JEPE Vol 2, 2016, p. 217-233

Orthogonal ⊥

A BOUNDARY VALUE PROBLEM FOR A NONLINEAR ELLIPTIC SYSTEM RELEVANT IN GENERAL RELATIVITY

GIOVANNI CIMATTI

ABSTRACT. A theorem of existence and non-existence of solutions for a boundary value problem for the equations of axially symmetric gravitational field in vacuum is given using the inverse function theorem in Banach spaces and the method of functional solutions. Conditions are given under which solutions exist or not exist.

1. Introduction

The Weyl's metric

$$ds^2 = -e^{2\psi}dt^2 + e^{2\gamma-2\psi}d\rho^2 + e^{2\gamma-2\psi}dz^2 + \rho^2 e^{-2\psi}d\varphi^2 \tag{1.1}$$

is one of the best studied in general relativity [9], [1]. The space-time is referred to the Weyl-Lewis-Papapetrou canonical coordinates [5], [6], [7] and [8] (t, ρ, z, φ), hereafter indexed as $(0, 1, 2, 3)$. The potentials ψ and γ are assumed to depend only on ρ and z. Thus (1.1) is compatible only with axially symmetric geometries. In this paper we study the so-called electrovac case. Therefore in the Einstein's equations, corresponding to (1.1),

$$R_{ik} - \frac{1}{2}g_{ik}R = 8\pi T_{ik}, \text{ where } \frac{k}{c^4} = 1 \text{ (k gravitational constant)} \tag{1.2}$$

the stress-energy tensor T_{ik} derives from the antisymmetric electro-magnetic tensor F_{ik} which is taken of the special form

$$\begin{pmatrix} 0 & -\phi_\rho & -\phi_z & 0 \\ \phi_\rho & 0 & 0 & 0 \\ \phi_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.3}$$

where $\phi(\rho, z)$ is the electrostatic potential. In (1.2) R_{ik} is the Ricci tensor, R its trace. Since [4]

$$T_{ik} = \frac{1}{4\pi}\big(F_{ik}F^{l}_{k.} - \frac{1}{2}g_{ik}F_{m\sigma}F^{m\sigma}\big) \tag{1.4}$$

2010 *Mathematics Subject Classification.* 83C10, 83C05.
Key words and phrases. Axially symmetric gravitational fields, boundary value problem, existence and uniqueness of solutions.
Received 04/11/2016, accepted 09/11/2016.

and $F^{ik} = e^{-2\gamma} F_{ik}$, we have, recalling (1.3)

$$T_{00} = \frac{1}{8\pi} e^{2\psi - 2\gamma} \big(\phi_\rho^2 + \phi_z^2\big),\ T_{01} = T_{02} = T_{03} = 0$$

$$T_{10} = 0,\ T_{11} = \frac{1}{8\pi} e^{-2\psi} \big(\phi_\rho^2 - \phi_z^2\big),\ T_{12} = -\frac{1}{4\pi} e^{-2\psi} \phi_\rho \phi_z,\ T_{13} = 0$$

$$T_{20} = 0,\ T_{21} = -\frac{1}{4\pi} e^{-2\psi} \phi_\rho \phi_z,\ T_{22} = \frac{1}{8\pi} e^{-2\psi} \big(\phi_\rho^2 - \phi_z^2\big),\ T_{23} = 0$$

$$T_{30} = 0,\ T_{31} = 0,\ T_{32} = 0,\ T_{33} = 0.$$

Since $T = g^{ik} T_{ik}$ the Einstein equation (1.2) reduces to

$$R_{ik} = 8\pi T_{ik}.$$

For the components of the Ricci tensor we have, in the case of the Weyl's metric,

$$R_{00} = A(\psi) e^{4\psi - 2\gamma},\ R_{01} = 0,\ R_{02} = 0,\ R_{03} = 0$$

$$R_{10} = 0,\ R_{11} = A(\psi) - \gamma_{\rho\rho} - \gamma_{zz} + \frac{1}{\rho}\gamma_\rho - 2\psi_\rho^2,\ R_{12} = -2\psi_\rho \psi_z + \frac{1}{\rho}\gamma_z,\ R_{13} = 0$$

$$R_{20} = 0,\ R_{21} = -2\psi_\rho \psi_z + \frac{1}{\rho}\gamma_z,\ R_{22} = A(\psi) - \gamma_{\rho\rho} - \gamma_{zz} - \frac{1}{\rho}\gamma_\rho - 2\psi_z^2,\ R_{23} = 0$$

$$R_{30} = 0,\ R_{31} = 0,\ R_{32} = 0,\ R_{33} = \rho^2 A(\psi) + e^{-2\gamma},$$

where

$$A(\psi) = \psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz}.$$

The operator A is singular for $\rho = 0$, i.e. on the z axis, and expressed in Cartesian coordinates reads

$$A(\psi) = \frac{1}{x^2 + y^2}\big(x^2 \psi_{xx} + 2xy\psi_{xy} + y^2 \psi_{yy} + x\psi_x + y\psi_y\big) + \psi_{zz}.$$

A has the immediately seen, but important, property of being "a piece" of the Laplace operator. For, we have

$$\Delta\psi = A(\psi) + \frac{1}{\rho^2}\psi_{\varphi\varphi}. \tag{1.5}$$

From the Einstein's equation $R_{33} = 8\pi T_{33}$ or $R_{00} = 8\pi T_{00}$ we obtain the equation

$$A(\psi) = e^{-2\psi}\big(\phi_\rho^2 + \phi_z^2\big). \tag{1.6}$$

and from $R_{11} = 8\pi T_{11}$

$$A(\psi) - \gamma_{\rho\rho} - \gamma_{zz} + \frac{1}{\rho}\gamma_\rho - 2\psi_\rho^2 = e^{-2\psi}\big(\phi_z^2 - \phi_\rho^2\big). \tag{1.7}$$

By difference (1.6) and (1.7) give

$$\gamma_\rho = \rho\big[\psi_\rho^2 - \psi_z^2 - e^{-2\psi}\big(\phi_\rho^2 - \phi_z^2\big)\big].$$

