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EXISTENCE AND REGULARITY RESULTS FOR FULLY NONLINEAR OPERATORS ON THE MODEL OF THE PSEUDO PUCCI'S OPERATORS

ISABEAU BIRINDELLI, FRANÇOISE DEMENGEL

Abstract. This paper is devoted to the existence and Lipschitz regularity of viscosity solutions for a class of very degenerate fully nonlinear operators, on the model of the pseudo p-Laplacian. We also prove a strong maximum principle.

1. Introduction

Recall that the pseudo-p-Laplacian, for $p > 1$ is defined by:

$$\tilde{\Delta}_p u := \sum_1^N \partial_i(|\partial_i u|^{p-2}\partial_i u).$$

When $p > 2$, it is degenerate elliptic at any point where even only one derivative $\partial_i u$ is zero.

Using classical methods in the calculus of variations, equation

$$\tilde{\Delta}_p u = f \tag{1.1}$$

has solutions in $W^{1,p}_{loc}$, when for example $f \in L^{p'}_{loc}$. Even if the existence results do not differ from the one for the usual p-Laplacian i.e. $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, the regularity raises high difficulties. For the usual p-Laplacian , the reader can look at [16], [10], in a non exhaustive manner . However, coming back to the pseudo p-Laplacian, when $p < 2$, Lipschitz regularity is a consequence of [11].

When $p > 2$ things are more delicate. Note that in [7], for some fixed non negative numbers δ_i, the following widely degenerate equation was considered

$$\sum_i \partial_i((|\partial_i u| - \delta_i)_+^{p-1}\frac{\partial_i u}{|\partial_i u|}) = f. \tag{1.2}$$

The authors proved that the solutions of (1.2) are in $W^{1,q}_{loc}$ for any $q < \infty$, when $f \in L^\infty_{loc}$. As a consequence, by the Sobolev Morrey's imbedding, the solutions are Hölder continuous for any exponent $\gamma < 1$.

The Lipschitz interior regularity for (1.1) has been recently proved by the second author in [9]. The regularity obtained concerns Lipschitz continuity for viscosity solutions. Since

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weak solutions are viscosity solutions, (see also [2]), she obtains Lipschitz continuity for weak solutions when the forcing term is in L^∞_{loc}.

At the same time, in [6], the local Lipschitz regularity of the solutions of (1.2) has been proved when either $N = 2$, $p \geq 2$ and $f \in W^{1,p'}_{loc}$ or $N \geq 3$, $p \geq 4$, and $f \in W^{1,\infty}_{loc}$. Remark that (1.2) can also be written formally as

$$\sum_i (|\partial_i u| - \delta_i)_+^{p-2} \partial_{ii} u = \frac{f}{(p-1)}.$$

This expression has an obvious meaning in the framework of viscosity solutions and with the methods used in [9], one can prove, in particular, that the solutions are locally Hölder's continuous for any exponent $\gamma < 1$, when $f \in L^\infty_{loc}$. Unfortunately the Lipschitz continuity for viscosity solutions of (1.2) cannot be obtained in the same way.

We now state the precise assumptions on the fully nonlinear operators that will be considered in this paper and we state our main result. Fix $\alpha > 0$ and, for any $q \in \mathbb{R}^N$, let $\Theta_\alpha(q)$ be the diagonal matrix with entries $|q_i|^{\frac{\alpha}{2}}$ on the diagonal, and let S be in the space of symmetric matrices in $\mathbb{R}^N$.

In the following the norm $|X|$ denotes for a symmetric matrix X, $|X| = \sum_i |\lambda_i(X)|$, sometimes for convenience of the computations we shall also use $||X|| = (\sum |\lambda_i|^2)^{\frac{1}{2}} \equiv tr({}^t XX))^{\frac{1}{2}}$.

Let F be defined on $\mathbb{R}^N \times \mathbb{R}^N \times S$, continuous in all its arguments, which satisfies $F(x, 0, M) = F(x, p, 0) = 0$ and

(H1) For any $M_1 \in S$ and $M_2 \in S$, $M_2 \geq 0$, for any $x \in \mathbb{R}^N$

$$\lambda tr(\Theta_\alpha(q) M_2 \Theta_\alpha(q)) \leq F(x, q, M_1 + M_2) - F(x, q, M_1) \leq \Lambda tr(\Theta_\alpha(q) M_2 \Theta_\alpha(q)). \tag{1.3}$$

(H2) There exist $\gamma_F \in]0, 1]$ and $c_{\gamma_F} > 0$ such that for any $(q, X) \in \mathbb{R}^N \times S$, for any $(x, y) \in (\mathbb{R}^N)^2$

$$|F(x, q, X) - F(y, q, X)| \leq c_{\gamma_F} |x - y|^{\gamma_F} |q|^\alpha |X|. \tag{1.4}$$

(H3) There exists ω_F a continuous function on $\mathbb{R}^+$ such that $\omega_F(0) = 0$, and as soon as (X, Y) satisfy for some $m > 0$

$$-m \begin{pmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq m \begin{pmatrix} \mathrm{I} & -\mathrm{I} \\ -\mathrm{I} & \mathrm{I} \end{pmatrix}$$

then

$$F(x, m(x-y), X) - F(y, m(x-y), Y) \leq \omega_F(m|x-y|^{\frac{\alpha+2}{\alpha+1}}) + o(m|x-y|^{\frac{\alpha+2}{\alpha+1}}).$$

(H4) There exists c_F such that for any $p, q \in \mathbb{R}^N$, for all $x \in \mathbb{R}^N$, $X \in S$

$$|F(x, p, X) - F(x, q, X)| \leq c_F (\sum_{i=1}^{i=N} ||p_i|^\alpha - |q_i|^\alpha|) |X|$$

Example of operators that satisfy (H1) to (H4) are

$$F(x, p, X) := tr(L(x) \Theta_\alpha(p) X \Theta_\alpha(p) L(x)),$$

when $L(x)$ is a Lipschitz and bounded matrix such that $\sqrt{\lambda}\mathrm{I} \leq L \leq \sqrt{\Lambda}\mathrm{I}$.

