varphi.\end{aligned}$$

Taking $\varphi = (u-\overline{u})_+ = \max\{(u-\overline{u}),0\}$ and recalling that $f_i(x,t) \geq 0$ in $[0,|\overline{u}|_{L^\infty}]$, $Tu=\overline{u}$ in $\{x \in \Omega : u(x) \geq \overline{u}(x)\}$ and $Tu \in [\underline{u},\overline{u}]$, we have

$$
\begin{aligned}
\|(u-\overline{u})_+\|^2 &\leq \int_\Omega \Big(\frac{f_1(x,Tu(x))|Tu|^{\alpha(x)}_{L^{q(x)}} - f_1(x,\overline{u}(x))|\overline{u}|^{\alpha(x)}_{L^{q(x)}}}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\Big)(u-\overline{u})_+ \\
&+ \int_\Omega \Big(\frac{f_2(x,Tu(x))|Tu|^{\gamma(x)}_{L^{s(x)}} - f_2(x,\overline{u}(x))|\overline{u}|^{\gamma(x)}_{L^{s(x)}}}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\Big)(u-\overline{u})_+ \\
&= \int_{\{x\in\Omega\,:\,u(x)\geq\overline{u}(x)\}} f_1(x,\overline{u}(x))\frac{(|Tu|^{\alpha(x)}_{L^{q(x)}} - |\overline{u}|^{\alpha(x)}_{L^{q(x)}})}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}(u-\overline{u}) \\
&+ \int_{\{x\in\Omega\,:\,u(x)\geq\overline{u}(x)\}} f_2(x,\overline{u}(x))\frac{(|Tu|^{\gamma(x)}_{L^{s(x)}} - |\overline{u}|^{\gamma(x)}_{L^{s(x)}})}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}(u-\overline{u}) \\
&\leq 0,
\end{aligned}
$$

which implies

$$\|(u-\overline{u})_+\|^2 \leq 0$$

and conclude

$$u \leq \overline{u} \ \text{ a.e in } \ \Omega.$$

Then, from definition of T, we derive $Tu = u$. Since u satisfies (3.3), we conclude the proof of Theorem 1. □

Remark 1. The weak positive solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem (P) is a strong solution. In fact, since $H(Tu) \in L^\infty(\Omega)$ and $Tu = u$, we have $H(u) \in L^\infty(\Omega)$ and u is a weak solution of $-\Delta u = H(u)$ in Ω with $u = 0$ on $\partial\Omega$. Then, by regularity results, we get $u \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$, $\forall\, p \geq 1$, such that

$$
\begin{cases}
-\mathcal{A}(x,|u|_{L^{r(x)}})\Delta u = f_1(x,u)|u|^{\alpha(x)}_{L^{q(x)}} + f_2(x,u)|u|^{\gamma(x)}_{L^{s(x)}} \ \text{ a.e in } \ \Omega, \\
u > 0 \text{ in } \Omega, \\
u = 0 \ \text{ on } \ \partial\Omega.
\end{cases}
$$

4. Applications of Theorem 1

In this section we make three applications of Theorem 1. From now on we denote by $e \in H_0^1(\Omega) \cap C^{2,\tau}(\overline{\Omega})$, for some $0 < \tau < 1$, the unique positive solution of problem $-\Delta e = 1$ in Ω, $e = 0$ on $\partial\Omega$ and by $\varphi_1 \in H_0^1(\Omega) \bigcap C^{2,\alpha}(\overline{\Omega})$ a positive eigenfunction associated with the first eigenvalue λ_1 of $(-\Delta, H_0^1(\Omega))$.

4.1. **Sublinear problem.** First of all, we study the nonlocal sublinear problem given by

$$
(Ps) \qquad \begin{cases}
-\mathcal{A}(x,|u|_{L^{r(x)}})\Delta u = u^{\beta(x)}|u|^{\alpha(x)}_{L^{q(x)}} \ \text{ in } \ \Omega, \\
u = 0 \ \text{ on } \ \partial\Omega.
\end{cases}
$$

Problem (Ps) is related with the problem studied in [19]. Let us study the above problem considering two cases: the first case is $\mathcal{A}(x,t) \geq a_0 > 0$ in $\overline{\Omega} \times [0,\infty)$ and the second case is $\mathcal{A}(x,0) = 0$.

The main result in this subsection is:

Theorem 3. *Suppose that $r(x), q(x) \in C_+(\overline{\Omega})$ and $0 \leq \alpha(x), \beta(x) \in C^0(\overline{\Omega})$ are such that*
$$0 < \alpha^+ + \beta^+ < 1.$$
Suppose also that $\mathcal{A} : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ is continuous and satisfies one of these two conditions:
(A_1) There is a positive constant $a_0 > 0$ such that
$$\mathcal{A}(x,t) \geq a_0 > 0 \ \ \textit{in} \ \overline{\Omega} \times [0,\infty).$$
(A_2) There are positive constants $a_1, a_\infty > 0$ such that
$$\mathcal{A}(x,0) = 0 < \mathcal{A}(x,t) \leq a_1 \ \ \textit{in} \ \overline{\Omega} \times (0,\infty)$$
and
$$\lim_{t\to\infty} \mathcal{A}(x,t) = a_\infty \ \ \textit{uniformly in} \ \overline{\Omega}.$$
Then, problem (Ps) has a weak positive solution.

Proof. We start with (A_1). Let us construct $\overline{u}$. Since $\alpha(x), \beta(x) \in C^0(\overline{\Omega})$, we have
$$\begin{array}{rcl} |e|^{\alpha(.)}_{L^{q(x)}}, \ |e|^{\beta(.)}_{L^\infty} \quad \overline{\Omega}: & \to & \mathbb{R}^+ \\ x & \mapsto & |e|^{\alpha(x)}_{L^{q(x)}}, \ |e|^{\beta(x)}_{L^\infty}, \end{array}$$
are continuous functions. Then, there are positive constants $C_1, C_2 > 0$, such that
$$|e|^{\alpha(x)}_{L^{q(x)}} \leq C_1 \quad \text{and} \quad |e|^{\beta(x)}_{L^\infty} \leq C_2, \ \ \forall \ x \in \overline{\Omega}.$$
Since $0 < \alpha^+ + \beta^+ < 1$, we choose $R > 0$, such that
$$R \geq \max\left\{ \left(\frac{C_1 C_2}{a_0}\right)^{\frac{1}{1-(\alpha^+ + \beta^+)}}, 1 \right\}.$$
For each $w \in L^\infty(\Omega)$ and setting $\overline{u} = Re$, by some straight forward algebraic manipulations we get
$$\begin{cases} -\Delta \overline{u} \geq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})} \overline{u}^{\beta(x)} |\overline{u}|^{\alpha(x)}_{L^{q(x)}} \ \text{ in } \ \Omega, \\ \hfill \overline{u} > 0 \ \text{ in } \ \Omega, \\ \hfill \overline{u} = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$
Now let us set $\underline{u}$. Considering $K = \max\big\{\mathcal{A}(x,t) \ : \ (x,t) \in \overline{\Omega} \times \big[0, \ |\overline{u}|_{L^{r(x)}}\big]\big\}$, given $w \in \big[0, \overline{u}\big]$, then $|w|_{L^{r(x)}} \leq |\overline{u}|_{L^{r(x)}}$, which implies
$$a_0 \leq \mathcal{A}(x, |w|_{L^{r(x)}}) \leq K \ \text{ in } \Omega, \ \forall \ w \in [0, \overline{u}].$$
Choosing
$$0 < \epsilon \leq \min\left\{ \left(\frac{|\varphi_1|^{\alpha^+}_{L^{q(x)}}}{\lambda_1 |\varphi_1|^{1-\beta^+}_{L^\infty} K} \right)^{\frac{1}{1-(\alpha^+ + \beta^+)}}, 1 \right\},$$
$\underline{u} = \epsilon\varphi_1$, for each $w \in [0, \overline{u}]$ and by some straight forward algebraic manipulations we have
$$\begin{cases} -\Delta \underline{u} \leq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})} \underline{u}^{\beta(x)} |\underline{u}|^{\alpha(x)}_{L^{q(x)}} \ \text{ in } \ \Omega, \\ \hfill \underline{u} > 0 \ \text{ in } \ \Omega, \\ \hfill \underline{u} = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

Now we are going to prove that $\underline{u} \leq \overline{u}$. For $\epsilon > 0$ sufficiently small we have

$$\lambda_1 \epsilon |\varphi_1|_{L^\infty} \leq R.$$

Then,

$$-\Delta(\epsilon\varphi_1) \leq -\Delta(Re) \text{ in } \Omega,$$

and by the Comparison Principle we get

$$\underline{u} := \epsilon\varphi_1 \leq Re =: \overline{u}.$$

Hence, $(\underline{u}, \overline{u})$ is a pair of sub-supersolution to problem (Ps). By Theorem 1, problem (Ps) has a weak positive solution $u \in H_0^1(\Omega) \bigcap L^\infty(\Omega)$ with

$$\underline{u} \leq u \leq \overline{u},$$

which proves that the result holds with (A_1).

Suppose now that (A_2) holds. We are going to set $\underline{u}$. We take $\epsilon > 0$ such that

$$0 < \epsilon \leq \min\left\{ \left(\frac{|\varphi_1|^{\alpha^+}_{L^{q(x)}}}{\lambda_1 |\varphi_1|^{1-\beta^+}_{L^\infty} a_1} \right)^{\frac{1}{1-(\alpha^+ + \beta^+)}}, 1 \right\}.$$

Then, defining $\underline{u} = \epsilon\varphi_1$ for $\epsilon > 0$ small, we get

$$\begin{cases} -\Delta \underline{u} \leq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})} \underline{u}^{\beta(x)} |\underline{u}|^{\alpha(x)}_{L^{q(x)}} \text{ in } \Omega, \\ \underline{u} > 0 \text{ in } \Omega, \\ \underline{u} = 0 \text{ on } \partial\Omega. \end{cases}$$

Now we are going to find $\overline{u}$. Since $\lim_{t\to\infty} \mathcal{A}(x,t) = a_\infty$ uniformly in $\overline{\Omega}$, there exists $M > 0$ sufficiently large such that

$$\mathcal{A}(x,t) \geq \frac{a_\infty}{2} \text{ in } \overline{\Omega} \times [M, \infty).$$

Let

$$m = \min\left\{\mathcal{A}(x,t) : (x,t) \in \overline{\Omega} \times \left[|\underline{u}|_{L^{r(x)}}, M\right]\right\} > 0.$$

Taking $k = \min\left\{m, \frac{a_\infty}{2}\right\}$, we have

$$\mathcal{A}(x,t) \geq k > 0 \text{ in } \overline{\Omega} \times \left[|\underline{u}|_{L^{r(x)}}, \infty\right).$$

Now consider $\overline{u} = Re$, with $R > 0$ such that

$$R \geq \max\left\{ \left(\frac{C_1 C_2}{k} \right)^{\frac{1}{1-(\alpha^+ + \beta^+)}}, 1 \right\},$$

where

$$|e|^{\alpha(x)}_{L^{q(x)}} \leq C_1 \quad \text{and} \quad |e|^{\beta(x)}_{L^\infty} \leq C_2, \ \forall \ x \in \overline{\Omega}.$$

Then, for each $w \in L^\infty(\Omega)$; $\underline{u} \leq w$, we have

$$\begin{cases} -\Delta \overline{u} \geq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})} \overline{u}^{\beta(x)} |\overline{u}|^{\alpha(x)}_{L^{q(x)}} \text{ in } \Omega, \\ \overline{u} > 0 \text{ in } \Omega, \\ \overline{u} = 0 \text{ on } \partial\Omega. \end{cases}$$

We are going to prove that $\underline{u} \leq \overline{u}$. Considering $R > 0$ such that

$$\lambda_1 \epsilon |\varphi_1|_{L^\infty} \leq R,$$

we get

$$-\Delta(\epsilon\varphi_1) \leq -\Delta(Re) \text{ in } \Omega.$$

By the Comparison Principle, we conclude that

$$\underline{u} := \epsilon\varphi_1 \leq Re =: \overline{u}.$$

Then, $(\underline{u}, \overline{u})$ is a pair of sub-supersolution to problem (Ps). By Theorem 1, problem (Ps) has a weak positive solution $u \in H_0^1(\Omega) \bigcap L^\infty(\Omega)$ with

$$\underline{u} \leq u \leq \overline{u},$$

and the proof is finished. □

4.2. **Concave and convex problem.** In this subsection we study the concave and convex problem

$$(P)_{\lambda,\mu} \qquad \begin{cases} -\mathcal{A}(x, |u|_{L^{r(x)}})\Delta u = \lambda |u|^{\beta(x)-1} u |u|^{\alpha(x)}_{L^{q(x)}} + \mu |u|^{\eta(x)-1} u |u|^{\gamma(x)}_{L^{s(x)}} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Let us study problem $(P)_{\lambda,\mu}$ for two classes of functions $\mathcal{A}$.