Finally, from $R_{12} = 8\pi T_{12}$ we have

$$\gamma_z = \rho\big(2\psi_\rho\psi_z - 2e^{-2\psi}\phi_\rho\phi_z\big).$$

All the others Einstein's equations are automatically satisfied. The Maxwell's equations, in the present electrovac case, reduce to the vector equation

$$\big(\sqrt{-g}F^{ik}\big)_{,k} = 0, \tag{1.8}$$

where $g = det\big(g_{ik}\big)$. By direct computation we find that the only non-vanishing component in (1.8) is

$$\big(\rho e^{-2\psi}\phi_\rho\big)_\rho + \big(\rho e^{-2\psi}\phi_z\big)_z = 0$$

which, if $\rho > 0$, can be equivalently written as

$$A(\phi) = 2\big(\phi_\rho\psi_\rho + \phi_z\psi_z\big).$$

In the end, we have for the determination of ψ, ϕ and γ the system of partial differential equations

$$A(\psi) = e^{-2\psi}\big(\phi_\rho^2 + \phi_z^2\big) \tag{1.9}$$

$$A(\phi) = 2\big(\phi_\rho\psi_\rho + \phi_z\psi_z\big) \tag{1.10}$$

$$\gamma_\rho = \rho\big[\psi_\rho^2 - \psi_z^2 - e^{-2\psi}\big(\phi_\rho^2 - \phi_z^2\big)\big] \tag{1.11}$$

$$\gamma_z = \rho\big(2\psi_\rho\psi_z - 2e^{-2\psi}\phi_\rho\phi_z\big). \tag{1.12}$$

If ϕ and ψ are obtained from the first two equations (1.9), (1.10) γ can be computed from the remaining equations (1.11), (1.12) by simple integration, since we have

$$\big[\rho\big(\psi_\rho^2 - \psi_z^2 - e^{-2\psi}\phi_\rho^2 + e^{-2\psi}\phi_z^2\big)\big]_z = \big[\rho\big(2\psi_\rho\psi_z - 2e^{-2\psi}\phi_\rho\phi_z\big)\big]_\rho. \tag{1.13}$$

Thus γ is determined apart for an arbitrary constant (which can be determined if γ is known in one point). In this paper we study the system

$$A(\psi) = e^{-2\psi}\big(\phi_\rho^2 + \phi_z^2\big) \tag{1.14}$$

$$A(\phi) = 2\big(\phi_\rho\psi_\rho + \phi_z\psi_z\big) \tag{1.15}$$

in an axially symmetric domain Ω of $\mathbf{R}^3$ with suitable prescribed boundary conditions. Usually it is assumed that Ω is unbounded and that at infinity the metric of the flat space, i.e $ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\varphi^2$ holds. In the last Section we show that this formulation leads in certain cases to a not well-posed boundary value problem. Therefore, we prefer to state the boundary value problem in a bounded subset of $\mathbf{R}^3$. In part of the boundary of Ω we could assume the conditions which correspond to the flat space solution. In a different part of the boundary we prescribe the values of the potentials which are determined by external masses and electric charges.

In Section 2 we prove, using the implicit function theorem in Banach spaces, that for arbitrary sufficiently small axially symmetric boundary data the corresponding boundary value problem has one and only one axially symmetric solution. Section 3 gives a theorem of existence and uniqueness for large, but special data. Section 4 deals with the class of functional solutions.

2. Existence and uniqueness of "small" solutions

Even if the basic equations (1.14) (1.15) have been derived on the assumption that the potentials ψ and ϕ do not depend on the axial variable φ, the problem in itself is three-dimensional and therefore we state it in a bounded, open and axially symmetric subset Ω of $\mathbf{R}^3$ not containing the z axis with a regular boundary Γ. We suppose Ω to be homeomorphic to the region G between two coaxial cylinders of radii R_1 and R_2 with $R_2 > R_1$ and of finite length t $2L$. The part of Γ corresponding to the bottom and upper part of the boundary of G shall be denoted Γ_4 and Γ_3 respectively. Whereas the parts of Γ corresponding to the internal and external parts of Γ are denoted Γ_1 and Γ_2.

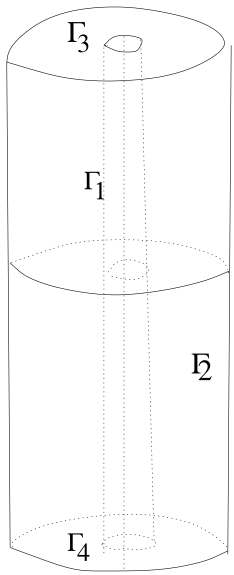

Figure 1. The domain G

We assume that inside the cylinder of radius R_1 a distribution of masses and electric charges exists which determines the axially symmetric value of ψ and ϕ on Γ_1. The space between the two cylinders is free from masses and charges. Taking R_2 much greater of R_1

it is not unreasonable to assume on the lateral surface Γ_2 of the external cylinder the values of ψ pertaining to the flat space solution or, more generally, to arbitrary axially symmetric value of ψ and ϕ determined by an external distributions of masses and charges. On both bases Γ_4 and Γ_3 we assume the vanishing on the normal derivatives of ψ and ϕ in accordance with the expected axial symmetry of the solutions. Therefore, we study the boundary value problem

$$A(\psi) = e^{-2\psi}\big(\phi_\rho^2 + \phi_z^2\big) \text{ in } \Omega \tag{2.1}$$

$$\psi = \psi_1 \text{ on } \Gamma_1, \ \psi = \psi_2 \text{ on } \Gamma_2 \tag{2.2}$$

$$\psi_\rho \nu_1 + \psi_z \nu_2 \text{ on } \Gamma_3 \cup \Gamma_4 \tag{2.3}$$

$$A(\phi) = 2\big(\phi_\rho \psi_\rho + \phi_z \psi_z\big) \text{ in } \Omega \tag{2.4}$$

$$\phi = \phi_1 \text{ on } \Gamma_1, \ \psi = \phi_2 \text{ on } \Gamma_2 \tag{2.5}$$

$$\phi_\rho \nu_1 + \phi_z \nu_2 \text{ on } \Gamma_3 \cup \Gamma_4, \tag{2.6}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is the exterior pointing normal unit vector to $\Gamma_3 \cup \Gamma_4$ and ψ_1, ψ_2, ϕ_1 and ϕ_2 are given axially symmetric $C^{2,\lambda}$ functions. We have