Other examples are the pseudo-Pucci's operators, for $0 < \lambda < \Lambda$

$$\begin{aligned}\mathcal{M}^+_\alpha(q,X) &= \Lambda tr((\Theta_\alpha(q)X\Theta_\alpha(q))^+) - \lambda tr((\Theta_\alpha(q)X\Theta_\alpha(q))^-) \\ &= \sup_{\lambda I\le A\le \Lambda I} tr(A\Theta_\alpha(q)X\Theta_\alpha(q)).\end{aligned}$$

and

$$\mathcal{M}^-_\alpha(q,X) = -\mathcal{M}^+_\alpha(q,-X).$$

satisfy all the assumptions above. The case $\alpha = 0$ reduces to the standard extremizing uniformly elliptic Pucci operators. In the appendix we shall check that $\mathcal{M}^+_\alpha(q,X)$ satisfies (H4).

We can also consider

$$F(x,p,X) := a(x)\mathcal{M}^\pm_\alpha(p,X),$$

where a is a Lipschitz function such that $a(x) \ge a_o > 0$.

We shall also consider equations with lower order terms. Precisely, let h be defined on $\mathbb{R}^N \times \mathbb{R}^N$, continuous with respect to its arguments, which satisfies on any bounded domain Ω

$$|h(x,q)| \le c_{h,\Omega}(|q|^{1+\alpha}+1) \tag{1.5}$$

Our main result is the following.

Theorem 1.1. *Let Ω be a bounded domain and f be continuous and bounded in Ω and suppose that (H1), (H2), (H4) and (1.5) hold. Let u be any viscosity solution of*

$$F(x,\nabla u, D^2u) + h(x,\nabla u) = f \quad in \;\; \Omega, \tag{1.6}$$

Then, for any $\Omega' \subset\subset \Omega$, there exists $C_{\Omega'}$, such that for any x and y in Ω'

$$|u(x) - u(y)| \le C_{\Omega'}|x-y|.$$

This will be a consequence of the more general result given in Theorem 2.1, Section 2.

We shall construct in Section 3 a super-solution of (1.6) which is zero on the boundary. Theorem 1.1, and the validity of the comparison principle, allows to prove, using Ishii's version of Perron's method, the following existence result :

Theorem 1.2. *Suppose that Ω is a bounded $\mathcal{C}^2$ domain and let F and h satisfy (H1), (H2), (H3), (H4) and (1.5). Then, for any $f \in \mathcal{C}(\overline{\Omega})$, there exists u a viscosity solution of*

$$\begin{cases} F(x,\nabla u, D^2u) + h(x,\nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore u is Lipschitz continuous in Ω .

Finally in the last section we prove that the strong maximum principle holds for solutions of equation (1.6) under the hypothesis of Theorem 1.2.

We end this introduction by recalling that many questions concerning these very degenerate operators are still open. For example it is not clear whether a sort of Alexandrov, Bakelman, Pucci 's inequality hold true, similarly to the cases treated by Imbert in [12]. Finally the next open question concerning the regularity of solutions would be to prove that the solutions are in fact $\mathcal{C}^1$. Even in the cases $f = 0$ and/or $N = 2$ it does not seem easy to do.

2. Proof of Lipschitz regularity.

Let F and Ω be as in Theorem 1.1. We shall now state and prove our main result:

Theorem 2.1. *Let f and k be continuous and bounded in a bounded open set Ω. Let F and h satisfy (H1), (H2), (H4) and (1.5). Suppose that u is a bounded USC sub-solution of*

$$F(x, \nabla u, D^2 u) + h(x, \nabla u) \geq f \text{ in } \Omega$$

and that v is a bounded LSC super-solution of

$$F(x, \nabla v, D^2 v) + h(x, \nabla v) \leq k \text{ in } \Omega.$$

Then, for any $\Omega' \subset\subset \Omega$, there exists $C_{\Omega'}$, such that for any $(x,y) \in (\Omega')^2$

$$u(x) \leq v(y) + \sup_{\Omega}(u - v) + C_{\Omega'}|x - y|.$$

We start by recalling some general facts.

If $\psi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, let $D_1\psi$ denotes the gradient in the first N variables and $D_2\psi$ the gradient in the last N variables.

In the proof of Theorem 2.1 we shall need the following technical lemma.

Lemma 2.2. *Suppose that u and v are respectively USC and LSC functions such that, for some constant $M > 1$ and for some $\mathcal{C}^2$ function Φ*

$$\psi(x, y) := u(x) - v(y) - M|x - x_o|^2 - M|y - x_o|^2 - M\Phi(x, y)$$

has a local maximum in $(\bar{x}, \bar{y})$.