Theorem 4. *Suppose that $r(x)$, $q(x)$, $s(x) \in C_+(\overline{\Omega})$ and $0 \leq \alpha(x)$, $\gamma(x)$, $\beta(x)$, $\eta(x) \in C^0(\overline{\Omega})$ such that*

$$0 < \alpha^- + \beta^- \leq \alpha^+ + \beta^+ < 1.$$

Suppose also that $\mathcal{A} : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ is a continuous function satisfying the following hypotheses:
(A_1) Suppose that $1 < \eta^- + \gamma^-$ and there are constants $a_0, b_0 > 0$ such that

$$\mathcal{A}(x, t) \geq a_0 > 0 \text{ in } \overline{\Omega} \times [0, b_0].$$

Then, given $\mu > 0$, there is $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$, problem $(P)_{\lambda,\mu}$ has a weak positive solution $u_{\lambda,\mu}$.
(A_2) Suppose that $1 < \eta^+ + \gamma^+$ and that there are constants $a_1, a_\infty > 0$, such that

$$\mathcal{A}(x, 0) = 0 < \mathcal{A}(x, t) \leq a_1 \text{ in } \overline{\Omega} \times (0, \infty)$$

and

$$\lim_{t \to \infty} \mathcal{A}(x, t) = a_\infty \text{ uniformly in } \overline{\Omega}.$$

Then, given $\lambda > 0$, there is $\mu_0 > 0$ such that, for each $\mu \in (0, \mu_0)$, problem $(P)_{\lambda,\mu}$ has a weak positive solution $u_{\lambda,\mu}$.

Proof. First of all, let us consider condition (A_1). We begin by constructing $\overline{u}$. Note that, for each $M > 0$, we get

$$-\Delta(Me) = M \text{ in } \Omega.$$

Then, we want to get $M > 0$ such that

$$M \geq \frac{1}{a_0}\left(\lambda (Me)^{\beta(x)} |Me|^{\alpha(x)}_{L^{q(x)}} + \mu (Me)^{\eta(x)} |Me|^{\gamma(x)}_{L^{s(x)}}\right) \text{ in } \Omega. \tag{4.1}$$

Note that, for $0 < M \leq 1$, inequality (4.1) is true when

$$M \geq \frac{1}{a_0}\left(\lambda M^{\beta^-+\alpha^-}|e|_{L^\infty}^{\beta(x)}|e|_{L^{q(x)}}^{\alpha(x)} + \mu M^{\eta^-+\gamma^-}|e|_{L^\infty}^{\eta(x)}|e|_{L^{s(x)}}^{\gamma(x)}\right) \text{ in } \Omega. \tag{4.2}$$

Considering

$$R = \max\Big\{|e|_{L^\infty},\ |e|_{L^{q(x)}},\ |e|_{L^{s(x)}},\ 1\Big\}, \tag{4.3}$$

inequality (4.2) is true when

$$1 \geq \frac{1}{a_0}\left(\lambda M^{\beta^-+\alpha^--1}R^{\eta^++\gamma^+} + \mu M^{\eta^-+\gamma^--1}R^{\eta^++\gamma^+}\right). \tag{4.4}$$

We are going to prove that (4.4) holds. Considering $\varphi : (0,\infty) \to \mathbb{R}$ given by

$$\varphi(M) = \frac{R^{\eta^++\gamma^+}\lambda}{a_0}M^{\beta^-+\alpha^--1} + \frac{R^{\eta^++\gamma^+}\mu}{a_0}M^{\eta^-+\gamma^--1},$$

since φ is a C^1 class function and $0 < \alpha^- + \beta^- < 1 < \eta^- + \gamma^-$, we get

$$\lim_{M\to 0^+}\varphi(M) = \lim_{M\to\infty}\varphi(M) = \infty.$$

Then, φ has a global minimum point $M_{\lambda,\mu} = M(\lambda,\mu) > 0$. Hence,

$$\varphi'(M_{\lambda,\mu}) = 0 \Leftrightarrow M_{\lambda,\mu} = c_1\left(\frac{\lambda}{\mu}\right)^{\frac{1}{(\eta^-+\gamma^-)-(\beta^-+\alpha^-)}},$$

where $c_1 = \left(\frac{1-(\beta^-+\alpha^-)}{(\eta^-+\gamma^-)-1}\right)^{\frac{1}{(\eta^-+\gamma^-)-(\beta^-+\alpha^-)}}$.
Note also that

$$\varphi(M_{\lambda,\mu}) = \frac{R^{\eta^++\gamma^+}c_2}{a_0}\left[\frac{\lambda^{(\eta^-+\gamma^-)-1}}{\mu^{(\beta^-+\alpha^-)-1}}\right]^{\frac{1}{(\eta^-+\gamma^-)-(\beta^-+\alpha^-)}},$$

where $c_2 = c_1^{(\beta^-+\alpha^-)-1} + c_1^{(\eta^-+\gamma^-)-1}$. Hence, given $\mu > 0$, there is $\lambda_0 = \lambda_0(\mu) > 0$, such that, for each $\lambda \in (0,\lambda_0]$, the pair (λ,μ) satisfies

$$0 < M_{\lambda,\mu} \leq 1 \quad\text{and}\quad \varphi(M_{\lambda,\mu}) \leq 1, \quad \forall\ 0 < \lambda \leq \lambda_0.$$

Note that $M_{\lambda,\mu}$ also satisfies (4.1). Since

$$\mathcal{A}(x,t) \geq a_0 > 0 \text{ in } \overline{\Omega}\times[0,b_0],$$

$M_{\lambda,\mu} \to 0$ as $\lambda \to 0$ and $\lambda \mapsto M_{\lambda,\mu}$ is increasing, we can choose $\lambda_0 > 0$ such that, for each $\lambda \in (0,\lambda_0)$, we have

$$\mathcal{A}(x,|w|_{L^{r(x)}}) \geq a_0 > 0, \ \ \forall\ w \in [0, M_{\lambda,\mu}e],$$

where $\big[0, M_{\lambda,\mu}e\big] := \big\{w \in L^\infty(\Omega) : 0 \leq w(x) \leq M_{\lambda,\mu}e \ \text{ a.e in } \ \Omega\big\}$. Now, defining $\overline{u} = \overline{u}(\lambda,\mu) := M_{\lambda,\mu}e$, for each $w \in [0,\overline{u}]$ we have

$$\begin{cases} -\Delta\overline{u} \geq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r(x)}})}\lambda\overline{u}^{\beta(x)}|\overline{u}|_{L^{q(x)}}^{\alpha(x)} + \dfrac{1}{\mathcal{A}(x,|w|_{L^{r(x)}})}\mu\overline{u}^{\eta(x)}|\overline{u}|_{L^{s(x)}}^{\gamma(x)} \text{ in } \Omega, \\ \overline{u} > 0 \text{ in } \Omega, \\ \overline{u} = 0 \text{ on } \partial\Omega. \end{cases}$$

We are going to construct $\underline{u}$. Considering $K = \max\left\{\mathcal{A}(x,t):\ (x,t) \in \overline{\Omega} \times \left[0,\ \ |\overline{u}|_{L^{r(x)}}\right]\right\}$, given $w \in [0, \overline{u}]$, we have $|w|_{L^{r(x)}} \leq |\overline{u}|_{L^{r(x)}}$. Then,

$$a_0 \leq \mathcal{A}(x, |w|_{L^{r(x)}}) \leq K \ \text{ in } \Omega,\ \forall\, w \in [0, \overline{u}].$$

Now consider $\varphi_1 > 0$ in Ω, $|\varphi_1|_{L^\infty} \leq 1$ and $|\varphi_1|_{L^{q(x)}} \leq 1$. Since $0 < \alpha^+ + \beta^+ < 1$, take $\epsilon = \epsilon(\lambda) > 0$, such that

$$0 < \epsilon \leq \min\left\{\left(\frac{\lambda|\varphi_1|^{\alpha^+}_{L^{q(x)}}}{\lambda_1|\varphi_1|^{1-\beta^+}_{L^\infty}K}\right)^{\frac{1}{1-(\alpha^+ + \beta^+)}}, 1\right\}.$$

If $\underline{u} = \underline{u}(\lambda) = \epsilon\varphi_1$, for each $w \in [0, \overline{u}]$, we get

$$\begin{cases} -\Delta \underline{u} \leq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})}\lambda \underline{u}^{\beta(x)}|\underline{u}|^{\alpha(x)}_{L^{q(x)}} + \dfrac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})}\mu \underline{u}^{\eta(x)}|\underline{u}|^{\gamma(x)}_{L^{q(x)}} \ \text{ in } \Omega, \\ \underline{u} > 0 \ \text{ in } \Omega, \\ \underline{u} = 0 \ \text{ on } \partial\Omega. \end{cases}$$

We are going to prove that $\underline{u} \leq \overline{u}$. For $\lambda \in (0, \lambda_0]$, taking $\epsilon > 0$ sufficiently small such that

$$\lambda_1 \epsilon |\varphi_1|_{L^\infty} \leq M_{\lambda,\mu},$$

we have,

$$-\Delta(\epsilon\varphi_1) \leq -\Delta(M_{\lambda,\mu} e).$$

Then, by the Comparison Principle,

$$\underline{u} := \epsilon\varphi_1 \leq M_{\lambda,\mu} e =: \overline{u},$$

that implies, $(\underline{u}, \overline{u})$ is a pair of sub-supersolution to problem $(P)_{\lambda,\mu}$. From Theorem 1, for each $0 < \lambda \leq \lambda_0$, there is a weak positive solution $u_{\lambda,\mu} \in H^1_0(\Omega) \bigcap L^\infty(\Omega)$ to $(P)_{\lambda,\mu}$ such that

$$\underline{u} \leq u_{\lambda,\mu} \leq \overline{u},$$

which finishes the prove for the case (A_1).

We consider the case (A_2) in order to construct $\underline{u}$. Given $\lambda > 0$ and making $\epsilon = \epsilon(\lambda) > 0$, such that

$$\epsilon \leq \min\left\{\left(\frac{\lambda|\varphi_1|^{\alpha^+}_{L^{q(x)}}}{\lambda_1|\varphi_1|^{1-\beta^+}_{L^\infty}a_1}\right)^{\frac{1}{1-(\alpha^+ + \beta^+)}}, 1\right\}$$

and considering $\underline{u} = \underline{u}(\lambda) := \epsilon\varphi_1$, for each $w \in L^\infty(\Omega)$, we have

$$\begin{cases} -\Delta \underline{u} \leq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})}\lambda \underline{u}^{\beta(x)}|\underline{u}|^{\alpha(x)}_{L^{q(x)}} + \dfrac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})}\mu \underline{u}^{\eta(x)}|\underline{u}|^{\gamma(x)}_{L^{q(x)}} \ \text{ in } \Omega, \\ \underline{u} > 0 \ \text{ in } \Omega, \\ \underline{u} = 0 \ \text{ on } \partial\Omega. \end{cases}$$

We construct $\overline{u}$. Since $\lim\limits_{t\to\infty} \mathcal{A}(x,t) = a_\infty$ uniformly in $\overline{\Omega}$, there is $M > 0$ sufficiently large such that

$$\mathcal{A}(x,t) \geq \frac{a_\infty}{2}, \ \text{ in } \overline{\Omega} \times [M, \infty).$$

Consider

$$m_\lambda = m(\lambda) = \min\left\{\mathcal{A}(x,t):\ (x,t) \in \overline{\Omega} \times \left[|\underline{u}|_{L^{r(x)}}, M\right]\right\} > 0.$$

Taking $k_\lambda = k(\lambda) = \min\left\{m_\lambda, \frac{a_\infty}{2}\right\}$, we have

$$\mathcal{A}(x,t) \geq k_\lambda > 0 \ \text{ in } \ \overline{\Omega} \times \left[|\underline{u}|_{L^{r(x)}}, \infty\right).$$

We want to obtain a positive constant $T > 0$ such that, for each $w \in L^\infty(\Omega)$ with $\underline{u} \leq w$, we have

$$T \geq \frac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})} \left(\lambda (Te)^{\beta(x)} |Te|^{\alpha(x)}_{L^{q(x)}} + \mu (Te)^{\eta(x)} |Te|^{\gamma(x)}_{L^{s(x)}}\right) \text{ in } \Omega.$$

But this relation is true if $T \geq 1$ and

$$1 \geq \frac{1}{k_\lambda}\left(\lambda T^{\beta^+ + \alpha^+ - 1} R^{\eta^+ + \gamma^+} + \mu T^{\eta^+ + \gamma^+ - 1} R^{\eta^+ + \gamma^+}\right), \tag{4.5}$$

where $R = \max\left\{|e|_{L^\infty}, |e|_{L^{q(x)}}, |e|_{L^{s(x)}}, 1\right\}$. Consider $\psi : (0,\infty) \to \mathbb{R}$ given by

$$\psi(t) = \frac{R^{\eta^+ + \gamma^+}\lambda}{k_\lambda} t^{\beta^+ + \alpha^+ - 1} + \frac{R^{\eta^+ + \gamma^+}\mu}{k_\lambda} t^{\eta^+ + \gamma^+ - 1}.$$