Lemma 1. *If in the problem (2.1)-(2.6) we take $\phi_1 = \phi_2 = 0$* [1] *the solution is unique and it is given by $(\psi, \phi) = (\bar\psi, 0)$, where $\bar\psi$ is the unique solution of the problem*

$$A(\bar\psi) = 0 \quad \text{in } \Omega \tag{2.7}$$

$$\bar\psi = \psi_1 \quad \text{on } \Gamma_1, \ \bar\psi = \psi_2 \quad \text{on } \Gamma_2 \tag{2.8}$$

$$\bar\psi_\rho \nu_1 + \bar\psi_z \nu_2 \ \text{on } \Gamma_3 \cup \Gamma_4. \tag{2.9}$$

Proof. We prove, as a first step, that the solution of problem (2.7)-(2.9) exists and is unique. Let $\bar\psi'$ and $\bar\psi''$ be two solutions and define $w = \bar\psi' - \bar\psi''$. We have

$$\frac{1}{\rho}\big(\rho w_\rho\big)_\rho + w_{zz} = 0 \text{ in } \Omega \tag{2.10}$$

$$w = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \tag{2.11}$$

$$w_\rho \nu_1 + w_z \nu_2 \text{ on } \Gamma_3 \cup \Gamma_4. \tag{2.12}$$

[1] We could with minor changes assume $\phi_1 = \phi_2 = constant$.

Multiplying (2.10) by ρw and integrating by parts over an arbitrary invariant cross-section D of Ω we obtain

$$\int_D \left(w_\rho^2 + w_z^2\right)\rho d\rho dz = 0. \tag{2.13}$$

Since D is arbitrary, we have $w_\rho(\rho, z, \varphi) = 0$, $w_z(\rho, z, \varphi) = 0$ in Ω. Thus w may depend only on φ, but this dependence is excluded in view of the axial symmetry of all the data. Moreover, by (2.11) we have $w = 0$ in Ω. To prove that (2.7)-(2.9) has a solution, we consider the auxiliary problem involving the "full" Laplace operator

$$U_{\rho\rho} + \frac{1}{\rho}U_\rho + U_{zz} + \frac{1}{\rho^2}U_{\varphi\varphi} = 0 \ \text{ in } \Omega \tag{2.14}$$

$$U = \psi_1 \ \text{ on } \Gamma_1, \ U = \psi_2 \ \text{ on } \Gamma_2, \ \frac{\partial U}{\partial \nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4. \tag{2.15}$$

By standard results on the theory of elliptic equation [2] the solution U exists and is unique. On the other hand, if we define $U^{(k)}(\rho, z, \varphi) = U(\rho, z, \varphi + k)$ in view of the uniqueness of the solution of problem (2.14), (2.15) and of the axial symmetry of all the data we have $U^{(k)} = U^{(0)} = U$. Hence U does not depend on φ and solves (2.7)-(2.9).

To prove that problem (2.1)-(2.6) has one and only one solution if $\phi_1 = \phi_2 = 0$, we rewrite (2.4) in the equivalent form [2]

$$\left(\rho e^{-2\psi}\phi_\rho\right)_\rho + \left(\rho e^{-2\psi}\phi_z\right)_z = 0. \tag{2.16}$$

We multiply (2.16) by ρ and integrate by parts over an invariant section D of Ω. Taking into account the boundary conditions we have

$$\int_D \rho e^{-2\psi}\left(\phi_\rho^2 + \phi_z^2\right) d\rho dz = 0.$$

Hence $\phi_\rho(\rho, z, \varphi) = 0$, $\phi_z(\rho, z, \phi) = 0$ in Ω. The possible dependence on ϕ is excluded since $\phi = 0$ on $\Gamma_1 \cup \Gamma_2$. Hence $\phi = 0$ in Ω. Thus from (2.1) we have

$$A(\psi) = 0 \ \text{ in } \Omega$$

$$\psi = \psi_1 \ \text{ on } \Gamma_1, \ \psi = \psi_2 \ \text{ on } \Gamma_2$$

$$\psi_\rho \nu_1 + \psi_z \nu_2 \text{ on } \Gamma_3 \cup \Gamma_4.$$

Hence, by uniqueness, $\psi = \bar{\psi}$. □

In the remaining part of this Section we show that from every solution of the form $(\psi, \phi) = (\bar{\psi}, 0)$ of problem (2.1)-(2.6) originates a branch of solutions of "small" electric potential of the same problem if the boundary data for ϕ is sufficiently small and the boundary data for ψ is sufficiently close to that of $\bar{\psi}$. To this end we use the following form of the implicit theorem in Banach spaces.

[2]Note that (2.15) is fully equivalent to (2.4) only if $\rho > 0$.

Theorem 1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $\mathcal{N}$ a given subset of $\mathcal{X}$, $u^* \in \mathcal{N}$ and $F : \mathcal{N} \to \mathcal{Y}$ with $F \in C^1$. Assume the Frechet's differential $F'(u^*)$ to be invertible. Then there exists a neighbourhood $\mathcal{U}$ of u^* in $\mathcal{X}$ and a neighbourhood $\mathcal{V}$ of $v^* = F(u^*)$ in $\mathcal{X}$ such that F is a diffeomorphism from $\mathcal{U}$ to $\mathcal{V}$.*

In order to overcome the difficulties inherent in the singular character of the operator A we shall consider, as suggested by (1.5), also the "full" problem

$$\Delta\psi = e^{-2\psi}|\nabla\phi|^2 \ \text{ in } \Omega, \qquad |\nabla\phi|^2 = \phi_\rho^2 + \phi_z^2 + \frac{1}{\rho^2}\phi_\varphi^2 \tag{2.17}$$

$$\psi = \psi_1 \ \text{ on } \Gamma_1, \ \psi = \psi_2 \ \text{ on } \Gamma_2 \tag{2.18}$$

$$\frac{\partial\psi}{\partial\nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4 \tag{2.19}$$

$$\Delta\phi = 2\nabla\phi \cdot \nabla\psi \ \text{ in } \Omega \tag{2.20}$$

$$\phi = \phi_1 \ \text{ on } \Gamma_1, \ \phi = \phi_2 \ \text{ on } \Gamma_2 \tag{2.21}$$

$$\frac{\partial\phi}{\partial\nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4. \tag{2.22}$$

We have the following

Lemma 2. *If the problem (2.17)-(2.22) has a unique solution (ψ, ϕ) this solution is axially symmetric and it gives the unique axially symmetric solution of problem (2.1)-(2.6).*