Then for any $\iota > 0$, there exist X_ι, Y_ι such that

$$(MD_1\Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), X_\iota) \in \bar{J}^{2,+}u(\bar{x}),$$
$$(-MD_2\Phi(\bar{x}, \bar{y}) - 2M(\bar{y} - x_o), -Y_\iota) \in \bar{J}^{2,-}v(\bar{y})$$

with

$$-(\frac{1}{\iota} + |A| + 1)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_\iota - 2MI & 0 \\ 0 & Y_\iota - 2MI \end{pmatrix} \leq (A + \iota A^2) + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

and $A = MD^2\Phi(\bar{x}, \bar{y})$.

This is a direct consequence of a famous Lemma by Ishii [14]. For the convenience of the reader the proof of Lemma 2.2 is given in the appendix. In the sequel, we will use Lemma 2.2 with $\Phi(x, y) := g(x-y)$, and g is some radial function $\mathcal{C}^2$ except at 0, that will be chosen later. Then

$$MD^2\Phi(\bar{x}, \bar{y}) = M\begin{pmatrix} D^2g(\bar{x} - \bar{y}) & -D^2g(\bar{x} - \bar{y}) \\ -D^2g(\bar{x} - \bar{y}) & D^2g(\bar{x} - \bar{y}) \end{pmatrix}.$$

Choosing $\iota = \frac{1}{1+4M|D^2g(x)|}$, and defining $\bar{H}(x) := D^2g(x) + 2\iota D^2g^2(x)$, one has

$$M(\ D^2\Phi + \iota(D^2\Phi)^2) = M\begin{pmatrix} \bar{H}(\bar{x} - \bar{y}) & -\bar{H}(\bar{x} - \bar{y}) \\ -\bar{H}(\bar{x} - \bar{y}) & \bar{H}(\bar{x} - \bar{y}) \end{pmatrix}.$$

Remark that $M|D^2\Phi(\bar{x}, \bar{y})| = 2M|D^2g(\bar{x} - \bar{y})|$. We give some precisions on the choice of g. We will assume that there exists $\omega \in \mathcal{C}(\mathbb{R}^+) \cup \mathcal{C}^2(\mathbb{R}^{+\star})$, such that $g(x) = \omega(|x|)$ and :

(2.1) $\quad \omega(0) = 0,\ \omega(s) > 0,\ \omega'(s) > 0$ and $\omega''(s) < 0$ on $]0, s_o[$, for some given $s_o \leq 1$.

For $x \neq 0$, it is well known that $Dg(x) = \omega'(|x|)\frac{x}{|x|}$ and

$$D^2 g(x) = \left(\omega''(|x|) - \frac{\omega'(|x|)}{|x|}\right)\frac{x \otimes x}{|x|^2} + \frac{\omega'(|x|)}{|x|}\mathrm{I}.$$

For $\iota \leq \frac{1}{4|D^2 g(x)|}$, defining $\gamma_H(r) = 1 + 2\iota\left(\frac{\omega'(r)}{r}\right) \in [\frac{1}{2}, \frac{3}{2}]$, and $\beta_H(r) = 1 + 2\iota\omega''(r) \in [\frac{1}{2}, \frac{3}{2}]$ then

$$D^2 g + 2\iota(D^2 g)^2(x) = \left(\beta_H(|x|)\omega''(|x|) - \gamma_H(|x|)\frac{\omega'(|x|)}{|x|}\right)\frac{x \otimes x}{|x|^2} + \gamma_H(|x|)\frac{\omega'(|x|)}{|x|}\mathrm{I}. \tag{2.2}$$

For $|x| < 1$ and $\epsilon > 0$, we shall use the following set:

$$I(x, \epsilon) := \{i \in [1, N], |x_i| \geq |x|^{1+\epsilon}\}$$

and the diagonal matrix $\Theta(x) := \Theta_\alpha(q)$ for $q = M\frac{\omega'(|x|)x_i}{|x|}$ i.e. with entries $\Theta_{ii}(x) = M^{\frac{\alpha}{2}}\left|\frac{\omega'(|x|)x_i}{|x|}\right|^{\frac{\alpha}{2}}$. From now on, if X is a symmetric matrix, $\mu_i(X)$ for $i = 1, \ldots, N$ indicate the ordered eigenvalues of X.

A consequence of (2.2) is the following Proposition proved in [9].

Proposition 2.3 ([9])**.** *Using the notations above,*

(1) *If $\alpha \leq 2$, for all $x \neq 0$, $|x| < s_o$,*

$$\mu_1\left(\Theta(x)\bar{H}(x)\Theta(x)\right) \leq \frac{M^{1+\alpha}}{2}N^{-\frac{\alpha}{2}}\omega''(|x|)(\omega'(|x|))^\alpha < 0. \tag{2.3}$$

(2) *If $\alpha > 2$, for all $x \neq 0$, $|x| < s_o$, for any $\epsilon > 0$ such that $I(x, \epsilon) \neq \emptyset$, and such that*

$$\beta_H(|x|)\omega''(|x|)(1 - N|x|^{2\epsilon}) + \gamma_H(|x|)N|x|^{2\epsilon}\frac{\omega'(|x|)}{|x|} \leq \frac{\omega''(|x|)}{4} < 0, \tag{2.4}$$

then

$$\mu_1\left(\Theta(x)\bar{H}(x)\Theta(x)\right) \leq M^{1+\alpha}\frac{1 - N|x|^{2\epsilon}}{\#I(x, \epsilon)}(\omega'(|x|))^\alpha\frac{\omega''(|x|)}{4}|x|^{(\alpha-2)\epsilon}. \tag{2.5}$$

[Proof of Theorem 2.1] It is clear that it is sufficient to prove the result when $\Omega = B_1$ is the ball of center 0 and radius 1 and $\Omega' = B_r$ for some $r < 1$.