Since $0 < \alpha^+ + \beta^+ < 1 < \eta^+ + \gamma^+$, ψ has a minimum point $T_{\lambda,\mu}$ given by

$$T_{\lambda,\mu} = c_3 \left(\frac{\lambda}{\mu}\right)^{\frac{1}{(\eta^+ + \gamma^+) - (\beta^+ + \alpha^+)}},$$

where $c_3 = \left(\frac{1 - (\beta^+ + \alpha^+)}{(\eta^+ + \gamma^+) - 1}\right)^{\frac{1}{(\eta^+ + \gamma^+) - (\beta^+ + \alpha^+)}}$.
Note that

$$T_{\lambda,\mu} \geq 1 \Leftrightarrow \lambda \geq \mu \left[\frac{(\eta^+ + \gamma^+) - 1}{1 - (\beta^+ + \alpha^+)}\right] > 0 \tag{4.6}$$

and

$$\psi(T_{\lambda,\mu}) \leq 1 \Leftrightarrow \frac{R^{\eta^+ + \gamma^+} c_4}{k_\lambda} \left[\frac{\lambda^{(\eta^+ + \gamma^+) - 1}}{\mu^{(\beta^+ + \alpha^+) - 1}}\right]^{\frac{1}{(\eta^+ + \gamma^+) - (\beta^+ + \alpha^+)}} \leq 1. \tag{4.7}$$

Then, given $\lambda > 0$, there exists $\mu_0 = \mu_0(\lambda) > 0$ such that, for each $\mu \in (0, \mu_0)$, the pair (λ, μ) satisfies

$$T_{\lambda,\mu} \geq 1 \ \text{ and } \ \psi(T_{\lambda,\mu}) \leq 1.$$

Hence, considering $\overline{u} = \overline{u}(\lambda,\mu) := T_{\lambda,\mu} e$, for each $w \in L^\infty(\Omega)$ with $\underline{u}_\lambda \leq w$ we have

$$\begin{cases} -\Delta \overline{u} \geq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})} \lambda \overline{u}^{\beta(x)} |\overline{u}|^{\alpha(x)}_{L^{q(x)}} + \dfrac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})} \mu \overline{u}^{\eta(x)} |\overline{u}|^{\gamma(x)}_{L^{q(x)}} \ \text{ in } \ \Omega, \\ \overline{u} > 0 \ \text{ in } \ \Omega, \\ \overline{u} = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

We are going to prove that $\underline{u} \leq \overline{u}$. Since $T_{\lambda,\mu} \to \infty$ as $\mu \to 0^+$, we can choose $\mu_0 = \mu_0(\lambda) > 0$, such that

$$\lambda_1 \epsilon |\varphi_1|_{L^\infty} \leq T_{\lambda,\mu_0}.$$

Then,

$$-\Delta(\epsilon\varphi_1) \leq -\Delta(T_{\lambda,\mu_0} e),$$

and by the Comparison Principle $\underline{u} := \epsilon\varphi_1 \le T_{\lambda,\mu_0}e$. Since the function $\mu \to T_{\lambda,\mu}$ is decreasing, we get

$$\underline{u} \le T_{\lambda,\mu_0}e \le T_{\lambda,\mu}e := \overline{u}, \ \ \forall\, \mu \in (0,\mu_0).$$

Hence, $(\underline{u},\overline{u})$ is a pair of sub-supersolution to $(P)_{\lambda,\mu}$ and from Theorem 1, there is a weak positive solution $u_{\lambda,\mu} \in H_0^1(\Omega)\bigcap L^\infty(\Omega)$ of $(P)_{\lambda,\mu}$, such that

$$\underline{u} \le u_{\lambda,\mu} \le \overline{u},$$

which finishes the proof for the case (A_2). □

4.3. **A generalized classical logistic equation.** In this application we consider the following class of problems given by

$$(P')_\lambda \qquad \begin{cases} -\mathcal{A}(x,|u|_{L^{r(x)}})\Delta u = \lambda f(u)|u|^{\alpha(x)}_{L^{q(x)}} \ \text{ in } \ \Omega, \\ \qquad\qquad u = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

We assume that there is $\theta > 0$ such that $f : [0,\infty) \to \mathbb{R}$ satisfies

- (f_1) $f \in C^0[0,\theta]$.
- (f_2) $f(0) = f(\theta) = 0$, $f'(0) > 0$ ($f'(0) \in \mathbb{R}$ or $f'(0) = \infty$) and $f(s) > 0\ \forall s \in (0,\theta)$.

Some prototypes of functions f that satisfy the above hypotheses are given by $f_1(t) = t(\gamma - t)$ and $f_2(t) = \mu t^q - t^p$; $0 < q < 1 < p$ e $\gamma,\mu > 0$. The main result in this subsection is:

Theorem 5. *Suppose that $r(x)$, $q(x) \in C_+(\overline{\Omega})$, $0 \le \alpha(x) \in C^0(\overline{\Omega})$ and $f : [0,\infty) \to \mathbb{R}$ satisfies (f_1) and (f_2). Suppose also that $\mathcal{A} : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ is a continuous function such that*

$$\mathcal{A}(x,t) > 0 \ \ in \ \overline{\Omega} \times \big(0, |\theta|_{L^{r(x)}}\big].$$

Then, there is $\lambda_0 > 0$ such that, for each $\lambda \ge \lambda_0$, problem $(P')_\lambda$ has a weak positive solution u_λ with

$$0 < u_\lambda \le \theta.$$

Proof. We are going to construct $\underline{u}$. Given $\delta > 0$ define $\underline{\lambda} := \dfrac{\lambda_1}{f'(0)} + \delta$ when $f'(0) \in \mathbb{R}$ and $\underline{\lambda} := \delta$ when $f'(0) = \infty$. We are going to prove that the solution of problem

$$(P_2)_{\underline{\lambda}} \qquad \begin{cases} -\Delta u = \underline{\lambda} f(u) \ \text{ in } \ \Omega, \\ \qquad u = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

is a subsolution of problem $(P')_\lambda$. Note that $\theta > 0$ is a supersolution to $(P_2)_{\underline{\lambda}}$, because $f(\theta) = 0$. Now, since $f'(0) > \dfrac{\lambda_1}{\underline{\lambda}}$ and $f(0) = 0$, there is $\tau > 0$ such that

$$\frac{f(t)}{t} \ge \frac{\lambda_1}{\underline{\lambda}}, \ \ \forall\, t \in (0,\tau].$$

Considering $\epsilon > 0$ such that $\epsilon|\varphi_1|_{L^\infty} \le \tau$, we have

$$\frac{f(\epsilon\varphi_1)}{\epsilon\varphi_1} \ge \frac{\lambda_1}{\underline{\lambda}} \ \text{ in } \ \Omega.$$

Then,

$$-\Delta(\epsilon\varphi_1) = \lambda_1(\epsilon\varphi_1) \le \underline{\lambda} f(\epsilon\varphi_1) \ \text{ in } \ \Omega.$$

Hence, $\epsilon\varphi_1$ is a subsolution of $(P_2)_{\underline{\lambda}}$. Making $\epsilon > 0$ small if necessary, we get $\epsilon\varphi_1 \leq \theta$. Then, using Theorem 1 with $\mathcal{A}(x,t) \equiv 1$, $\alpha(x) \equiv 0$, $f_2 \equiv 0$ and $f_1 \equiv \underline{\lambda} f$, we obtain a weak positive solution $\varphi = \varphi_{\underline{\lambda}}$ of $(P_2)_{\underline{\lambda}}$ verifying

$$\epsilon\varphi_1 \leq \varphi \leq \theta. \tag{4.8}$$

Note that

$$\begin{aligned} |\varphi|^{\alpha(.)}_{L^{q(x)}} : \quad \overline{\Omega} \quad &\to \mathbb{R}^+ \\ x \quad &\mapsto |\varphi|^{\alpha(x)}_{L^{q(x)}} \end{aligned}$$

is a continuous function. Then, there is a constant $C > 0$, such that

$$|\varphi|^{\alpha(x)}_{L^{q(x)}} \geq C \quad \text{in } \overline{\Omega}.$$

Considering $K = \max\left\{\mathcal{A}(x,t) : (x,t) \in \overline{\Omega} \times [|\varphi|_{L^{r(x)}}, |\theta|_{L^{r(x)}}]\right\}$, $\mu = \frac{K}{C}$ and $\varphi = \varphi_{\underline{\lambda}}$ the solution of $(P_2)_{\underline{\lambda}}$, we have

$$-\Delta\varphi = \underline{\lambda} f(\varphi) = \frac{\underline{\lambda}\mu f(\varphi)|\varphi|^{\alpha(x)}_{L^{q(x)}}}{K} \frac{K}{\mu|\varphi|^{\alpha(x)}_{L^{q(x)}}}.$$

Since $\frac{K}{\mu|\varphi|^{\alpha(x)}_{L^{q(x)}}} \leq 1$, we get

$$-\Delta\varphi \leq \underline{\lambda}\mu \frac{f(\varphi)|\varphi|^{\alpha(x)}_{L^{q(x)}}}{K}.$$

Then, for each $\lambda \geq \underline{\lambda}\mu$, we conclude that

$$-\Delta\varphi \leq \lambda \frac{f(\varphi)|\varphi|^{\alpha(x)}_{L^{q(x)}}}{\mathcal{A}(x, |w|_{L^{r(x)}})}, \quad \forall\, w \in [\varphi, \theta].$$

Defining $\underline{u} := \varphi = \varphi_{\underline{\lambda}}$ we have proved that $\underline{u}$ is a subsolution of $(P')_\lambda$, for each $\lambda \geq \underline{\lambda}\mu$. Since $f(\theta) = 0$, the function $\overline{u} := \theta$ is a supersolution to $(P')_\lambda$, because

$$\begin{cases} -\Delta\overline{u} = 0 = \lambda \frac{f(\overline{u})}{\mathcal{A}(x,|w|_{L^{r(x)}})}|\overline{u}|^{\alpha(x)}_{L^{q(x)}} \text{ in } \Omega, \ \forall\, w \in [\varphi, \theta], \\ \overline{u} > 0 \text{ in } \Omega, \\ \overline{u} > 0 \text{ on } \partial\Omega. \end{cases}$$

Note that from (4.8) we get

$$\underline{u} := \varphi_{\underline{\lambda}} \leq \theta =: \overline{u}.$$

Hence, $(\underline{u}, \overline{u})$ is a pair of sub-supersolution to $(P')_\lambda$ and by Theorem 1, for each $\lambda \geq \lambda_0 := \underline{\lambda}\mu$, there is a weak positive solution $u_\lambda \in H^1_0(\Omega) \bigcap L^\infty(\Omega)$ of $(P')_\lambda$ such that

$$\varphi_{\underline{\lambda}} \leq u_\lambda \leq \theta.$$

□

5. The system case

We start with the definition of weak solutions and sub-supersolutions to (S).