Proof. Let $(\psi(\rho, z, \varphi), \phi(\rho, z, \varphi))$ be the unique solution of problem (2.17)-(2.22). All the geometric data Ω, Γ_1, Γ_2, Γ_3, Γ_4 are axially symmetric and also the boundary data ψ_1, ψ_2, ϕ_1, ϕ_2 do not depend on φ by assumption. Define $\psi^{(k)}(\rho, z, \varphi) = \psi(\rho, z, \varphi + k)$ and $\phi^{(k)}(\rho, z, \varphi) = \phi(\rho, z, \varphi + k)$. In view of the axial symmetry of all the data $(\psi^{(k)}, \phi^{(k)})$ is also a solution of problem (2.17)-(2.22) for every $k \in \mathbf{R}^1$. Therefore (ψ, ϕ) does not depend on φ and is also the unique solution of problem (2.1)-(2.6). □

Theorem 2. *Let $\bar{\psi}$ be the unique solution of the problem*

$$A(\bar{\psi}) = 0 \quad \text{in } \Omega$$

$$\bar{\psi} = \bar{\psi}_1 \quad \text{on } \Gamma_1, \ \bar{\psi} = \bar{\psi}_2 \quad \text{on } \Gamma_2$$

$$\bar{\psi}_\rho\nu_1 + \bar{\psi}_z\nu_2 \ \text{ on } \Gamma_3 \cup \Gamma_4.$$

There exists a constant $\delta > 0$ such that if

$$\|\psi_1 - \bar{\psi}_1\|_{C^{2,\alpha}(\Gamma_1)} \leq \delta, \ \|\psi_2 - \bar{\psi}_2\|_{C^{2,\alpha}(\Gamma_2)} \leq \delta$$

$$\|\phi_1\|_{C^{2,\alpha}(\Gamma_1)} \leq \delta, \ \|\phi_2\|_{C^{2,\alpha}(\Gamma_2)} \leq \delta$$

the problem

$$A(\psi) = e^{-2\psi}\big(\phi_\rho^2 + \phi_z^2\big) \quad in\ \Omega$$

$$\psi = \psi_1 \quad on\ \Gamma_1, \ \psi = \psi_2 \quad on\ \Gamma_2$$

$$\psi_\rho \nu_1 + \psi_z \nu_2 \ \ on\ \Gamma_3 \cup \Gamma_4$$

$$A(\phi) = 2\big(\phi_\rho \psi_\rho + \phi_z \psi_z\big) \quad in\ \Omega$$

$$\phi = \phi_1 \quad on\ \Gamma_1, \ \psi = \phi_2 \quad on\ \Gamma_2$$

$$\phi_\rho \nu_1 + \phi_z \nu_2 \ \ on\ \Gamma_3 \cup \Gamma_4$$

has one and only one axially symmetric solution.

Proof. We start by studying the "full" problem (2.17)-(2.22) which, by Lemma 2.1, has only the trivial solution $(\psi, \phi) = (\bar{\psi}, 0)$ if $\phi_1 = \phi_2 = 0$. Referring to the notations of Theorem 2.2 we define $\mathcal{X} = \mathcal{A} \times \mathcal{A}$ where $\mathcal{A} = \big\{\eta \in C^{2,\alpha}(\bar{\Omega}),\ \frac{\partial \eta}{\partial \nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4\big\}$ and $\mathcal{Y} = \big(\mathcal{B} \times \mathcal{C}_1 \times \mathcal{C}_2\big) \times \big(\mathcal{B} \times \mathcal{C}_1 \times \mathcal{C}_2\big)$ where $\mathcal{B} = C^{0,\alpha}(\bar{\Omega})$, $\mathcal{C}_1 = C^{2,\alpha}(\Gamma_1)$, $\mathcal{C}_2 = C^{2,\alpha}(\Gamma_2)$ and $u^* = (\bar{\psi}, 0)$. Let $F : \mathcal{X} \to \mathcal{Y}$ be defined by

$$F((\psi, \phi)) = \big(\big(\Delta\psi - e^{-2\psi}|\nabla\phi|^2,\ \psi_{|\Gamma_1},\ \psi_{|\Gamma_2}\big),\ \big(\Delta\phi - 2\nabla\phi \cdot \nabla\psi,\ \phi_{|\Gamma_1},\ \phi_{|\Gamma_2}\big)\big).$$ [3]

The differential of F in $(\bar{\psi}, 0)$ is easily computed and it is given by

$$F'((\bar{\psi}, 0))[\Psi, \Phi] = \big(\big(\Delta\Psi,\ \Psi_{|\Gamma_1},\ \Psi_{|\Gamma_2}\big),\ \big(\Delta\Phi - 2\nabla\bar{\psi} \cdot \nabla\Phi,\ \Phi_{|\Gamma_1},\ \Phi_{|\Gamma_2}\big)\big).$$

To prove that F', a linear operator from $\mathcal{X}$ to $\mathcal{Y}$, is invertible, as required by Theorem 2.2, we simply note that the linear elliptic boundary value problem

$$\Delta\Psi = a \ \text{ in } \Omega, \quad \Psi = b \ \text{ on } \Gamma_1,\ \Psi = c \ \text{ on } \Gamma_2, \quad \frac{\partial \Psi}{\partial \nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4$$

$$\Delta\Phi - 2\nabla\bar{\psi} \cdot \nabla\Phi = e \ \text{ in } \Omega, \quad \Phi = f \text{ on } \Gamma_1,\ \Phi = g \ \text{ on } \Gamma_2, \quad \frac{\partial \Phi}{\partial \nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4$$

has, by standard results [2], one and only one solution if $((a, b, c), (e, f, g)) \in \mathcal{Y}$. Thus the "full" problem has a unique small solution which, by uniqueness and by the axial symmetry of all the data, is axially symmetric. By Lemma 2.3 this solution is also the unique axially symmetric solution of the "truncated" problem. □