Borrowing ideas from [1], [5], [15], [13], for some $x_o \in B_r$ we define the function

$$\psi(x, y) = u(x) - v(y) - \sup(u - v) - M\omega(|x - y|) - M|x - x_o|^2 - M|y - x_o|^2;$$

M is a large constant and ω is a function satisfying (2.1), both to be defined more precisely later .

If there exists M, independent of $x_o \in B_r$, such that $\psi(x, y) \leq 0$ in B_1^2, by taking $x = x_o$ and, using $|x_o - y| \leq 2$, one gets

$$u(x_o) - v(y) \leq \sup(u - v) + 3M\omega(|x_o - y|).$$

So making x_o vary we obtain that, for any $(x, y) \in B_r^2$,

$$u(x) - v(y) \leq \sup(u - v) + 3M\omega(|x - y|).$$

This proves the theorem when $\omega(s)$ behaves like s near zero. Note that this will be obtained once the case where $\omega(s) = s^\gamma$ is treated for $\gamma \in]0, 1[$, i.e the Hölder's analogous result.

In order to prove that $\psi(x,y) \le 0$ in B_r^2, suppose by contradiction that the supremum of ψ is positive and achieved on $(\bar{x},\bar{y}) \in \overline{B_1}^2$. For some $\delta > 0$, with $\delta < s_o$, we choose M such that

$$M(1-r)^2 > 8(|u|_\infty + |v|_\infty), \text{ and } M > 1 + \frac{2|u|_\infty + 2|v|_\infty}{\omega(\delta)}. \tag{2.6}$$

This implies that $|\bar{x}-x_o|, |\bar{y}-x_o| < \frac{1-r}{2}$. Hence, by (2.6), $\bar{x}$ and $\bar{y}$ are in $B_{\frac{1+r}{2}}$ in particular they are in B_1. Furthermore, always using (2.6), the positivity of the supremum of ψ, the value chosen for M and the increasing behaviour of ω before s_o, lead to $|\bar{x}-\bar{y}| < \delta$.

As it will be shown later the contradiction will be found by choosing δ small enough depending on $(r,\alpha,\lambda,\Lambda,N)$.

We proceed using Lemma 2.2 and so, for all $\iota > 0$ there exist X_ι and Y_ι such that

$$(q^x, X_\iota) \in \overline{J}^{2,+}u(\bar{x}) \text{ and } (q^y, -Y_\iota) \in \overline{J}^{2,-}v(\bar{y})$$

with $q^x = q + 2M(\bar{x}-x_o)$, $q^y = q - 2M(\bar{x}-x_o)$, $q = M\omega'(|\bar{x}-\bar{y}|)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}$. Furthermore, still using the above notations i.e. $g(x) = \omega(|x|)$, and recalling that we have chosen $\iota \le \frac{1}{1+4M|D^2g(\bar{x}-\bar{y})|}$, for $\bar{H} = (D^2g(\bar{x}-\bar{y})) + 2\iota D^2g(\bar{x}-\bar{y}))^2)$, we have that

$$\begin{aligned} -(\tfrac{1}{\iota} + 2M|\bar{H}|)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &\le \begin{pmatrix} X_\iota - (2M+1)\mathrm{I} & 0 \\ 0 & Y_\iota - (2M+1)\mathrm{I} \end{pmatrix} \\ &\le M\begin{pmatrix} \bar{H} & -\bar{H} \\ -\bar{H} & \bar{H} \end{pmatrix}. \end{aligned} \tag{2.7}$$

From now on we will drop the ι for X and Y. Recall that $\Theta(q) := \Theta_\alpha(q)$ is the diagonal matrix such that $(\Theta)_{ii}(q) = (|q_i|)^{\frac{\alpha}{2}}$.

In order to end the proof we will prove the following claims.

Claims. There exists $c > 0$ depending only on $\alpha, N, \lambda, \Lambda, r$ and there exists $\hat{\tau} > 0$, such that, if δ is small enough and $|\bar{x}-\bar{y}| < \delta$, the matrix $\Theta(X+Y)\Theta$ satisfies

$$\mu_1(\Theta(X+Y)\Theta) \le -cM^{\alpha+1}|\bar{x}-\bar{y}|^{-\hat{\tau}} \tag{2.8}$$

There exist $\tau_i < \hat{\tau}$ and c_i for $i = 1,\dots,4$ depending on $\alpha, N, \lambda, \Lambda, r$ such that the four following assertions hold :

$$\text{for all } j \ge 2,\ \mu_j(\Theta(X+Y)\Theta) \le c_1 M^{\alpha+1}|\bar{x}-\bar{y}|^{-\tau_1}, \tag{2.9}$$

$$|F(\bar{x},q^x,X) - F(\bar{x},q,X)| \le c_2 M^{\alpha+1}|\bar{x}-\bar{y}|^{-\tau_2} \tag{2.10}$$

(similarly $|F(\bar{y},q^y,-Y) - F(\bar{y},q,-Y)| \le c_2 M^{\alpha+1}|\bar{x}-\bar{y}|^{-\tau_2}$)

$$|F(\bar{x},q,X) - F(\bar{y},q,X)| \le c_3 M^{\alpha+1}|\bar{x}-\bar{y}|^{-\tau_3}, \tag{2.11}$$