Definition 4. The pair (u, v) is a weak positive solution of (S) if $u, v \in H_0^1(\Omega)\bigcap L^\infty(\Omega)$ with $u, v > 0$ a.e in Ω,

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega \left(\frac{f_1(x,u,v)|v|_{L^{q_1(x)}}^{\alpha_1(x)}}{\mathcal{A}(x,|v|_{L^{r_1(x)}})} + \frac{f_2(x,u,v)|v|_{L^{s_1(x)}}^{\gamma_1(x)}}{\mathcal{A}(x,|v|_{L^{r_1(x)}})} \right) \varphi, \quad \forall\, \varphi \in H_0^1(\Omega)$$

and

$$\int_\Omega \nabla v \nabla \psi = \int_\Omega \left(\frac{g_1(x,u,v)|u|_{L^{q_2(x)}}^{\alpha_2(x)}}{\mathcal{A}(x,|u|_{L^{r_2(x)}})} + \frac{g_2(x,u,v)|u|_{L^{s_2(x)}}^{\gamma_2(x)}}{\mathcal{A}(x,|u|_{L^{r_2(x)}})} \right) \psi, \quad \forall\, \psi \in H_0^1(\Omega).$$

Definition 5. Given $z, \theta \in L^\infty(\Omega)$, with $z \le \theta$ a.e in Ω, we define

$$[z,\theta] := \big\{ w \in L^\infty(\Omega) : \ z(x) \le w(x) \le \theta(x) \ \text{ a.e in } \ \Omega \big\}$$

and

$$[z,\infty) := \big\{ w \in L^\infty(\Omega) : \ z(x) \le w(x) \ \text{ a.e in } \ \Omega \big\}.$$

Definition 6. The pairs $(\underline{u}, \underline{v}), (\overline{u}, \overline{v})$ are sub-supersolution to (S), if $\underline{u}, \underline{v} \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\overline{u}, \overline{v} \in H^1(\Omega)\bigcap L^\infty(\Omega)$ with $\underline{u} \le \overline{u}$, $\underline{v} \le \overline{v}$ a.e in Ω and $\underline{u} = 0 \le \overline{u}$, $\underline{v} = 0 \le \overline{v}$ a.e on $\partial\Omega$ and given $\varphi, \psi \in H_0^1(\Omega)$ with $\varphi, \psi \ge 0$, we have

$$(5.1) \quad \begin{cases} \int_\Omega \nabla \underline{u} \nabla \varphi \le \int_\Omega \left(\frac{f_1(x,\underline{u},w)|\underline{v}|_{L^{q_1(x)}}^{\alpha_1(x)}}{\mathcal{A}(x,|w|_{L^{r_1(x)}})} + \frac{f_2(x,\underline{u},w)|\underline{v}|_{L^{s_1(x)}}^{\gamma_1(x)}}{\mathcal{A}(x,|w|_{L^{r_1(x)}})} \right) \varphi \quad \forall\, w \in [\underline{v},\ \overline{v}], \\ \\ \int_\Omega \nabla \underline{v} \nabla \psi \le \int_\Omega \left(\frac{g_1(x,w,\underline{v})|\underline{u}|_{L^{q_2(x)}}^{\alpha_2(x)}}{\mathcal{A}(x,|w|_{L^{r_2(x)}})} + \frac{g_2(x,w,\underline{v})|\underline{u}|_{L^{s_2(x)}}^{\gamma_2(x)}}{\mathcal{A}(x,|w|_{L^{r_2(x)}})} \right) \psi \quad \forall\, w \in [\underline{u},\ \overline{u}] \end{cases}$$

and

$$(5.2) \quad \begin{cases} \int_\Omega \nabla \overline{u} \nabla \varphi \ge \int_\Omega \left(\frac{f_1(x,\overline{u},w)|\overline{v}|_{L^{q_1(x)}}^{\alpha_1(x)}}{\mathcal{A}(x,|w|_{L^{r_1(x)}})} + \frac{f_2(x,\overline{u},w)|\overline{v}|_{L^{s_1(x)}}^{\gamma_1(x)}}{\mathcal{A}(x,|w|_{L^{r_1(x)}})} \right) \varphi \quad \forall\, w \in [\underline{v},\ \overline{v}], \\ \\ \int_\Omega \nabla \overline{v} \nabla \psi \ge \int_\Omega \left(\frac{g_1(x,w,\overline{v})|\overline{u}|_{L^{q_2(x)}}^{\alpha_2(x)}}{\mathcal{A}(x,|w|_{L^{r_2(x)}})} + \frac{g_2(x,w,\overline{v})|\overline{u}|_{L^{s_2(x)}}^{\gamma_2(x)}}{\mathcal{A}(x,|w|_{L^{r_2(x)}})} \right) \psi \quad \forall\, w \in [\underline{u},\ \overline{u}]. \end{cases}$$

5.1. **Proof of Theorem 2.**

Proof. Consider the truncation operators given by

$$T, S : L^2(\Omega) \to L^\infty(\Omega)$$

$$Tz(x) = \begin{cases} \underline{u}(x) & \text{if } \quad z(x) \le \underline{u}(x), \\ z(x) & \text{if } \ \underline{u}(x) \le z(x) \le \overline{u}(x), \\ \overline{u}(x) & \text{if } \quad z(x) \ge \overline{u}(x) \end{cases}$$

and

$$Sw(x) = \begin{cases} \underline{v}(x) & \text{if } \quad w(x) \le \underline{v}(x), \\ w(x) & \text{if } \ \underline{v}(x) \le w(x) \le \overline{v}(x), \\ \overline{v}(x) & \text{if } \quad w(x) \ge \overline{v}(x). \end{cases}$$

By definition of T and S, we have

$$\underline{u} \leq Tz \leq \overline{u} \quad \text{and} \quad \underline{v} \leq Sw \leq \overline{v} \ \text{ in } \Omega, \ \ \forall\, z, w \in L^2(\Omega).$$

Since $\underline{w} = \min\{\underline{u}, \underline{v}\}$ and $\overline{w} = \max\{\overline{u}, \overline{v}\}$, then

$$\underline{w} \leq Tz, Sw \leq \overline{w} \ \text{ in } \Omega, \ \ \forall\, z, w \in L^2(\Omega)$$

which implies $Tz, Sw \in L^\infty(\Omega)$. Hence T and S are well-defined,

$$|\underline{w}|_{L^\infty} \leq |Tz|_{L^\infty}, |Sw|_{L^\infty} \leq |\overline{w}|_{L^\infty}, \ \ \forall\, z, w \in L^2(\Omega).$$

and

$$|\underline{w}|_{L^{m(x)}} \leq |Tz|_{L^{m(x)}}, |Sw|_{L^{m(x)}} \leq |\overline{w}|_{L^{m(x)}}, \ \forall \ \ z, w \in L^2(\Omega), \ m(x) \in C_+(\overline{\Omega}).$$

Let us consider the operators $H_1, H_2 : [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}] \to L^2(\Omega)$ given by

$$H_1(u,v)(x) := \frac{f_1(x, u(x), v(x))|v|_{L^{q_1(x)}}^{\alpha_1(x)}}{\mathcal{A}(x, |v|_{L^{r_1(x)}})} + \frac{f_2(x, u(x), v(x))|v|_{L^{s_1(x)}}^{\gamma_1(x)}}{\mathcal{A}(x, |v|_{L^{r_1(x)}})}$$

and

$$H_2(u,v)(x) := \frac{g_1(x, u(x), v(x))|u|_{L^{q_2(x)}}^{\alpha_2(x)}}{\mathcal{A}(x, |u|_{L^{r_2(x)}})} + \frac{g_2(x, u(x), v(x))|u|_{L^{s_2(x)}}^{\gamma_2(x)}}{\mathcal{A}(x, |u|_{L^{r_2(x)}})}.$$

We are proving that $H_i, i = 1,2$ are well-defined and $(z,w) \mapsto H_i(Tz, Sw)$; $(z,w) \in L^2(\Omega) \times L^2(\Omega)$ are continuous from $L^2(\Omega) \times L^2(\Omega)$ in $L^2(\Omega)$. In fact, since $\mathcal{A}(x,t) > 0$ in $\overline{\Omega} \times [\underline{\sigma}, \overline{\sigma}]$ there are positive constants $k, K > 0$, such that

$$k \leq \mathcal{A}(x,t) \leq K, \ \forall\, (x,t) \in \overline{\Omega} \times [\underline{\sigma}, \overline{\sigma}].$$

Given $(u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$, we have $u, v \in [\underline{w}, \overline{w}]$. Then, $|u|_{L^{r_i(x)}}, |v|_{L^{r_i(x)}} \in [\underline{\sigma}, \overline{\sigma}]$ and

$$0 < k \leq \mathcal{A}(x, |u|_{L^{r_i(x)}}), \mathcal{A}(x, |v|_{L^{r_i(x)}}) \leq K \ \text{ in } \ \overline{\Omega}, \ \ \forall\, (u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}], i = 1,2.$$

Since $f_i(x,t,s), g_i(x,t,s)$ are continuous in $\overline{\Omega} \times [0, |\overline{u}|_{L^\infty}] \times [0, |\overline{v}|_{L^\infty}]$, there are $c_1, c_2, c_3, c_4 > 0$ such that

$$|H_1(u,v)| \leq \frac{c_1(|\overline{w}|_{L^{q_1(x)}}^{\alpha_1^-} + |\overline{w}|_{L^{q_1(x)}}^{\alpha_1^+}) + c_2(|\overline{w}|_{L^{s_1(x)}}^{\gamma_1^-} + |\overline{w}|_{L^{s_1(x)}}^{\gamma_1^+})}{k} \ \text{ in } \ \overline{\Omega},$$

$\forall\, (u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ and

$$|H_2(u,v)| \leq \frac{c_3(|\overline{w}|_{L^{q_2(x)}}^{\alpha_2^-} + |\overline{w}|_{L^{q_2(x)}}^{\alpha_2^+}) + c_4(|\overline{w}|_{L^{s_2(x)}}^{\gamma_2^-} + |\overline{w}|_{L^{s_2(x)}}^{\gamma_2^+})}{k} \ \text{ in } \ \overline{\Omega},$$

$\forall\, (u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ that implies that H_i are well-defined. Now we prove the continuity of the functions $(z,w) \mapsto H_i(Tz, Sw)$ from $L^2(\Omega) \times L^2(\Omega)$ to $L^2(\Omega)$. Let us consider $L^2(\Omega) \times L^2(\Omega)$ with the norm

$$|(u,v)|_{L^2 \times L^2} = |u|_{L^2} + |v|_{L^2}$$

and $(z_n, w_n) \to (z, w)$ in $L^2(\Omega) \times L^2(\Omega)$. Then, up to a subsequence,

$$z_n(x) \to z(x) \ \text{ and } \ w_n(x) \to w(x) \text{ a.e in } \Omega,$$

$$Tz_n(x) \to Tz(x) \ \text{ and } \ Sw_n(x) \to Sw(x) \text{ a.e in } \Omega.$$

Hence,

$$|Tz_n(x) - Tz(x)|^{m(x)}, \ |Sw_n(x) - Sw(x)|^{m(x)} \to 0 \ \text{ a.e in } \ \Omega$$

and

$$|Tz_n(x) - Tz(x)|^{m(x)} \leq 2|\overline{u}|_{L^\infty}^{m(x)} \leq C \ \text{ a.e in } \ \Omega,$$
$$|Sw_n(x) - Sw(x)|^{m(x)} \leq 2|\overline{v}|_{L^\infty}^{m(x)} \leq C \ \text{ a.e in } \ \Omega.$$

From Lebesgue's Dominated Convergence Theorem

$$\int_\Omega |Tz_n - Tz|^{m(x)} \to 0 \ \text{ and } \ \int_\Omega |Sw_n - Sw|^{m(x)} \to 0,$$

we conclude

$$Tz_n \to Tz \ \text{ and } \ Sw_n \to Sw \ \text{ in } \ L^{m(x)}(\Omega).$$

Since $f_i(x,t)$, $g_i(x,t)$ and $\mathcal{A}(x,t)$, $i = 1, 2$, are continuous, we have

$$H_i(Tz_n, Sw_n) \to H_i(Tz, Sw) \ \text{ a.e in } \Omega.$$

By Lebesgue's Dominated Convergence Theorem we get

$$H_i(Tu_n, Sw_n) \to H_i(Tu, Sw) \text{ in } \ L^2(\Omega),$$

which proves that $H_i(Tu, Sw)$ are continuous from $L^2(\Omega) \times L^2(\Omega)$ to $L^2(\Omega)$. Now, given $(z, w) \in L^2(\Omega) \times L^2(\Omega)$, consider the linear system

$$(S_L) \qquad \begin{cases} -\Delta u = H_1(Tz, Sw) \ \text{ in } \ \Omega, \\ -\Delta v = H_2(Tz, Sw) \ \text{ in } \ \Omega, \\ \qquad\quad u = v = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

Then, each equation of (S_L) has a unique weak solution in $H_0^1(\Omega)$ and we can define the operator

$$\Phi : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$$
$$(z, w) \mapsto \Phi(z, w) = (u, v),$$

where $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a weak solution of (S_L). We are going to prove that Φ has a fixed point using Schaefer's Fixed Point Theorem. Arguing as in the proof of Theorem 1, we can prove that the operator Φ is compact and continuous. Now we are going to prove that there is $R > 0$ such that, if

$$(u, v) = \theta\Phi(u, v) \text{ with } \theta \in [0, 1],$$

then we have

$$|(u, v)|_{L^2 \times L^2} < R.$$

Indeed, if $\theta = 0$, we get $(u, v) = (0, 0)$. If $\theta \neq 0$, we obtain

$$\Phi(u, v) = \Big(\frac{u}{\theta}, \frac{v}{\theta}\Big),$$

which implies

$$\int_\Omega \nabla\Big(\frac{u}{\theta}\Big)\nabla\varphi = \int_\Omega H_1(Su, Tv)\varphi, \ \forall\, \varphi \ \in \ H_0^1(\Omega)$$

and

$$\int_\Omega \nabla\Big(\frac{v}{\theta}\Big)\nabla\psi = \int_\Omega H_2(Su, Tv)\psi, \ \forall\, \psi \ \in \ H_0^1(\Omega).$$