[3] ψ_{Γ_1} denotes the restriction of ψ to Γ_1

3. Existence and uniqueness of "large" axially symmetric solutions

The results of the previous Section are purely local in nature. Here we prove a theorem of existence, uniqueness and of non-existence of axially symmetric solutions, not necessarily small, but assuming constant boundary data. More precisely we consider the problem

$$A(\psi) = e^{-2\psi}\left(\phi_\rho^2 + \phi_z^2\right) \text{ in } \Omega \tag{3.1}$$

$$\psi = \psi_1 \text{ on } \Gamma_1,\ \psi = \psi_2 \text{ on } \Gamma_2 \tag{3.2}$$

$$\psi_\rho \nu_1 + \psi_z \nu_2 \text{ on } \Gamma_3 \cup \Gamma_4 \tag{3.3}$$

$$\left(\rho e^{-2\psi}\phi_\rho\right)_\rho + \left(\rho e^{-2\psi}\phi_z\right)_z = 0 \text{ in } \Omega \tag{3.4}$$

$$\phi = \phi_1 \text{ on } \Gamma_1,\ \psi = \phi_2 \text{ on } \Gamma_2 \tag{3.5}$$

$$\phi_\rho \nu_1 + \phi_z \nu_2 \text{ on } \Gamma_3 \cup \Gamma_4, \tag{3.6}$$

where ϕ_1, ϕ_2, ψ_1 and ψ_2 are given constants. Together with the "truncated" problem (3.1)-(3.6) we consider the corresponding "full" problem i.e

$$\nabla \cdot \left(e^{-2\psi}\nabla\phi\right) = 0 \text{ in } \Omega \tag{3.7}$$

$$\phi = \phi_1 \text{ on } \Gamma_1,\ \phi = \phi_2 \text{ on } \Gamma_2 \tag{3.8}$$

$$\frac{\partial \phi}{\partial \nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4 \tag{3.9}$$

$$\Delta\psi = e^{-2\psi}|\nabla\phi|^2 \text{ in } \Omega \quad [4] \tag{3.10}$$

$$\psi = \psi_1 \text{ on } \Gamma_1,\ \psi = \psi_2 \text{ on } \Gamma_2 \tag{3.11}$$

$$\frac{\partial \psi}{\partial \nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4, \tag{3.12}$$

[4] It is interesting to note that the equations (3.7) and (3.10) are the Euler's system of the functional $J = \int_\Omega \left(e^{-2\psi}|\nabla\phi|^2 + |\nabla\psi|^2\right) dV$.

again assuming ϕ_1, ϕ_2, ψ_1 and ψ_2 as constants.[5] Since the case $\phi_1 = \phi_2$ is analogous to the case $\phi_1 = \phi_2 = 0$ which as been treated in Section 2, we assume

$$\psi_1 < \psi_2, \quad \phi_1 < \phi_2. \tag{3.13}$$

Two "a priori" estimates for the solutions of problem (3.7)-(3.12) follow from the maximum principle which gives

$$\phi_1 \leq \phi \leq \phi_2 \quad \text{in } \Omega \tag{3.14}$$

$$\psi \leq \psi_2 \quad \text{in } \Omega. \tag{3.15}$$

We want to show that the solutions of problem (3.7)-(3.12) can be expressed in terms of the solutions of the problem

$$\Delta w = 0 \;\text{ in } \Omega, \quad w = 0 \;\text{ on } \Gamma_1, \quad w = w_2 \;\text{ on } \Gamma_2, \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma_3 \cup \Gamma_4 \tag{3.16}$$

where w_2 is a constant suitably chosen. To see heuristically how this reduction of problem (3.7)-(3.12) to problem (3.16) is possible, let us define

$$F(\psi) = \frac{1}{2}\left(e^{2\psi} - e^{2\psi_1}\right) \tag{3.17}$$

and

$$\theta = \frac{\phi^2}{2} - F(\psi). \tag{3.18}$$

If (ψ, ϕ) is a solution of (3.7)-(3.12), we have, from (3.18)

$$e^{-2\psi}\nabla\theta = \phi e^{-2\psi}\nabla\phi - \nabla\psi. \tag{3.19}$$

Taking into account (3.7) and (3.10) we obtain, from (3.19),

$$\nabla \cdot \left(e^{-2\psi}\nabla\theta\right) = 0 \quad \text{in } \Omega \tag{3.20}$$

and by (3.8), (3.9), (3.11), (3.12) and (3.18)

$$\theta = \frac{\phi_1^2}{2} \;\text{ on } \Gamma_1, \; \theta = \frac{\phi_2^2}{2} - F(\psi_2) \;\text{ on } \Gamma_2, \; \frac{\partial\theta}{\partial\nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4. \tag{3.21}$$

The equations (3.8) and (3.21) imply the existence, between θ and ϕ, of a functional relation of the form

[5]There is a curious similarity between problem (3.7)-(3.12) and a problem of electrical heating of conductor. For, if we interpret ψ as the temperature inside the conductor Ω and $\sigma(\psi) = e^{-2\psi}$ as temperature dependent electrical conductivity, the problem of finding the electric potential and the temperature inside the conductor is modelled by the equations $\nabla \cdot (\sigma(\psi)\nabla\phi) = 0$, $-\Delta\psi = \sigma(\psi)|\nabla\phi|^2$. The crucial difference from (3.7), (3.10) is in the sign in the second equation. Thus our original problem would be modeled assuming a density flow of potential of the form $\mathbf{q} = e^{-2\psi}\nabla\phi$.

$$\theta = k_1\phi + k_2 \tag{3.22}$$

since θ and ϕ satisfy the same equation with different, but constant, boundary conditions. The constants k_1 and k_2 are easily computed from the conditions (3.8) and (3.21). We find

$$\theta = \Big[\frac{\phi_1+\phi_2}{2} - \frac{F(\psi_2}{\phi_1-\phi_2}\Big]\phi - \frac{\phi_1\phi_2}{2} + \frac{F(\psi_2)\phi_1}{\phi_2-\phi_1}. \tag{3.23}$$

If we define

$$H(\phi) = \frac{\phi^2}{2} - \frac{1}{2}\big(\phi_1+\phi_2\big)\phi - F(\psi_2)\frac{\phi_1-\phi}{\phi_2-\phi_1} + \frac{\phi_1\phi_2}{2} \tag{3.24}$$

we have from (3.23), (3.18) and (3.24)

$$F(\psi) = H(\phi). \tag{3.25}$$

If (3.25) can be solved with respect to ψ we can write

$$\psi = F^{-1}(H(\phi)). \tag{3.26}$$

Hence the equation $\nabla\cdot(e^{-2\psi}\nabla\phi) = 0$ becomes

$$\nabla\cdot\big(e^{-2F^{-1}(H(\phi))}\nabla\phi\big) = 0$$

to which we add the boundary conditions

$$\phi = \phi_1 \ \text{ on } \Gamma_1, \quad \phi = \phi_2 \ \text{ on } \Gamma_2, \quad \frac{\partial\phi}{\partial\nu} = 0 \text{ on } \Gamma_3\cup\Gamma_4.$$