(similarly $|F(\bar{x},q,-Y) - F(\bar{y},q,-Y)| \le c_3 M^{\alpha+1}|\bar{x}-\bar{y}|^{-\tau_3}$)

$$|h(\bar{x},q^x)| + |h(\bar{y},q^y)| \le c_4 M^{\alpha+1}|\bar{x}-\bar{y}|^{-\tau_4}. \tag{2.12}$$

From all these claims, by taking δ small enough such that for $c > 0$ defined in (2.8), $c_2\delta^{-\hat{\tau}_2+\hat{\tau}} + c_3\delta^{\hat{\tau}-\tau_3} + c_4\delta^{\hat{\tau}-\tau_4} + \Lambda c_1\delta^{\hat{\tau}-\tau_1} < \frac{\lambda c}{2}$, one gets

$$F(\bar{x}, q^x, X) - F(\bar{y}, q^y, -Y) + h(\bar{x}, q^x) - h(\bar{y}, q^y) \leq -\frac{\lambda c}{2} M^{\alpha+1} |\bar{x} - \bar{y}|^{-\hat{\tau}}.$$

Observe that δ depends only on $\lambda, \Lambda, \alpha, N, r$. Finally, one can conclude as follows

$$\begin{aligned} f(\bar{x}) &\leq F(\bar{x}, q^x, X) + h(\bar{x}, q^x) \\ &\leq F(\bar{y}, q^y, -Y) + h(\bar{y}, q^y) - \frac{\lambda c}{2} M^{\alpha+1} |\bar{x} - \bar{y}|^{-\hat{\tau}} \\ &\leq -\frac{\lambda c}{2} M^{\alpha+1} |\bar{x} - \bar{y}|^{-\hat{\tau}} + k(\bar{y}). \end{aligned}$$

This contradicts the fact that f and k are bounded, as soon as δ is small or M is large enough.

In conclusion, in order to end the proof it is sufficient to prove (2.8), (2.9), (2.10), (2.11), (2.12). But we will need to distinguish the cases $\omega(s) = s^\gamma$ and $\omega(s) \simeq s$ both when $\alpha \leq 2$ and when $\alpha \geq 2$.

To prove the claims, we will use inequality (2.7) which has three important consequences for $\Theta(X + Y - 2(2M+1)\mathrm{I})\Theta$:

(1) As it is well known the second inequality in (2.7) gives $(X + Y - 2(2M+1)\mathrm{I}) \leq 0$, then also $\Theta(X + Y - 2(2M+1)\mathrm{I})\Theta \leq 0$. In particular, for any $j = 1, \ldots, N$

$$\mu_j(\Theta(X+Y)\Theta) \leq 6M|\Theta|^2. \tag{2.13}$$

(2) By Proposition 2.3, $\Theta(\bar{H})\Theta$ has a large negative eigenvalue, given respectively by (2.3) in the case $\alpha \leq 2$ and by (2.5) when $\alpha \geq 2$. Let e be a corresponding eigenvector. Multiplying by $\Theta \begin{pmatrix} e \\ -e \end{pmatrix}$ on the right and by its transpose on the left of (2.7), one gets, that

$${}^t e\Theta(X + Y - 2(2M+1)\mathrm{Id})\Theta e \leq 4 {}^t e(\Theta(\bar{H})\Theta)e.$$

In particular, using (2.3), one obtains that when $\alpha \leq 2$,

$$\mu_1(\Theta(X + Y - 2(2M+1)\mathrm{I})\Theta) \leq 2N^{-1} M^{1+\alpha} \omega''(|\bar{x} - \bar{y}|)(\omega'(|\bar{x} - \bar{y}|))^\alpha; \tag{2.14}$$

which in turn implies that

$$\mu_1(\Theta(X+Y)\Theta) \leq 2N^{-1} M^{1+\alpha} \omega''(|\bar{x} - \bar{y}|)(\omega'(|\bar{x} - \bar{y}|))^\alpha + 6M|\Theta|^2. \tag{2.15}$$

When $\alpha > 2$, if (2.4) holds, using (2.5), one obtains

$$\mu_1(\Theta(X+Y)\Theta) \leq \frac{1 - N|\bar{x} - \bar{y}|^{2\epsilon}}{\#I(\bar{x} - \bar{y}, \epsilon)} M^{1+\alpha} \omega''(|\bar{x} - \bar{y}|)(\omega'(|\bar{x} - \bar{y}|))^\alpha |\bar{x} - \bar{y}|^{(\alpha-2)\epsilon} + 6M|\Theta|^2. \tag{2.16}$$

(3) Finally, using (2.7), we obtain an upper bound for $|X|$, $|Y|$ i.e.

$$|X|, |Y| \leq 6M(|D^2 g(\bar{x} - \bar{y})| + 1), \tag{2.17}$$

remarking that $|\bar{H}| \leq \frac{3}{2} |D^2 g(\bar{x} - \bar{y})|$.

Proofs of the claims when $\omega(s) = s^\gamma$ and $\alpha \le 2$.