Taking $\varphi = u$ and $\psi = v$, we conclude that

$$\|u\|^2 \leq \theta K_1 |u|_{L^1} \quad \text{and} \quad \|v\|^2 \leq \theta K_1 |v|_{L^1},$$

and using Poincaré's inequality we obtain $R > 0$, such that

$$|u|_{L^2} + |v|_{L^2} < R.$$

From Schaefer's Fixed Point Theorem, there exists $(u, v) \in L^2(\Omega) \times L^2(\Omega)$, such that

$$\Phi(u, v) = (u, v) \quad \text{and} \quad |(u, v)|_{L^2 \times L^2} < R.$$

By the definition of Φ, we have proved that the pair (u, v) satisfies
(5.3)

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega \left(\frac{f_1(x, Tu, Sv)|Sv|^{\alpha_1(x)}_{L^{q_1(x)}}}{\mathcal{A}(x, |Sv|_{L^{r_1(x)}})} + \frac{f_2(x, Tu, Sv)|Sv|^{\gamma_1(x)}_{L^{s_1(x)}}}{\mathcal{A}(x, |Sv|_{L^{r_1(x)}})} \right) \varphi, \ \ \forall\, \varphi \in H^1_0(\Omega)$$

and
(5.4)

$$\int_\Omega \nabla v \nabla \psi = \int_\Omega \left(\frac{g_1(x, Tu, Sv)|Tu|^{\alpha_2(x)}_{L^{q_2(x)}}}{\mathcal{A}(x, |Tu|_{L^{r_2(x)}})} + \frac{g_2(x, Tu, Sv)|Tu|^{\gamma_2(x)}_{L^{s_2(x)}}}{\mathcal{A}(x, |Tu|_{L^{r_2(x)}})} \right) \psi, \ \ \forall\, \psi \in H^1_0(\Omega).$$

We are going to prove that

$$\underline{u} \le u \le \overline{u} \quad \text{and} \quad \underline{v} \le v \le \overline{v}.$$

Making $w = Sv \in [\underline{v}, \overline{v}]$ in the definition of subsolution, for each $\varphi \in H^1_0(\Omega), \varphi \ge 0$, we have

$$\begin{aligned} \int_\Omega \nabla(\underline{u} - u)\nabla\varphi \ &\le \ \int_\Omega \Big(\frac{f_1(x, \underline{u}(x), Sv(x))|\underline{v}|^{\alpha_1(x)}_{L^{q_1(x)}} - f_1(x, Tu(x), Sv(x))|Sv|^{\alpha_1(x)}_{L^{q_1(x)}}}{\mathcal{A}(x, |Sv|_{L^{r_1(x)}})} \Big)\varphi \\ &+ \ \int_\Omega \Big(\frac{f_2(x, \underline{u}(x), Sv(x))|\underline{v}|^{\gamma_1(x)}_{L^{s_1(x)}} - f_2(x, Tu(x), Sv(x))|Sv|^{\gamma_1(x)}_{L^{s_1(x)}}}{\mathcal{A}(x, |Sv|_{L^{r_1(x)}})} \Big)\varphi. \end{aligned}$$

Taking $\varphi = (\underline{u} - u)_+ := \max\{(\underline{u} - u), 0\}$, since $f_i(x, t, s) \ge 0$ in $\overline{\Omega} \times [0, |\overline{u}|_{L^\infty}] \times [0, |\overline{v}|_{L^\infty}]$, $Tu = \underline{u}$ in $\{x \in \Omega : \underline{u}(x) \ge u(x)\}$ and $Sv \in [\underline{v}, \overline{v}]$, we get

$$\begin{aligned} \|(\underline{u} - u)_+\|^2 \ &\le \ \int_{\{x\in\Omega:\ \underline{u}(x)\ge u(x)\}} f_1(x, \underline{u}(x), Sv(x)) \frac{(|\underline{v}|^{\alpha_1(x)}_{L^{q_1(x)}} - |Sv|^{\alpha_1(x)}_{L^{q_1(x)}})}{\mathcal{A}(x, |Sv|_{L^{r_1(x)}})} (\underline{u} - u) \\ &+ \ \int_{\{x\in\Omega:\ \underline{u}(x)\ge u(x)\}} f_2(x, \underline{u}(x), Sv(x)) \frac{(|\underline{v}|^{\gamma_1(x)}_{L^{s_1(x)}} - |Sv|^{\gamma_1(x)}_{L^{s_1(x)}})}{\mathcal{A}(x, |Sv|_{L^{r_1(x)}})} (\underline{u} - u) \\ &\le \ 0. \end{aligned}$$

Then, $(\underline{u} - u)_+ = 0$, that implies $\underline{u} \le u$. Considering again $Sw = v \in [\underline{v}, \overline{v}]$, for each $\varphi \in H^1_0(\Omega); \varphi \ge 0$, we have

$$\begin{aligned} \int_\Omega \nabla(u - \overline{u})\nabla\varphi \ &\le \ \int_\Omega \Big(\frac{f_1(x, Tu(x), Sv(x))|Sv|^{\alpha_1(x)}_{L^{q_1(x)}} - f_1(x, \overline{u}(x), Sv(x))|\overline{v}|^{\alpha_1(x)}_{L^{q_1(x)}}}{\mathcal{A}(x, |Sv|_{L^{r_1(x)}})} \Big)\varphi \\ &+ \ \int_\Omega \Big(\frac{f_2(x, Tu(x), Sv(x))|Sv|^{\gamma_1(x)}_{L^{s_1(x)}} - f_2(x, \overline{u}(x), Sv(x))|\overline{v}|^{\gamma_1(x)}_{L^{s_1(x)}}}{\mathcal{A}(x, |Sv|_{L^{r_1(x)}})} \Big)\varphi. \end{aligned}$$

Taking $\varphi = (u-\overline{u})_+ := \max\{(u-\overline{u}), 0\}$, since que $f_i(x,t,s) \geq 0$ in $\overline{\Omega}\times[0,|\overline{u}|_{L^\infty}]\times[0,|\overline{v}|_{L^\infty}]$, $Tu = \overline{u}$ in $\{x\in\Omega : u(x)\geq \overline{u}(x)\}$ and $Sv \in [\underline{v}, \overline{v}]$, we get

$$\begin{aligned}\|(u-\overline{u})_+\|^2 &\leq \int_{\{x\in\Omega:\ u(x)\geq\overline{u}(x)\}} f_1(x,\overline{u}(x),Sv(x))\frac{(|Sv|^{\alpha_1(x)}_{L^{q_1(x)}} - |\overline{v}|^{\alpha_1(x)}_{L^{q_1(x)}})}{\mathcal{A}(x,|Sv|_{L^{r_1(x)}})}(u-\overline{u})\\ &+ \int_{\{x\in\Omega:\ u(x)\geq\overline{u}(x)\}} f_2(x,\overline{u}(x),Sv(x))\frac{(|Sv|^{\gamma_1(x)}_{L^{s_1(x)}} - |\overline{v}|^{\gamma_1(x)}_{L^{s_1(x)}})}{\mathcal{A}(x,|Sv|_{L^{r_1(x)}})}(u-\overline{u})\\ &\leq 0.\end{aligned}$$

Then, $(u-\overline{u})_+ = 0$ which implies $u \leq \overline{u}$. Using the same arguments we can prove

$$\underline{v} \leq v \leq \overline{v}.$$

By the definition of T and S, we conclude $Tu = u$ and $Sv = v$. Then the pair (u,v) is a weak positive solution of (S) with

$$\underline{u} \leq u \leq \overline{u} \ \text{ and } \ \underline{v} \leq v \leq \overline{v}.$$

□

Remark 2. As in the scalar case, the weak solution (u,v) found is, in fact, a strong solution and satisfies

$$\begin{cases} -\mathcal{A}(x,|v|_{L^{r_1(x)}})\Delta u = f_1(x,u,v)|v|^{\alpha_1(x)}_{L^{q_1(x)}} + f_2(x,u,v)|v|^{\gamma_1(x)}_{L^{s_1(x)}} & \text{a.e in} \quad \Omega,\\ -\mathcal{A}(x,|u|_{L^{r_2(x)}})\Delta v = g_1(x,u,v)|u|^{\alpha_2(x)}_{L^{q_2(x)}} + g_2(x,u,v)|u|^{\gamma_2(x)}_{L^{s_2(x)}} & \text{a.e in} \quad \Omega,\\ u,v>0 & \text{in} \quad \Omega,\\ u=v=0 & \text{on} \quad \partial\Omega.\end{cases}$$

6. Applications of Theorem 2

In this section we make three applications of Theorem 2. From now on we denote by $e \in H^1_0(\Omega)\cap C^{2,\tau}(\overline{\Omega})$ for some $0<\tau<1$, the unique positive solution of problem $-\Delta e = 1$ in Ω, $e = 0$ on $\partial\Omega$ and by $\varphi_1 \in H^1_0(\Omega)\bigcap C^{2,\alpha}(\overline{\Omega})$ a positive eigenfunction associated to the first eigenvalue λ_1 of $(-\Delta, H^1_0(\Omega))$.

6.1. **The sublinear system.** In this subsection we study

$$(Ss) \qquad \begin{cases} -\mathcal{A}(x,|v|_{L^{r_1(x)}})\Delta u = (u^{\beta_1(x)} + v^{\gamma_1(x)})|v|^{\alpha_1(x)}_{L^{q_1(x)}} & \text{in} \quad \Omega,\\ -\mathcal{A}(x,|u|_{L^{r_2(x)}})\Delta v = (u^{\beta_2(x)} + v^{\gamma_2(x)})|u|^{\alpha_2(x)}_{L^{q_2(x)}} & \text{in} \quad \Omega,\\ u=v=0 & \text{on} \quad \partial\Omega.\end{cases}$$

The main result is:

Theorem 6. *Suppose that $r_i(x), q_i(x) \in C_+(\overline{\Omega})$ and $0 \leq \alpha_i(x), \beta_i(x), \gamma_i(x) \in C^0(\overline{\Omega})$ such that*

$$0 < \alpha_i^+ + \beta_i^+, \alpha_i^+ + \gamma_i^+ < 1, \ \ i = 1,2.$$

Suppose also $\mathcal{A} : \overline{\Omega}\times[0,\infty) \to \mathbb{R}$ is continuous and satisfies one of these two conditions:
(A_1) There is a constant $a_0 > 0$, such that

$$\mathcal{A}(x,t) \geq a_0 > 0 \ \text{ in } \ \overline{\Omega}\times[0,\infty).$$

(A_2) *There are constants* $a_1, a_\infty > 0$, *such that*

$$\mathcal{A}(x,0) = 0 < \mathcal{A}(x,t) \leq a_1 \ \ in \ \overline{\Omega} \times (0,\infty)$$

and

$$\lim_{t\to\infty} \mathcal{A}(x,t) = a_\infty \ \ uniformly\ in\ \overline{\Omega}.$$

Then (Ss) *has a weak positive solution* (u,v).

Proof. Let us assume first that (A_1) is true. We are going to construct $(\overline{u},\overline{v})$. Since the functions

$$\begin{array}{rcl} |e|^{\alpha_i(.)}_{L^{q_i(x)}},\ |e|^{\beta_i(.)}_{L^\infty},\ |e|^{\gamma_i(.)}_{L^\infty}: & \overline{\Omega} & \to \mathbb{R}^+ \\ & x & \mapsto |e|^{\alpha_i(x)}_{L^{q_i(x)}},\ |e|^{\beta_i(x)}_{L^\infty},\ |e|^{\gamma_i(.)}_{L^\infty} \end{array}$$

are continuous, there are $C_1, C_2, C_3 > 0$, such that

$$|e|^{\alpha_i(x)}_{L^{q_i(x)}} \leq C_1,\ \ |e|^{\beta_i(x)}_{L^\infty} \leq C_2 \ \ \text{and} \ \ |e|^{\gamma_i(x)}_{L^\infty} \leq C_3,\ \ \forall\ x \in \overline{\Omega},\ i=1,2.$$

Recalling that $0 < \alpha_i^+ + \beta_i^+, \alpha_i^+ + \gamma_i^+ < 1$, with $i=1,2$, we have

$$\lim_{R\to\infty} R^{\alpha_i+\beta_i^+-1} = 0 = \lim_{R\to\infty} R^{\alpha_i+\gamma_i^+-1}, i=1,2.$$

Hence, we can choose $R>0$, sufficiently large such that

$$\begin{cases} 1 \geq \dfrac{C_1C_2}{a_0} R^{\alpha_1+\beta_1^+-1} + \dfrac{C_1C_3}{a_0} R^{\alpha_1+\gamma_1^+-1}, \\ \\ 1 \geq \dfrac{C_1C_2}{a_0} R^{\alpha_2+\beta_2^+-1} + \dfrac{C_1C_3}{a_0} R^{\alpha_2+\gamma_2^+-1}. \end{cases}$$