Thus we succeeded in reducing the original problem to a form to which the Kirchhoff's transformation is applicable. For, if we define

$$w = L(\phi) = \int_{\phi_1}^{\phi} e^{-2F^{-1}(H(t))}dt \tag{3.27}$$

we have by (3.27) and (3.26),

$$\nabla w = e^{-2F^{-1}(H(\phi))}\nabla\phi = e^{-2\psi}\nabla\phi \tag{3.28}$$

and, by (3.7)

$$\Delta w = 0. \tag{3.29}$$

To this equation we add the boundary conditions

$$w = 0 \ \text{ on } \Gamma_1, \quad w = L(\phi_2) \ \text{ on } \Gamma_2, \quad \frac{\partial w}{\partial\nu} = 0 \text{ on } \Gamma_3\cup\Gamma_4. \tag{3.30}$$

Thus, if (3.26) and (3.27) are invertible, we obtain as solution of problem (3.7)-(3.12)

$$(\psi(\mathbf{x}), \phi(\mathbf{x})) = (F^{-1}(H(\phi(\mathbf{x})), L^{-1}(w(\mathbf{x})))). \tag{3.31}$$

To validate this procedure we must ascertain under which conditions the inverse functions involved do really exist. This is done in the following

Theorem 3. *Let ψ_1, ψ_2, ϕ_1 and ϕ_2 be given constants with $\psi_2 > \psi_1$ and $\phi_2 > \phi_1$. Define*

$$F(\psi) = \frac{1}{2}\big(e^{2\psi} - e^{2\psi_1}\big). \tag{3.32}$$

If

$$F(\psi_2) \geq \frac{1}{2}\big(\phi_2 - \phi_1\big)^2 \tag{3.33}$$

the problem (3.1)-(3.6) has one and only one solution. If

$$F(\psi_2) < \frac{1}{2}\big(\phi_2 - \phi_1\big)^2 \tag{3.34}$$

the problem (3.1)-(3.6) has no solution.

Proof. We prove, as a first step, the result for the "full" problem associated with (3.1)-(3.6), i.e.

$$\nabla \cdot \big(e^{-2\psi}\nabla\phi\big) = 0 \ \text{ in } \Omega \tag{3.35}$$

$$\phi = \phi_1 \ \text{ on } \Gamma_1, \ \phi = \phi_2 \ \text{ on } \Gamma_2 \tag{3.36}$$

$$\frac{\partial\phi}{\partial\nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4 \tag{3.37}$$

$$\Delta\psi = e^{-2\psi}|\nabla\phi|^2 \ \text{ in } \Omega \tag{3.38}$$

$$\psi = \psi_1 \ \text{ on } \Gamma_1, \ \psi = \psi_2 \ \text{ on } \Gamma_2 \tag{3.39}$$

$$\frac{\partial\phi}{\partial\nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4. \tag{3.40}$$

Define

$$\theta = \frac{\phi^2}{2} - F(\psi) \tag{3.41}$$

and the parabola

$$H(\phi) = \frac{1}{2}\phi^2 - \frac{1}{2}\big(\phi_1 + \phi_2\big)\phi - F(\psi_2)\frac{\phi_1 - \phi}{\phi_2 - \phi_1} + \frac{1}{2}\phi_1\phi_2. \tag{3.42}$$

We have $H(\phi_1) = 0$, $H(\phi_2) = F(\psi_2)$. Since $\frac{dH}{d\phi}(\phi_1) = 0$ if $F(\psi_2) = \frac{1}{2}\big(\phi_2 - \phi_1\big)^2$, we conclude that, if

$$F(\psi_2) \geq \frac{1}{2}\big(\phi_2 - \phi_1\big)^2, \tag{3.43}$$

we have $H(\phi) > 0$ in $\big(\phi_1, \phi_2]$ and $0 \leq H(\phi) \leq F(\psi_2)$ in $[\phi_1, \phi_2]$. Therefore, the inverse function $\psi = F^{-1}(H(\phi))$, when $\phi \in [\phi_1, \phi_2]$, is well-defined and we have the functional relation

$$F(\psi) = H(\phi). \tag{3.44}$$

Let us consider the mixed problem for the laplacian

$$\Delta w = 0, \quad w = 0 \text{ on } \Gamma_1, \quad w = L(\phi_2) \text{ on } \Gamma_2, \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4, \tag{3.45}$$

where

$$L(\phi) = \int_{\phi_1}^{\phi} e^{-2F^{-1}(H(t))} dt. \tag{3.46}$$

By the maximum principle applied to (3.45) we have

$$0 \leq w(\mathbf{x}) \leq L(\phi_2) \quad \text{in } \Omega. \tag{3.47}$$

Therefore, $w = L(\phi)$ defines a one-to-one mapping from $[\phi_1, \phi_2]$ onto $[0, L(\phi_2]$. By (3.47) the functions

$$\phi(\mathbf{x}) = L^{-1}(w(\mathbf{x})), \quad \psi(\mathbf{x}) = F^{-1}(H(\phi(\mathbf{x}))) \tag{3.48}$$

are well-defined. We prove that they give a solution to problem (3.35)-(3.40). Since $\nabla w = e^{-2\psi}\nabla\phi$, by (3.45) we have

$$\nabla \cdot \big(e^{-2\psi}\nabla\phi\big) = 0. \tag{3.49}$$

Moreover, $\phi(\mathbf{x})$ satisfies the required boundary conditions. For,

$$\phi = L^{-1}(0) = \phi_1 \text{ on } \Gamma_1, \quad \phi = L^{-1}(L(\phi_2) = \phi_2 \text{ on } \Gamma_2, \quad \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4.$$