In this case, $\omega'(s) = \gamma s^{\gamma-1}$ and $\omega''(s) = -\gamma(1-\gamma)s^{\gamma-2}$,

$$q = M\gamma|\bar{x}-\bar{y}|^{\gamma-1}\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}, \quad q^x = q + 2M(\bar{x}-x_o), \quad q^y = q - 2M(\bar{y}-x_o).$$

By (2.15), since $\gamma \in (0,1)$,

$$\mu_1(\Theta(X+Y)\Theta) \le -2\gamma(1-\gamma)N^{-1}M^{\alpha+1}|\bar{x}-\bar{y}|^{\gamma-2+(\gamma-1)\alpha} + 6M|\Theta|^2,$$

while $6|\Theta|^2 \le 6M^\alpha\gamma^\alpha|\bar{x}-\bar{y}|^{(\gamma-1)\alpha}$. Consequently, as soon as δ is small enough,

$$\begin{aligned}\mu_1(\Theta(X+Y)\Theta) &\le -\frac{2\gamma(1-\gamma)}{N}M^{\alpha+1}|\bar{x}-\bar{y}|^{\gamma-2+(\gamma-1)\alpha} + 6M^{1+\alpha}\gamma^\alpha|\bar{x}-\bar{y}|^{(\gamma-1)\alpha}\\ &\le -\gamma\frac{1-\gamma}{N}M^{\alpha+1}|\bar{x}-\bar{y}|^{\gamma-2+(\gamma-1)\alpha}.\end{aligned}$$

This proves (2.8) with $\hat{\tau} = 2-\gamma+(1-\gamma)\alpha$, and $c = \gamma\frac{1-\gamma}{N}$.

Now using (2.13) and the above estimate on $M|\Theta|^2$, (2.9) holds with $\tau_1 = (1-\gamma)\alpha < \hat{\tau}$. Recall that by (2.17),

$$|X|, |Y| \le 6M(\gamma(N-\gamma)+1)|\bar{x}-\bar{y}|^{\gamma-2}. \tag{2.18}$$

Consequently (2.11) holds with $\tau_3 = (2-\gamma)+(1-\gamma)\alpha-\gamma_F$ and $c_3 = 6c_{\gamma_F}\gamma^\alpha(\gamma(N-\gamma)+1)$ using hypothesis (1.4).

To prove (2.10) we will use the following universal inequality : For any z and t in $\mathbb{R}$

$$||z|^\alpha - |t|^\alpha| \le \sup(1,\alpha)|z-t|^{\inf(1,\alpha)}(|z|+|t|)^{(\alpha-1)^+}$$

in the form (for any $i \in [1,N]$),

$$||q_i^x|^\alpha - |q_i|^\alpha| \le \sup(1,\alpha)M^\alpha|\bar{x}_i-\bar{y}_i|^{(\gamma-1)(\alpha-1)^+}. \tag{2.19}$$

Hence using (H4) and (2.18), (2.10) holds with $\tau_2 = (2-\gamma)+(1-\gamma)(\alpha-1)^+$, and $c_2 = 6c_F N\sup(1,\alpha)\gamma^\alpha(\gamma(N-\gamma)+1)$. Finally, (2.12) holds with $\tau_4 = (1-\gamma)(1+\alpha)$ and $c_4 = 2c_{h,\Omega}((\gamma+3)^{1+\alpha}+1)$.

Proofs of the claims when $\omega(s) = s^\gamma$ and $\alpha \ge 2$.

The function ω is the same than in the previous case. In order to use the result in Proposition 2.3 we need (2.4) to be satisfied. For that aim we take $\epsilon > 0$ such that $\epsilon < \inf\left(\frac{\gamma_F}{2}, \frac{1-\gamma}{2}\right)$. Let

$$\delta_N := \left[\frac{(1-\gamma)}{2(4-\gamma)N}\right]^{\frac{1}{2\epsilon}} \tag{2.20}$$

and assume $\delta < \delta_N$. In particular, for $\alpha \ge 2$, using the definition of δ_N in (2.20), for $|\bar{x}-\bar{y}| < \delta \le \delta_N$ the set $I(\bar{x}-\bar{y},\epsilon) \neq \emptyset$, indeed observe that there exists $i \in [1,N]$ such that

$$|\bar{x}_i-\bar{y}_i|^2 \ge \frac{|\bar{x}-\bar{y}|^2}{N} \ge |\bar{x}-\bar{y}|^{2+2\epsilon}.$$

Furthermore,

$$\begin{aligned}\frac{1}{2}\omega''(|\bar{x}-\bar{y}|)(1-N|\bar{x}-\bar{y}|^{2\epsilon}) &+ \frac{3N}{2}|\bar{x}-\bar{y}|^{2\epsilon}\frac{\omega'(|\bar{x}-\bar{y}|)}{|\bar{x}-\bar{y}|}\\ &\leq \frac{1}{2}\omega''(|\bar{x}-\bar{y}|)\\ &\quad +\frac{N}{2}|\bar{x}-\bar{y}|^{2\epsilon}(\gamma(1-\gamma)+3\gamma)|\bar{x}-\bar{y}|^{\gamma-2}\\ &\leq \frac{1}{4}\gamma(\gamma-1)|\bar{x}-\bar{y}|^{\gamma-2}\\ &= \frac{\omega''(|\bar{x}-\bar{y}|)}{4},\end{aligned}$$

and then (2.4) is satisfied. We are in a position to apply (2.16), and $\Theta(X+Y)\Theta$ satisfies

$$\mu_1(\Theta(X+Y)\Theta) \leq -\left(\frac{1-N|\bar{x}-\bar{y}|^{2\epsilon}}{\#I(|\bar{x}-\bar{y},\epsilon)}\right)\gamma(1-\gamma)M^{\alpha+1}|\bar{x}-\bar{y}|^{\gamma-2+(\gamma-1)\alpha+\epsilon}+6M|\Theta|^2,$$