Now, for each $w \in L^\infty(\Omega)$, we get

$$\begin{cases} R \geq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r_1(x)}})}\big((Re)^{\beta_1(x)} + (Re)^{\gamma_1(x)}\big)|Re|^{\alpha_1(x)}_{L^{q_1(x)}} \ \ \text{in}\ \ \Omega, \\ \\ R \geq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r_2(x)}})}\big((Re)^{\beta_2(x)} + (Re)^{\gamma_2(x)}\big)|Re|^{\alpha_2(x)}_{L^{q_2(x)}} \ \ \text{in}\ \ \Omega. \end{cases}$$

Considering $\overline{u} = Re$ and $\overline{v} = Re$, we derive

$$\begin{cases} -\Delta\overline{u} \geq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r_1(x)}})}(\overline{u}^{\beta_1(x)} + w^{\gamma_1(x)})|\overline{v}|^{\alpha_1(x)}_{L^{q_1(x)}} \ \ \text{in}\ \ \Omega,\ \ \forall\ w \in [0,\overline{v}], \\ \\ -\Delta\overline{v} \geq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r_2(x)}})}(w^{\beta_2(x)} + \overline{v}^{\gamma_2(x)})|\overline{u}|^{\alpha_2(x)}_{L^{q_2(x)}} \ \ \text{in}\ \ \Omega,\ \ \forall\ w \in [0,\overline{u}], \\ \hfill \overline{u},\overline{v} > 0 \ \ \text{in}\ \ \Omega, \\ \hfill \overline{u} = \overline{v} = 0 \ \ \text{on}\ \ \partial\Omega. \end{cases}$$

We are going to construct $(\underline{u},\underline{v})$. Using the notation of Theorem 2, we have, $\overline{w} = Re$, where $\overline{w} = \max\{\overline{u},\overline{v}\}$. Let $K = \max\big\{\mathcal{A}(x,t) : (x,t) \in \overline{\Omega}\times[0,\ \overline{\sigma}]\big\}$, where $\overline{\sigma} = \max\{|\overline{w}|_{L^{r_i(x)}} : i = 1,2\}$. Given $w \in [0, \overline{w} := Re]$, we get $|w|_{L^{r_i(x)}} \leq \overline{\sigma}$ and conclude,

$$a_0 \leq \mathcal{A}(x,|w|_{L^{r_i(x)}}) \leq K \ \ \text{in}\ \Omega,\ \forall\ w \in [0,\overline{w}],\ \ i=1,2.$$

Let $\varphi_1 > 0$ in Ω, $|\varphi_1|_{L^\infty} \leq 1$ and $|\varphi_1|_{L^{q(x)}} \leq 1$. Considering

$$0 < \epsilon \leq \min \left\{ \left(\frac{|\varphi_1|^{\alpha_1^+}_{L^{q_1(x)}}}{\lambda_1 |\varphi_1|^{1-\beta_1^+}_{L^\infty} K} \right)^{\frac{1}{1-(\alpha_1^+ + \beta_1^+)}}, \left(\frac{|\varphi_1|^{\alpha_2^+}_{L^{q_2(x)}}}{\lambda_1 |\varphi_1|^{1-\gamma_2^+}_{L^\infty} K} \right)^{\frac{1}{1-(\alpha_2^+ + \gamma_2^+)}}, 1 \right\}; \ i = 1, 2,$$

for each $w \in [0, \overline{w}]$, we have

$$\begin{cases} \lambda_1 \epsilon \varphi_1(x) \leq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r_1(x)}})} (\epsilon\varphi_1(x))^{\beta_1(x)} |\epsilon\varphi_1|^{\alpha_1(x)}_{L^{q_1(x)}} \ \text{ in } \Omega, \\ \\ \lambda_1 \epsilon \varphi_1(x) \leq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r_2(x)}})} (\epsilon\varphi_1(x))^{\gamma_2(x)} |\epsilon\varphi_1|^{\alpha_2(x)}_{L^{q_2(x)}} \ \text{ in } \Omega. \end{cases}$$

Taking $\underline{u} = \epsilon\varphi_1$ and $\underline{v} = \epsilon\varphi_1$, we get

$$\begin{cases} -\Delta \underline{u} \leq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r_1(x)}})} (\underline{u}^{\beta_1(x)} + w^{\gamma_1(x)}) |\underline{v}|^{\alpha_1(x)}_{L^{q_1(x)}} \ \text{ in } \Omega, \ \ \forall\, w \in [0, \overline{v}], \\ \\ -\Delta \underline{v} \leq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r_2(x)}})} (w^{\beta_2(x)} + \underline{v}^{\gamma_2(x)}) |\underline{u}|^{\alpha_2(x)}_{L^{q_2(x)}} \ \text{ in } \Omega, \ \ \forall\, w \in [0, \overline{u}], \\ \hfill \underline{u}, \underline{v} > 0 \ \text{ in } \Omega, \\ \hfill \underline{u}, \underline{v} = 0 \ \text{ on } \partial\Omega. \end{cases}$$

We are going to prove $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$. For $\epsilon > 0$ sufficiently small, we derive $\lambda_1 \epsilon |\varphi_1|_{L^\infty} \leq R$. Then,

$$-\Delta(\epsilon\varphi_1) \leq -\Delta(Re)$$

and by the Comparison Principle, we obtain

$$\underline{u} := \epsilon\varphi_1 \leq Re =: \overline{u} \ \text{ and } \ \underline{v} := \epsilon\varphi_1 \leq Re =: \overline{v}.$$

Hence, $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are a sub-supersolution to (Ss). From Theorem 2, there is a weak positive solution (u, v) to (Ss) with

$$\underline{u} \leq u \leq \overline{u} \ \text{ and } \ \underline{v} \leq v \leq \overline{v},$$

with finishes the proof of the case (A_1). Now we consider the hypothesis (A_2) and we are going to construct $(\underline{u}, \underline{v})$. Considering

$$0 < \epsilon \leq \min \left\{ \left(\frac{|\varphi_1|^{\alpha_1^+}_{L^{q_1(x)}}}{\lambda_1 |\varphi_1|^{1-\beta_1^+}_{L^\infty} a_1} \right)^{\frac{1}{1-(\alpha_1^+ + \beta_1^+)}}, \left(\frac{|\varphi_1|^{\alpha_2^+}_{L^{q_2(x)}}}{\lambda_1 |\varphi_1|^{1-\gamma_2^+}_{L^\infty} a_1} \right)^{\frac{1}{1-(\alpha_2^+ + \gamma_2^+)}}, 1 \right\}; \ i = 1, 2$$

and defining $\underline{u} = \epsilon\varphi_1$ and $\underline{v} = \epsilon\varphi_1$, we have

$$\begin{cases} -\Delta \underline{u} \leq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r_1(x)}})} (\underline{u}^{\beta_1(x)} + w^{\gamma_1(x)}) |\underline{v}|^{\alpha_1(x)}_{L^{q_1(x)}} \ \text{ in } \Omega, \ \ \forall\, w \in [\underline{v}, \infty), \\ \\ -\Delta \underline{v} \leq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r_2(x)}})} (w^{\beta_1(x)} + \underline{v}^{\gamma_2(x)}) |\underline{u}|^{\alpha_2(x)}_{L^{q_2(x)}} \ \text{ in } \Omega, \ \ \forall\, w \in [\underline{u}, \infty), \\ \hfill \underline{u}, \underline{v} > 0 \ \text{ in } \Omega, \\ \hfill \underline{u}, \underline{v} = 0 \ \text{ on } \partial\Omega. \end{cases}$$

We are going to construct $(\overline{u},\overline{v})$. Since $\lim_{t\to\infty}\mathcal{A}(x,t)=a_\infty$ uniformly in $\overline{\Omega}$, we have

$$k:=\min\big\{\mathcal{A}(x,t):\ (x,t)\in\overline{\Omega}\times[\underline{\sigma},\infty)\big\}>0,$$

where $\underline{\sigma}:=\min\{|\underline{w}|_{r_i}: i=1,2\}$ and $\underline{w}=\epsilon\varphi_1$. Choosing $R>0$ such that

$$\begin{cases} 1\geq \dfrac{C_1C_2}{k}R^{\alpha_1+\beta_1^+-1}+\dfrac{C_1C_3}{k}R^{\alpha_1+\gamma_1^+-1},\\[2ex] 1\geq \dfrac{C_1C_2}{k}R^{\alpha_2+\beta_2^+-1}+\dfrac{C_1C_3}{k}R^{\alpha_2+\gamma_2^+-1},\end{cases}$$

and defining $\overline{u}=\overline{v}:=Re$, where the constants C_1, C_2 and C_3 are as in the first case, we get

$$\begin{cases} -\Delta\overline{u}\geq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r_1(x)}})}(\overline{u}^{\beta_1(x)}+w^{\gamma_1(x)})|\overline{v}|^{\alpha_1(x)}_{L^{q_1(x)}} \ \ in\ \ \Omega,\ \ \forall\, w\in[\underline{v},\overline{v}]\\[2ex] -\Delta\overline{v}\geq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r_2(x)}})}(w^{\beta_2(x)}+\overline{v}^{\gamma_2(x)})|\overline{u}|^{\alpha_2(x)}_{L^{q_2(x)}} \ \ in\ \ \Omega,\ \ \forall\, w\in[\underline{u},\overline{u}]\\ \overline{u},\overline{v}>0\ \text{ in }\ \Omega,\\ \overline{u},\overline{v}=0\ \text{ on }\ \partial\Omega.\end{cases}$$

Now we are going to prove that $\underline{u}\leq\overline{u}$ and $\underline{v}\leq\overline{v}$. In fact, taking $R>0$ sufficiently large, such that $\lambda_1\epsilon|\varphi_1|_{L^\infty}\leq R$, we have

$$-\Delta(\epsilon\varphi_1)\leq-\Delta(Re).$$

By the Comparison Principle,

$$\underline{u}:=\epsilon\varphi_1\leq Re=:\overline{u}\ \text{ and }\ \underline{v}:=\epsilon\varphi_1\leq Re=:\overline{v}.$$

Then, $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ are a sub-supersolution to (Ss). From Theorem 2, there is a weak positive solution (u,v) to (Ss) with

$$\underline{u}\leq u\leq\overline{u}\ \text{ and }\ \underline{v}\leq v\leq\overline{v},$$

which finishes the proof. □

6.2. **Concave and convex system.** In this subsection we study

$$(S)_{\lambda,\mu}\qquad \begin{cases} -\mathcal{A}(x,|v|_{L^{r_1(x)}})\Delta u=\lambda u^{\beta_1(x)-1}u|v|^{\alpha_1(x)}_{L^{q_1(x)}}+\mu v^{\eta_1(x)-1}v|v|^{\gamma_1(x)}_{L^{s_1(x)}}\ \text{ in }\ \Omega,\\ -\mathcal{A}(x,|u|_{L^{r_2(x)}})\Delta v=\lambda v^{\beta_2(x)-1}v|u|^{\alpha_2(x)}_{L^{q_2(x)}}+\mu u^{\eta_2(x)-1}u|u|^{\gamma_2(x)}_{L^{s_2(x)}}\ \text{ in }\ \Omega,\\ u,v=0\ \text{ on }\ \partial\Omega.\end{cases}$$

The main result is:

Theorem 7. *Suppose $r_i(x)$, $q_i(x)$, $s_i(x)\in C_+(\overline{\Omega})$ and $0\leq\alpha_i(x)$, $\gamma_i(x)$, $\beta_i(x)$, $\eta_i(x)\in C^0(\overline{\Omega})$, such that*

$$0<\alpha_i^-+\beta_i^-\leq\alpha_i^++\beta_i^+<1.$$

Suppose also that $\mathcal{A}:\overline{\Omega}\times[0,\infty)\to\mathbb{R}$ is continuous and satisfies one of these two conditions:
(A_1) Suppose that $1<\eta_i^-+\gamma_i^-$ and that there are constants $a_0,b_0>0$ such that

$$\mathcal{A}(x,t)\geq a_0>0\ \ in\ \ \overline{\Omega}\times[0,b_0].$$

Then, given $\mu > 0$, there exists $\lambda_0 > 0$ such that, for each $\lambda \in (0, \lambda_0)$, $(S)_{\lambda,\mu}$ has a weak positive solution $(u_{\lambda,\mu}, v_{\lambda,\mu})$.
(A_2) *Suppose that $1 < (\eta_1^+ + \gamma_1^+)(\eta_2^+ + \gamma_2^+)$ and there are constants $a_1, a_\infty > 0$ such that*

$$\mathcal{A}(x,0) = 0 < \mathcal{A}(x,t) \le a_1 \ \ in \ \ \overline{\Omega} \times (0,\infty)$$

and

$$\lim_{t\to\infty} \mathcal{A}(x,t) = a_\infty \ \ uniformly\ in \ \ \overline{\Omega}.$$

Then, given $\lambda > 0$, there exists $\mu_0 > 0$ such that, for each, $\mu \in (0, \mu_0)$, $(S)_{\lambda,\mu}$ has a weak positive solution $(u_{\lambda,\mu}, v_{\lambda,\mu})$.