By (3.44) and (3.42), recalling (3.41), we obtain the following functional relation between ϕ and θ

$$\theta = \Big[\frac{\phi_1 + \phi_2}{2} - \frac{F(\psi_2}{\phi_2 - \phi_1}\Big]\phi + \frac{F(\psi_2)\phi_1}{\phi_2 - \phi_1} - \frac{\phi_1\phi_2}{2}.$$

Hence

$$\nabla\theta = \Big[\frac{\phi_1 + \phi_2}{2} - \frac{F(\psi_2}{\phi_2 - \phi_1}\Big]\nabla\phi. \tag{3.50}$$

Multiplying (3.50) by $e^{-2\psi}$ and recalling (3.49) we obtain

$$\nabla \cdot \left(e^{-2\psi} \nabla \theta\right) = 0. \tag{3.51}$$

On the other hand, from (3.41) we have, after multiplication by $e^{-2\psi}$

$$e^{-2\psi} \nabla \theta = e^{-2\psi} \phi \nabla \phi - \nabla \psi \tag{3.52}$$

and, by (3.51) and (3.49), finally we get

$$\Delta \psi = e^{-2\psi} |\nabla \phi|^2 \quad \text{in } \Omega. \tag{3.53}$$

Moreover, ψ satisfies the correct boundary conditions, in fact

$$\psi = F^{-1}(H(\phi_1)) = F^{-1}(0) = \psi_1 \quad \text{on } \Gamma_1$$

$$\psi = F^{-1}(H(\phi_2)) = F^{-1}(F(\psi_2)) = \psi_2 \quad \text{on } \Gamma_2$$

$$\frac{\partial \psi}{\partial \nu} = 0 \text{ on } \Gamma_3 \cup \Gamma_4.$$

We conclude that (3.48) gives a solution to the "full" problem. We claim that this solution is in fact the only solution. By contradiction, let (ψ', ϕ') be a second solution. Proceeding exactly as before, we find between ψ' and ϕ' the functional relation

$$F(\psi') = H(\phi').$$

Thus, if $w(\mathbf{x})$ is the (unique) solution of problem (3.45) we have

$$\phi'(\mathbf{x}) = L^{-1}(w(\mathbf{x})) = \phi(\mathbf{x})$$

and

$$\psi'(\mathbf{x}) = F^{-1}(H(\phi'(\mathbf{x})) = F^{-1}(H(L^{-1}(w(\mathbf{x})))) = \psi(\mathbf{x}).$$

Thus the solution of the "full" problem exists and is unique and in view of the axial symmetry of all data it is axially symmetric. Hence by Lemma 2.3 it is also the unique axially symmetric solution of the truncated problem.

We prove now that if (3.34) holds the "full" problem has no solution. Suppose the contrary and let (ψ, ϕ) be a solution. We have again the functional relation

$$F(\psi) = H(\phi),$$

but this time the point of minimum $\tilde{\phi}$ of the parabola $H(\phi)$ belongs to the interval (ϕ_1, ϕ_2) and $H(\tilde{\phi}) < 0$. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be arbitrary points of Γ_1 and Γ_2 respectively. If $\mathbf{x} = \tilde{\mathbf{x}}(t)$ is a curve connecting $\mathbf{x}_1$ and $\mathbf{x}_2$, the function $\phi(\tilde{\mathbf{x}}(t))$ takes all the value between ϕ_1 and ϕ_2. Thus there exists t^* such that $\phi(\tilde{\mathbf{x}}(t^*)) = \tilde{\phi}$ and

$$0 \leq F(\psi(\tilde{\mathbf{x}}(t^*))) = H(\phi(\tilde{\mathbf{x}}(t^*))) = H(\tilde{\phi}) < 0.$$

A contradiction. Therefore the "full" problem and, by the usual argument, also the truncated problem have no solution if (3.34) holds. □

Remark 1. Theorem 3.1 gives also a further estimate on $\psi(\mathbf{x})$ in addition to (3.15) i.e.

$$\psi \geq \psi_1 \quad \text{in } \Omega, \tag{3.54}$$

which does not follow from the maximum principle.

4. THE ORDINARY DIFFERENTIAL EQUATIONS OF THE FUNCTIONAL SOLUTIONS

The functional relation (3.25) can also be seen as the first integral of two ordinary differential equations which form the object of the present Section, where the point of view of the functional solutions is adopted in the sense of the following definition.

Definition 1. We say that $(\psi(\rho, z, \varphi), \phi(\rho, z, \varphi))$ is a functional solution of the system of partial equations

$$\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz} = e^{-2\psi}\left(\phi_\rho^2 + \phi_z^2\right) \tag{4.1}$$

$$\phi_{\rho\rho} + \frac{1}{\rho}\phi_\rho + \phi_{zz} = 2\left(\phi_r\psi_\rho + \phi_z\psi_z\right) \tag{4.2}$$

if a regular function $\psi = \Psi(\phi)$ exists such that

$$\psi(\rho, z, \phi) = \Psi(\phi(\rho, z, \varphi)) \tag{4.3}$$

or, as an alternative, if there is a function $\phi = \Phi(\psi)$ such that

$$\phi(\rho, z, \phi) = \Phi(\psi(\rho, z, \varphi)). \tag{4.4}$$

Since the (4.3) or (4.4) hold up to the boundary, functional solutions are useful to study the system (4.1), (4.2) only with special boundary conditions. This is what has been done in the previous Section. On the positive side, we have the fact that the functions $\Phi(\psi)$ or $\Psi(\phi)$ entering in the definition above can be explicitly computed as solutions of two ordinary differential equations. In fact, we have

Lemma 3. *If (ψ, ϕ) is a functional solution of (4.1), (4.2) such that*

$$\phi_\rho^2 + \phi_z^2 \neq 0 \tag{4.5}$$

the function $\Psi(\phi)$ of Definition 4.1 is a solution of the autonomous differential equation

$$\frac{d^2\Psi}{d\phi^2} + 2\Big(\frac{d\Psi}{d\phi}\Big)^2 = e^{-2\Psi}. \tag{4.6}$$

If $\psi_\rho^2 + \psi_z^2 \neq 0$ the function $\Phi(\psi)$ is a solution of the Riccati equation

$$\frac{d^2\Phi}{d\psi^2} + e^{-2\psi}\left(\frac{d\Phi}{d\psi}\right)^3 - 2\frac{d\Phi}{d\psi} = 0. \tag{4.7}$$