hence remarking that $\frac{1-N||\bar{x}-\bar{y}|^{2\epsilon}}{\#I(|\bar{x}-\bar{y},\epsilon)} \geq \frac{1}{2N}$

$$\begin{aligned}\mu_1(\Theta(X+Y)\Theta) &\leq -(\frac{\gamma(1-\gamma)}{2N})M^{\alpha+1}|\bar{x}-\bar{y}|^{\gamma-2+(\gamma-1)\alpha+\epsilon}\\ &\quad +6M^{1+\alpha}\gamma^{\alpha}|\bar{x}-\bar{y}|^{(\gamma-1)\alpha}\\ &\leq -(\frac{\gamma(1-\gamma)}{4N})M^{\alpha+1}|\bar{x}-\bar{y}|^{\gamma-2+(\gamma-1)\alpha+\epsilon}\end{aligned}$$

for $|\bar{x}-\bar{y}| \leq \delta$ small enough. Hence (2.8) holds with $\hat{\tau} = 2-\gamma+(1-\gamma)\alpha-\epsilon$.

Note that (2.9), (2.11) (2.10) and (2.12) have already been proved in the previous case, since the sign of $\alpha-2$ does not play a role. Recall then that $\tau_1 = (-\gamma+1)\alpha$, while $\tau_3 = (2-\gamma)+(1-\gamma)\alpha-\gamma_F < \hat{\tau}$ by the choice of ϵ.

Finally $\tau_2 = (2-\gamma)+(\alpha-1)(\gamma-1)$ and (2.12) still holds with $\tau_4 = (1-\gamma)(1+\alpha)$.

Let us observe that in the hypothesis of Theorem 2.1 we have proved that u and v satisfy, for any $\gamma \in (0,1)$,

$$u(x) \leq v(y) + \sup_{\Omega}(u-v) + c_{\gamma,r}|x-y|^{\gamma}. \tag{2.21}$$

This will be used in the next cases.

Proofs of the claims when $\omega(s) \simeq s$ and $\alpha \leq 2$.

We choose $\tau \in (0, \inf(\gamma_F, \frac{1}{2}, \frac{\alpha}{2}))$ and $\gamma \in]\frac{\tau}{\inf(\frac{1}{2},\frac{\alpha}{2})}, 1[$. We define, for $s \leq s_o$, $\omega(s) = s-\omega_o s^{1+\tau}$ and, for $s > s_o$, $\omega(s) = \frac{s_o\tau}{1+\tau}$, ω_o is chosen so that ω is extended continuously.

We suppose that $\delta < 1$ and $\delta^{\tau}\omega_o(1+\tau) < \frac{1}{2}$, which ensures that

$$\text{for } s<\delta,\ \frac{1}{2} \leq \omega'(s) < 1,\ \omega(s) \geq \frac{s}{2}. \tag{2.22}$$

We suppose that

$$M = \sup\left(\frac{(1+\tau)2(|u|_\infty+|v|_\infty)}{\delta\tau}, 1+\frac{4(|u|_\infty+|v|_\infty)}{(1-r)^2}\right) \tag{2.23}$$

which implies in particular (2.6). So we derive that $|\bar{x}-\bar{y}|\le\delta$ and $\bar{x},\bar{y}\in B_{\frac{1+r}{2}}$.

Here

$$|D^2g(\bar{x}-\bar{y})| = \frac{N-1}{|\bar{x}-\bar{y}|} + \omega_o\tau(1+\tau)|\bar{x}-\bar{y}|^{-1+\tau} \le (N-1+\omega_o\tau(1+\tau))|\bar{x}-\bar{y}|^{-1},$$

$$\text{and } |\bar{H}| \le \frac{3}{2}|D^2g(\bar{x}-\bar{y})|.$$

Then (2.17) is nothing else but

$$|X|,|Y| \le 6M(|D^2g(\bar{x}-\bar{y})|+1) \le 6M(N+\omega_o\tau(1+\tau))|\bar{x}-\bar{y}|^{-1}. \tag{2.24}$$

Furthermore $q = M\omega'(|\bar{x}-\bar{y}|)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}$, $q^x = q+2M(\bar{x}-x_o)$, $q^y = q-2M(\bar{y}-x_o)$.

Using (2.21) in $B_{\frac{1+r}{2}}$, for all $\gamma<1$,

$$M|\bar{x}-x_o|^2 + M|\bar{y}-x_o|^2 + \sup(u-v) \le u(\bar{x})-v(\bar{y}) \le \sup(u-v) + c_{\gamma,r}|\bar{x}-\bar{y}|^\gamma$$

and then

$$|\bar{y}-x_o| + |\bar{x}-x_o| \le 2\left(\frac{c_{\gamma,r}|\bar{x}-\bar{y}|^\gamma}{M}\right)^{\frac{1}{2}}. \tag{2.25}$$

Then taking δ small enough, more precisely if $(c_{\gamma,r}\delta^\gamma)^{\frac{1}{2}} < \frac{1}{64}$ by (2.22),

$$\frac{M}{2} \le |q| \le M,\ \frac{M}{4} \le |q^x|,|q^y| \le \frac{5M}{4}. \tag{2.26}$$

Then we derive from (2.15) that

$$\mu_1(\Theta(X+Y)\Theta) \le -\frac{\omega_o\tau(1+\tau)}{N}M^{\alpha+1}|\bar{x}-\bar{y}|^{\tau-1} + 6M|\Theta|^2.$$