Proof. Firstly, we assume (A_1). We are going to construct $(\overline{u}, \overline{v})$. Note that, for each $M > 0$, we have

$$-\Delta(Me) = M \ \text{ in } \ \Omega.$$

We are going to prove that there exists $M > 0$ such that

$$(S)_{a_0} \qquad \begin{cases} M \ge \dfrac{1}{a_0}\left(\lambda(Me)^{\beta_1(x)}|Me|_{L^{q_1(x)}}^{\alpha_1(x)} + \mu(Me)^{\eta_1(x)}|Me|_{L^{s_1(x)}}^{\gamma_1(x)}\right) \text{ in } \Omega, \\ \\ M \ge \dfrac{1}{a_0}\left(\lambda(Me)^{\beta_2(x)}|Me|_{L^{q_2(x)}}^{\alpha_2(x)} + \mu(Me)^{\eta_2(x)}|Me|_{L^{s_2(x)}}^{\gamma_2(x)}\right) \text{ in } \Omega. \end{cases}$$

Considering

$$R = \max\left\{|e|_{L^\infty},\ |e|_{L^{q_i(x)}},\ |e|_{L^{s_i(x)}},\ 1\right\},\ i = 1,2,$$

$\varrho = \min\{(\beta_i^- + \alpha_i^-) : i = 1,2\}$, $\tau = \min\{(\eta_i^- + \gamma_i^-) : i = 1,2\}$ and $p = \max\{(\eta_i^+ + \gamma_i^+) : i = 1,2\}$, note that, for

$$M \ge \frac{1}{a_0}\left(\lambda M^\varrho R^p + \mu M^\tau R^p\right) \quad \text{and} \quad 0 < M \le 1,$$

the constant M is a solution of $(S)_{a_0}$. But this inequality is equivalent to

$$1 \ge \frac{1}{a_0}\left(\lambda M^{\varrho-1} R^p + \mu M^{\tau-1} R^p\right) \text{ and } \quad 0 < M \le 1.$$

Since $0 < \varrho < 1 < \tau$, given $\mu > 0$, there exists $\lambda_0 > 0$ such that, for each $\lambda \in (0, \lambda_0)$, there exists $M_{\lambda,\mu} > 0$, given by

$$M_{\lambda,\mu} = c_1\left(\frac{\lambda}{\mu}\right)^{\frac{1}{\tau-\varrho}},$$

such that

$$0 < M_{\lambda,\mu} \le 1 \ \text{ and } \ 1 \ge \frac{1}{a_0}\left(\lambda M_{\lambda,\mu}^{\varrho-1} R^p + \mu M_{\lambda,\mu}^{\tau-1} R^p\right).$$

Then, $M_{\lambda,\mu}$ is a solution of $(S)_{a_0}$. Since $M_{\lambda,\mu} \to 0$ as $\lambda \to 0$, we can choose $\lambda_0 > 0$ such that

$$\sigma_0 := \max\{|M_{\lambda_0,\mu} e|_{L^{r_i(x)}} : i = 1,2.\} \le b_0.$$

Note also that $\lambda \mapsto M_{\lambda,\mu}$ is increasing, then

$$|M_{\lambda,\mu} e|_{L^{r_i(x)}} \le \sigma_0 \le b_0, \ \ \forall\, \lambda \in (0, \lambda_0).$$

Using

$$\mathcal{A}(x,t) \ge a_0 > 0 \ \text{ in } \ \overline{\Omega} \times [0, b_0],$$

we have

$$\mathcal{A}(x,|w|_{L^{r_i(x)}}) \geq a_0 > 0, \ \ \forall \ \ w \in [0, M_{\lambda,\mu}e].$$

From this relation, the fact that $M_{\lambda,\mu}$ is a solution of $(S)_{a_0}$, and considering $\overline{u} = \overline{v} := M_{\lambda,\mu}e$, we get

$$\begin{cases} -\Delta\overline{u} \geq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r_1(x)}})}\left(\lambda\overline{u}^{\beta_1(x)}|\overline{v}|^{\alpha_1(x)}_{L^{q_1(x)}} + \mu w^{\eta_1(x)}|\overline{v}|^{\gamma_1(x)}_{L^{s_1(x)}}\right) \text{ in } \Omega, \ \ \forall\, w \in [0,\overline{v}], \\ -\Delta\overline{v} \geq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r_2(x)}})}\left(\lambda\overline{v}^{\beta_2(x)}|\overline{u}|^{\alpha_2(x)}_{L^{q_2(x)}} + \mu w^{\eta_2(x)}|\overline{u}|^{\gamma_2(x)}_{L^{s_2(x)}}\right) \text{ in } \Omega, \ \ \forall\, w \in [0,\overline{u}], \\ \overline{u},\overline{v} > 0 \ \text{ in } \ \Omega, \\ \overline{u},\overline{v} = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

We are going to construct $(\underline{u},\underline{v})$. Let $K = \max\left\{\mathcal{A}(x,t) : (x,t) \in \overline{\Omega} \times [0, \ \overline{\sigma}]\right\}$ be a constant, where $\overline{\sigma} = \max\{|\overline{w}|_{L^{r_i(x)}} : i = 1,2\}$ and $\overline{w} = M_{\lambda,\mu}e$. We can suppose $\varphi_1 > 0$ in Ω, $|\varphi_1|_{L^\infty} \leq 1$ and $|\varphi_1|_{L^{q(x)}} \leq 1$. Considering

$$0 < \epsilon \leq \min\left\{\left(\frac{\lambda|\varphi_1|^{\alpha_i^+}_{L^{q_i(x)}}}{\lambda_1|\varphi_1|^{1-\beta_i^+}_{L^\infty}K}\right)^{\frac{1}{1-(\alpha_i^+ + \beta_i^+)}}, 1\right\}; \ i = 1,2,$$

we have, for each $w \in [0, M_{\lambda,\mu}e]$,

$$\begin{cases} \lambda_1\epsilon\varphi_1(x) \leq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r_1(x)}})}(\epsilon\varphi_1(x))^{\beta_1(x)}|\epsilon\varphi_1|^{\alpha_1(x)}_{L^{q_1(x)}} \ \text{ in } \ \Omega, \\ \lambda_1\epsilon\varphi_1(x) \leq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r_2(x)}})}(\epsilon\varphi_1(x))^{\beta_2(x)}|\epsilon\varphi_1|^{\alpha_2(x)}_{L^{q_2(x)}} \ \text{ in } \ \Omega. \end{cases}$$

Taking $\underline{u} = \epsilon\varphi_1$ and $\underline{v} = \epsilon\varphi_1$, we get

$$\begin{cases} -\Delta\underline{u} \leq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r_1(x)}})}\left(\lambda\underline{u}^{\beta_1(x)}|\underline{v}|^{\alpha_1(x)}_{L^{q_1(x)}} + \mu w^{\eta_1(x)}|\underline{v}|^{\gamma_1(x)}_{L^{s_1(x)}}\right) \text{ in } \ \Omega, \ \ \forall\, w \in [0,\overline{v}] \\ -\Delta\underline{v} \leq \dfrac{1}{\mathcal{A}(x,|w|_{L^{r_2(x)}})}\left(\lambda\underline{v}^{\beta_2(x)}|\underline{u}|^{\alpha_2(x)}_{L^{q_2(x)}} + \mu w^{\eta_2(x)}|\underline{v}|^{\gamma_2(x)}_{L^{s_2(x)}}\right) \text{ in } \ \Omega, \ \ \forall\, w \in [0,\overline{u}] \\ \underline{u},\underline{v} > 0 \ \text{ in } \ \Omega, \\ \underline{u},\underline{v} = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

We are going to prove $\underline{u} \leq \overline{u}$ e $\underline{v} \leq \overline{v}$. Given $\lambda \in (0,\lambda_0]$, considering ϵ sufficiently small, we get

$$\lambda_1\epsilon|\varphi_1|_{L^\infty} \leq M_{\lambda,\mu}.$$

Then,

$$-\Delta(\epsilon\varphi_1) \leq -\Delta(M_{\lambda,\mu}e)$$

and by the Comparison Principle we get

$$\underline{u} = \underline{v} := \epsilon\varphi_1 \leq M_{\lambda,\mu}e =: \overline{v} = \overline{u}.$$

Hence, $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ are sub-supersolution to $(S)_{\lambda,\mu}$. By Theorem 2, for each $\lambda \in (0,\lambda_0)$, there is $(u_{\lambda,\mu}, v_{\lambda,\mu})$ a weak positive solution of $(S)_{\lambda,\mu}$ with

$$\underline{u} \leq u_{\lambda,\mu} \leq \overline{u} \ \ \text{ and } \ \ \underline{v} \leq v_{\lambda,\mu} \leq \overline{v}.$$

We consider the hypothesis (A_2). We are going to construct $(\underline{u},\underline{v})$. Consider

$$0<\epsilon\leq\min\left\{\left(\frac{\lambda|\varphi_1|^{\alpha_i^+}_{L^{q_i(x)}}}{\lambda_1|\varphi_1|^{1-\beta_i^+}_{L^\infty}a_1}\right)^{\frac{1}{1-(\alpha^++\beta^+)}},1\right\};\ i=1,2.$$

Taking $\underline{u}=\underline{u}(\lambda):=\epsilon\varphi_1$ and $\underline{v}=\underline{v}(\lambda)=:\epsilon\varphi_1$, we have

$$\begin{cases}-\Delta\underline{u}\leq\dfrac{1}{\mathcal{A}(x,|w|_{L^{r_1(x)}})}\left(\lambda\underline{u}^{\beta_1(x)}|\underline{v}|^{\alpha_1(x)}_{L^{q_1(x)}}+\mu w^{\eta_1(x)}|\underline{v}|^{\gamma_1(x)}_{L^{s_1(x)}}\right)\ in\ \ \Omega,\ \ \forall\, w\in[\underline{v},\infty),\\ -\Delta\underline{v}\leq\dfrac{1}{\mathcal{A}(x,|w|_{L^{r_2(x)}})}\left(\lambda\underline{v}^{\beta_2(x)}|\underline{u}|^{\alpha_2(x)}_{L^{q_2(x)}}+\mu w^{\eta_2(x)}|\underline{v}|^{\gamma_2(x)}_{L^{s_2(x)}}\right)\ in\ \ \Omega,\ \ \forall\, w\in[\underline{u},\infty),\\ \underline{u},\underline{v}>0\ \text{ in }\ \Omega,\\ \underline{u}=\underline{v}=0\ \text{ on }\ \partial\Omega.\end{cases}$$

We are going to construct $(\overline{u},\overline{v})$. Since $\mathcal{A}(x,t)>0$ in $\overline{\Omega}\times[\underline{\sigma},\infty)$ and $\lim\limits_{t\to\infty}\mathcal{A}(x,t)=a_\infty$ uniformly in $\overline{\Omega}$, we have

$$k_\lambda:=\min\big\{\mathcal{A}(x,t):\ (x,t)\in\overline{\Omega}\times[\underline{\sigma},\infty)\big\}>0,$$

where $\underline{\sigma}:=\min\{|\underline{w}|_{r_i}\ :\ i=1,2\}$ and $\underline{w}=\epsilon\varphi_1$. We are going to prove that, for each $w\in[\epsilon\varphi,\infty)$, there exists $T\geq 1$, such that

$$(S)_{a_\infty}\qquad\begin{cases}T\geq\dfrac{1}{\mathcal{A}(x,|w|_{L^{r_1(x)}})}\left(\lambda(Te)^{\beta_1(x)}|Te|^{\alpha_1(x)}_{L^{q_1(x)}}+\mu(Te)^{\eta_1(x)}|Te|^{\gamma_1(x)}_{L^{s_1(x)}}\right)\ \text{in}\ \Omega,\\ T\geq\dfrac{1}{\mathcal{A}(x,|w|_{L^{r_2(x)}})}\left(\lambda(Te)^{\beta_2(x)}|Te|^{\alpha_2(x)}_{L^{q_2(x)}}+\mu(Te)^{\eta_2(x)}|Te|^{\gamma_2(x)}_{L^{s_2(x)}}\right)\ \text{in}\ \Omega.\end{cases}$$

Considering

$$R=\max\Big\{|e|_{L^\infty},\ |e|_{L^{q_i(x)}},\ |e|_{L^{s_i(x)}},\ 1\Big\},\ i=1,2,$$

$\zeta=\max\{(\beta_i^++\alpha_i^+):i=1,2\}$ and $p=\max\{(\eta_i^++\gamma_i^+):i=1,2\}$, for T sufficiently large, system $((S)_{a_\infty})$ has a solution when T satisfies

$$T\geq\frac{1}{k_\lambda}\left(\lambda T^\zeta R^p+\mu T^pR^p\right)\quad\text{and}\quad T\geq 1,$$

which is equivalent to

$$1\geq\frac{1}{k_\lambda}\left(\lambda T^{\zeta-1}R^p+\mu T^{p-1}R^p\right)\quad\text{and}\quad T\geq 1.$$