Proof. We consider the first case. We have $\psi_\rho = \Psi'(\phi)\phi_\rho$, $\psi_z = \Psi'(\phi)\phi_z$, $\psi_{\rho\rho} = \Psi''(\phi)\phi_\rho^2 + \Psi'(\phi)\phi_{\rho\rho}$, $\psi_{zz} = \Psi''(\phi)\phi_z^2 + \Psi'(\phi)\phi_{zz}$. Substituting in (4.1) we have

$$\Psi''\big(\phi_\rho^2+\phi_z^2\big)+\Psi'\big(\phi_{\rho\rho}+\frac{1}{\rho}\phi_\rho+\phi_{zz}\big)= e^{-2\psi}\big(\phi_\rho^2+\phi_z^2\big).$$

Using (4.2) and recalling (4.5) we obtain (4.6). The Riccati equation for the determination of $\Phi(\psi)$ is obtained in a similar manner. □

The equations (4.6) and (4.7) have the same first integral, see [3]

$$e^{2\psi} = \phi^2 - 2C\phi + B. \tag{4.8}$$

If we solve (4.8) with respect to ψ we find the solutions of (4.6) and if we solve (4.8) with respect to ϕ we find the solutions of (4.7).

5. Remark on the Exterior Boundary Value Problem

At first sight it may appear more natural to state the problem (2.1)-(2.6) not in a bounded domain, but as an exterior Dirichlet's problem, prescribing at infinity the condition on ψ pertaining to the flat space solution which corresponds to the metric of flat space.This would imply the boundary condition

$$\lim_{\rho\to\infty} \psi = 0 \text{ uniformly with respect to z.}$$

Similarly one would like to assume on the electric potential

$$\lim_{\rho\to\infty} \phi = 0 \text{ uniformly with respect to z.}$$

The corresponding boundary value problem, however, in general has no solutions, at least in the class of functional solutions as shown in the following example. Let us take

$$\Omega = \{(\rho, z, \varphi);\ \rho > 1,\ |z| < \infty,\ 0 < \varphi \le 2\pi\}$$

and

$$\Gamma_1 = \{(\rho, z, \varphi);\ \rho = 1,\ |z| < \infty,\ 0 < \varphi \le 2\pi\}.$$

Given the special geometry we search for solutions depending only on ρ on which we prescribe the boundary conditions

$$\psi(1) = \psi_1, \quad \phi(1) = \phi_1$$

with ϕ_1 and ψ_1 given constants, $\phi_1 \neq \psi_1$ and

$$\lim_{\rho\to\infty} \psi(\rho) = 0, \qquad \lim_{\rho\to\infty} \phi(\rho) = 0. \tag{5.1}$$

If we search functional solutions (4.8) becomes, by (5.1),

$$e^{2\psi} = \phi^2 - 2C\phi + 1. \tag{5.2}$$

Moreover, if we assume in (5.2) ψ as a function of ϕ i.e. $\psi = \Psi(\phi)$, we have

$$\frac{d\Psi}{d\phi} = \frac{\phi - C}{\phi^2 - 2C\phi + 1} \tag{5.3}$$

and from (2.4), since $\psi_\rho = \Psi'(\phi)\phi_\rho$, $\psi_z = \Psi'(\phi)\phi_z$, we have

$$\rho\frac{d^2\phi}{d\rho^2} + \frac{d\phi}{d\rho} = 2\Psi'(\phi)\Big(\frac{d\phi}{d\rho}\Big)^2.$$

Hence, from (5.3) we arrive to the equation

$$\rho\frac{d^2\phi}{d\rho^2} + \frac{d\phi}{d\rho} = \frac{2(\phi - C)\big(\frac{d\phi}{d\rho}\big)^2}{\phi^2 - 2C\phi + 1}. \tag{5.4}$$

The solution of (5.4) is given by

$$\phi(\rho; C, C_1, C_2) = C - \sqrt{C^2-1}\tanh\big(C_1\sqrt{C^2-1}\ln\rho + C_2\sqrt{C^2-1}\big). \tag{5.5}$$

Now, whatever the choice of the constants C, C_1 and C_2, we can never satisfy the second condition in (5.1). For, if $C^2 - 1 = 0$ we have $\phi(\rho) = C$ which is only compatible with the trivial solution $\phi(\rho) = 0$. If $C^2 - 1 \neq 0$ and $C_1 > 0$ we have

$$\lim_{\rho\to\infty}\phi(\rho) = C - \sqrt{C^2-1}.$$

However, the equation $C - \sqrt{C^2-1} = 0$ has no real solutions. If $C^2 - 1 \neq 0$ and $C_1 < 0$ we have

$$\lim_{\rho\to\infty}\phi(\rho) = C + \sqrt{C^2-1}$$

and again $C + \sqrt{C^2-1} = 0$ has no real solutions.

References

[1] J. Bicak, *Selected solutions of Einstein's field equations: their role in general relativity and astrophysics*, in "Einstein's field equations and their physical implications", Bern G. Schmidt (Ed.), Lecture Notes in Physics, Springer-Verlag, Berlin, 2000.

[2] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of the second Order*, Springer-Verlag, New York, 1983.

[3] G.A. Gonzalez, A.C. Gutierrez-Pineros and P.A. Ospina, *Finite axisymmetric charged dust disks in conformastatic spacetimes*, arXiv:0806.4285v1.

[4] L. Landau and E. Lifchitz, *Thorie des Champs*, Editions MIR Moscow, 1970.

[5] T. Lewis, *Solutions of the equations of axially symmetric gravitational fields*, Proc. Roy. Soc. London A, **136**, (1932), 176-192.

[6] A. Papapetrou, *Eine rotationssymmetrische Lösung in der Allgemeinen Relativitäts Theorie*, Ann. Physik, **12**, (1953), 309-315.

[7] A. Papapetrou, *Champs gravitationnels stationnaires à symétrie axiale*, Ann. Inst. Henry Poincar **4**, (1966), 83-105.

[8] H. Weyl, *Zur Gravitationstheorie*, Ann. Phys., **54**, (1917), 117-145.

[9] H. Stephani, D. Kramer, M. McCallun, C. Hoenselaers and E. Herlt, *Exact Solutions to Einstein's Field Equations*, University Press, Cambridge, 2003.

Department of Mathematics, University of Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa Italy
E-mail address: cimatti@dm.unipi.it