Since $0<\zeta<1<p$, given $\lambda>0$, there exists $\mu_0>0$, such that, for each $\mu\in(0,\mu_0)$, there exits

$$T_{\lambda,\mu}=c_3\Big(\frac{\lambda}{\mu}\Big)^{\frac{1}{p-\zeta}},$$

such that

$$T_{\lambda,\mu}\geq 1\ \text{ and }\ 1\geq\frac{1}{k_\lambda}\left(\lambda T^{\zeta-1}_{\lambda,\mu}R^p+\mu T^{p-1}_{\lambda,\mu}R^p\right).$$

Moreover, $\mu \mapsto T_{\lambda,\mu},\ \mu \in (0,\mu_0]$ is a decreasing function with $T_{\lambda,\mu} \to \infty$ as $\mu \to 0^+$. Then, $T_{\lambda,\mu}$ is a solution of $(S)_{a_\infty}$ and we can consider $\overline{u} = \overline{v} := T_{\lambda,\mu} e$ because

$$\begin{cases} -\Delta \overline{u} \geq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r_1(x)}})} \left(\lambda \overline{u}^{\beta_1(x)} |\overline{v}|^{\alpha_1(x)}_{L^{q_1(x)}} + \mu w^{\eta_1(x)} |\overline{v}|^{\gamma_1(x)}_{L^{s_1(x)}} \right) \text{ in } \Omega, \ \ \forall\, w \in [\underline{v}, \overline{v}], \\ -\Delta \overline{v} \geq \dfrac{1}{\mathcal{A}(x, |w|_{L^{r_2(x)}})} \left(\lambda \overline{v}^{\beta_2(x)} |\overline{u}|^{\alpha_2(x)}_{L^{q_2(x)}} + \mu w^{\eta_2(x)} |\overline{u}|^{\gamma_2(x)}_{L^{s_2(x)}} \right) \text{ in } \Omega, \ \ \forall\, w \in [\underline{u}, \overline{u}], \\ \hfill \overline{u}, \overline{v} > 0 \ \text{ in } \ \Omega, \\ \hfill \overline{u}, \overline{v} = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

We are going to prove that $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$. Since $T_{\lambda,\mu} \to \infty$ as $\mu \to 0^+$, we can choose $\mu_0 > 0$, such that

$$-\Delta(\epsilon\varphi_1) \leq \lambda_1 \epsilon |\varphi_1|_{L^\infty} \leq T_{\lambda,\mu_0} = -\Delta(T_{\lambda,\mu_0} e).$$

By the Comparison Principle $\epsilon\varphi_1 \leq T_{\lambda,\mu_0} e$. Since $\mu \to T_{\lambda,\mu}$ is decreasing, we get

$$\epsilon\varphi_1 \leq T_{\lambda,\mu_0} e \leq T_{\lambda,\mu} e, \ \ \forall\, \mu \in (0,\mu_0),$$

which implies,

$$\underline{u} := \epsilon\varphi_1 \leq T_{\lambda,\mu} e =: \overline{u} \quad \text{e} \ \ \underline{v} := \epsilon\varphi_1 \leq T_{\lambda,\mu} e =: \overline{v}.$$

Hence, $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ is a sub-supersolution to $(S)_{\lambda,\mu}$ and from Theorem 2, there exists $(u_{\lambda,\mu}, v_{\lambda,\mu})$, a weak positive solution of $(S)_{\lambda,\mu}$ with

$$\underline{u} \leq u_{\lambda,\mu} \leq \overline{u} \ \text{ and } \ \underline{v} \leq v_{\lambda,\mu} \leq \overline{v},$$

which finishes the proof in the case that (A_2) holds. □

6.3. **A generalized classical logistic system.** In this subsection we study the system

$$(S')_{\lambda,\mu} \qquad \begin{cases} -\mathcal{A}(x, |v|_{L^{r_1(x)}}) \Delta u = \lambda f_1(u) |v|^{\alpha_1(x)}_{L^{q_1(x)}} \ \text{ in } \ \Omega, \\ -\mathcal{A}(x, |u|_{L^{r_2(x)}}) \Delta v = \mu f_2(v) |u|^{\alpha_2(x)}_{L^{q_2(x)}} \ \text{ in } \ \Omega, \\ \hfill u = v = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

We assume there are constants $\theta_i > 0$, such that $f_i : [0,\infty) \to \mathbb{R}, i = 1,2$ satisfy

- (f_1) $f_i \in C^0[0,\theta_i]$.
- (f_2) $f_i(0) = f_i(\theta) = 0,\ \ f_i'(0) > 0\ (f_i'(0) \in \mathbb{R}$ or $f_i'(0) = \infty)$ and $f_i(s) > 0\ \forall s \in (0,\theta_i)$, where i=1,2.

The main result in this subsection is:

Theorem 8. *Suppose that $r_i(x), q_i(x) \in C_+(\overline{\Omega})$ $0 \leq \alpha_i(x) \in C^0(\overline{\Omega})$ and $f_i : [0,\infty) \to \mathbb{R}$ satisfy (f_1) and (f_2). Suppose also that $\mathcal{A} : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ is continuous and*

$$\mathcal{A}(x,t) > 0 \ \text{ in } \ \overline{\Omega} \times (0, \overline{\sigma}],$$

where $\overline{\sigma} = \max\{|\theta_i|_{L^{r_i(x)}} : i = 1,2\}$. Then, there are constants $\lambda_0, \mu_0 > 0$ such that, for each $\lambda \geq \lambda_0$ and $\mu \geq \mu_0$, system $(S')_{\lambda,\mu}$ has a weak positive solution $(u_{\lambda,\mu}, v_{\lambda,\mu})$, such that

$$0 < u_{\lambda,\mu} \leq \theta_1 \ \text{ and } \ 0 < v_{\lambda,\mu} \leq \theta_2 \ \text{ in } \ \Omega.$$

Proof. We are going to construct $(\underline{u}, \underline{v})$. Since $f_i(0) = f_i(\theta_i) = 0$ and $f_i'(0) > 0$, by Theorem 5, there are $\eta_0, \nu_0 > 0$ such that, for each $\eta \geq \eta_0$ and $\nu \geq \nu_0$, problems

$$(P)_\eta \qquad \begin{cases} -\Delta u = \eta f_1(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(P)_\nu \qquad \begin{cases} -\Delta v = \nu f_2(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

have solutions φ_η and φ_ν, respectively, such that

$$0 < \varphi_\eta \leq \theta_1 \text{ and } 0 < \varphi_\nu \leq \theta_2 \text{ in } \Omega. \tag{6.1}$$

Taking $\varphi_1 := \varphi_{\eta_0}$ and $\varphi_2 := \varphi_{\nu_0}$ as solutions of $(P)_{\eta_0}$ and $(P)_{\nu_0}$, respectively, we consider $\overline{w} := \max\{\theta_i : i = 1, 2\}$, $\underline{w} := \min\{\varphi_i : i = 1, 2\}$ and $K := \max\left\{\mathcal{A}(x, t) : (x, t) \in \overline{\Omega} \times [\underline{\sigma}, \overline{\sigma}]\right\}$, where $\underline{\sigma} = \min\{|\underline{w}|_{L^{r_i(x)}} : i = 1, 2\}$ and $\overline{\sigma} = \max\{|\overline{w}|_{L^{r_i(x)}} : i = 1, 2\}$. Note that

$$\begin{array}{rccl} |\varphi_1|^{\alpha_2(.)}_{L^{q_2(x)}}, |\varphi_2|^{\alpha_1(.)}_{L^{q_1(x)}} : & \overline{\Omega} & \to & \mathbb{R}^+ \\ & x & \mapsto & |\varphi_1|^{\alpha_2(x)}_{L^{q_2(x)}}, |\varphi_2|^{\alpha_1(x)}_{L^{q_1(x)}} \end{array}$$

are continuous. Then, there is a constant $C > 0$, such that

$$|\varphi_1|^{\alpha_2(x)}_{L^{q_2(x)}}, |\varphi_2|^{\alpha_1(x)}_{L^{q_1(x)}} \geq C \quad \text{in } \overline{\Omega}.$$

Now consider $\tau = \frac{K}{C}$ and note that

$$-\Delta\varphi_1 = \eta_0 f_1(\varphi_1) = \frac{\eta_0 \tau f_1(\varphi_1) |\varphi_2|^{\alpha_1(x)}_{L^{q_1(x)}}}{K} \frac{K}{\tau |\varphi_2|^{\alpha_1(x)}_{L^{q_1(x)}}}.$$

Since $\tau = \frac{K}{C}$, we derive $\dfrac{K}{\tau |\varphi_2|^{\alpha_1(x)}_{L^{q_1(x)}}} \leq 1$, that implies,

$$-\Delta\varphi_1 \leq \frac{\eta_0 \tau f_1(\varphi_1) |\varphi_2|^{\alpha_1(x)}_{L^{q_1(x)}}}{K}.$$

By the definition of K, we have

$$-\Delta\varphi_1 \leq \frac{\eta_0 \tau f_1(\varphi_1) |\varphi_2|^{\alpha_1(x)}_{L^{q_1(x)}}}{\mathcal{A}(x, |w|_{L^{r_1(x)}})}, \quad \forall \ w \in [\underline{w}, \overline{w}]$$

and

$$-\Delta\varphi_2 \leq \frac{\nu_0 \tau f_2(\varphi_2) |\varphi_1|^{\alpha_2(x)}_{L^{q_2(x)}}}{\mathcal{A}(x, |w|_{L^{r_2(x)}})}, \quad \forall \ w \in [\underline{w}, \overline{w}].$$

Defining $\underline{u} := \varphi_1$ and $\underline{u} := \varphi_2$, we get, for each $\lambda \geq \lambda_0$ and for each $\mu \geq \mu_0$,

$$\begin{cases} -\Delta \underline{u} \leq \lambda \dfrac{f_1(\underline{u})|\underline{v}|^{\alpha_1(x)}_{L^{q_1(x)}}}{\mathcal{A}(x, |w|_{L^{r_1(x)}})} \text{ in } \Omega, \ \forall \ w \in [\underline{w}, \overline{w}], \\ -\Delta \underline{v} \leq \mu \dfrac{f_2(\underline{v})|\underline{u}|^{\alpha_2(x)}_{L^{q_2(x)}}}{\mathcal{A}(x, |w|_{L^{r_2(x)}})} \text{ in } \Omega, \ \forall \ w \in [\underline{w}, \overline{w}], \\ \underline{u}, \underline{v} > 0 \text{ in } \Omega, \\ \underline{u}, \underline{v} = 0 \text{ on } \partial\Omega. \end{cases}$$

Now we are going to construct $(\overline{u}, \overline{v})$. Since $f_i(\theta_i) = 0$, the pair $(\overline{u}, \overline{v})$, where $\overline{u} := \theta_1$ and $\overline{v} := \theta_2$, is a supersolution to $(S')_{\lambda,\mu}$, because

$$\begin{cases} -\Delta \overline{u} = 0 = \lambda \frac{f_1(\overline{u})|\overline{v}|^{\alpha_1(x)}_{L^{q_1(x)}}}{\mathcal{A}(x, |w|_{L^{r_1(x)}})} \text{ in } \Omega, \ \forall \ w \in [\underline{w}, \overline{w}], \\ -\Delta \overline{v} = 0 = \mu \frac{f_2(\overline{v})|\overline{u}|^{\alpha_2(x)}_{L^{q_2(x)}}}{\mathcal{A}(x, |w|_{L^{r_2(x)}})} \text{ in } \Omega, \ \forall \ w \in [\underline{w}, \overline{w}], \\ \overline{u}, \overline{v} > 0 \text{ in } \Omega, \\ \overline{u}, \overline{v} > 0 \text{ on } \partial\Omega. \end{cases}$$

We are going to prove that $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$. But

$$\underline{u} := \varphi_1 \leq \theta_1 =: \overline{u} \text{ and } \underline{v} := \varphi_2 \leq \theta_2 =: \overline{v}.$$

Then, $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are sub-supersolution to $(S')_{\lambda,\mu}$ and by Theorem 2, for each $\lambda \geq \lambda_0 := \eta_0\tau$ and for each $\mu \geq \mu_0 := \nu_0\tau$, there is a weak positive solution $(u_{\lambda,\mu}, v_{\lambda,\mu})$ to $(S')_{\lambda,\mu}$ with

$$\varphi_1 \leq u_{\lambda,\mu} \leq \theta_1 \text{ and } \varphi_2 \leq v_{\lambda,\mu} \leq \theta_2 \text{ in } \Omega.$$

□

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