Since $M|\Theta|^2 \le M^{1+\alpha}$, (2.8) holds (as soon as δ is small enough) with $\hat{\tau} = 1-\tau$, and $c = \frac{\omega_o\tau(1+\tau)}{2N}$, (2.9) holds with

$$\tau_1 = 0 < 1-\tau, \quad \text{and} \quad c_1 = 6,$$

while (2.11) is satisfied with

$$\tau_3 = -\gamma_F + 1 < 1-\tau, \quad \text{and} \quad c_3 = 12c_{\gamma_F}(N+\omega_o\tau(1+\tau)).$$

To check (2.10), we use (2.19), (2.24) , (2.25) and (2.26)

$$|\,|q_i^x|^\alpha - |q_i|^\alpha|\,|X| \le 6(N+\omega_o\tau(1+\tau))M^{1+\frac{\inf(\alpha,1)}{2}}c_{\gamma,r}^{\frac{\inf(1,\alpha)}{2}}|\bar{x}-\bar{y}|^{\frac{\inf(1,\alpha)\gamma}{2}}|\bar{x}-\bar{y}|^{-1}.$$

Hence, for $\inf(1,\alpha)\gamma > 2\tau$, (2.10) holds with

$$\tau_2 = 1-\frac{\inf(1,\alpha)}{2}\gamma \quad \text{and} \quad c_2 = 6c_F\sup(1,\alpha)N(N-1+\omega_o\tau(1+\tau))(c_{\gamma,r})^{\frac{\alpha}{2}}$$

if $\alpha\le 1$ and

$$c_2 = \alpha c_F 6N(N-1+\omega_o\tau(1+\tau))(c_{\gamma,r})^{\frac{\alpha}{2}}3^{\alpha-1}), \quad \text{if} \quad \alpha\ge 1.$$

Finally $\tau_4 = 0$ and $c_4 = c_{h,\Omega}(2^{1+\alpha}+1)$ are convenient for (2.12).

Proofs of the claims when $\omega(s)\simeq s$ and $\alpha>2$.

In order to use the result in Proposition 2.3 we need (2.4) to be satisfied. For that aim we take τ, $\epsilon>0$ and γ such that

$$(2.27)\qquad 0<\tau<\frac{\gamma_F}{\alpha},\ 1>\gamma>\tau\alpha,\ \text{and}\ \frac{\tau}{2}<\epsilon<\inf\left(\frac{\frac{\gamma}{2}-\tau}{\alpha-2},\frac{\gamma_F-\tau}{\alpha-2}\right).$$

Let us define ω, s_o, as in the case $\alpha\le 2$. We suppose $\delta<\delta_N$ where

$$(2.28)\qquad \delta_N \quad := \quad \left(\frac{\omega_o(1+\tau)\tau}{2N(3+\omega_o\tau(1+\tau))}\right)^{\frac{1}{2\epsilon-\tau}}.$$

In particular, since there exists i such that $|\bar{x}_i-\bar{y}_i|^2\ge\frac{1}{N}|\bar{x}-\bar{y}|^2\ge|\bar{x}-\bar{y}|^{2+2\epsilon}$, by (2.28), $I(\bar{x}-\bar{y},\epsilon)\ne\emptyset$. Furthermore, recall that by (2.28), $1\ge\omega'(|\bar{x}-\bar{y}|)\ge\frac{1}{2}$ and

$$\begin{aligned}\frac{1}{2}\omega''(|\bar{x}-\bar{y}|) \quad &+ \quad \frac{N}{2}\omega_o\tau(1+\tau)|\bar{x}-\bar{y}|^{\tau-1+2\epsilon}+\frac{3}{2}N|\bar{x}-\bar{y}|^{2\epsilon-1}\omega'(|\bar{x}-\bar{y}|)\\ &\le \quad \frac{1}{2}\omega''(|\bar{x}-\bar{y}|)+\frac{N}{2}(\omega_o\tau(1+\tau)+3)|\bar{x}-\bar{y}|^{2\epsilon-1}\\ &\le \quad -\frac{1}{4}\omega_o(1+\tau)\tau|\bar{x}-\bar{y}|^{-1+\tau}=\frac{\omega''(|\bar{x}-\bar{y}|)}{4},\end{aligned}$$

and then (2.4) holds. We still assume that (2.23) holds.

As in the case $\alpha\le 2$, using (2.21), for δ small enough, (2.26) is still true.

The hypothesis (2.28) ensures, using also (2.5) that

$$\mu_1(\Theta(X+Y)\mathrm{I})\Theta)\le-\frac{\omega_o\tau(1+\tau)}{2N}M^{1+\alpha}|\bar{x}-\bar{y}|^{-1+\tau+(\alpha-2)\epsilon}+6M|\Theta|^2$$

and then, by (2.13) and $6M|\Theta|^2\le 6M^{1+\alpha}$, by (2.27) and for δ small enough, (2.8) holds with $\hat{\tau}=(2-\alpha)\epsilon+1-\tau$ and $c=\frac{\omega_o\tau(1+\tau)}{4N}$. Furthermore (2.9) holds with $\tau_1=0$, and $c_1=6$.

As in the previous case, (2.24) is true, and then, (2.11) holds with

$$\tau_3=1-\gamma_F<1-\tau+(2-\alpha)\epsilon \text{ and } c_3=6c_{\gamma_F}(N-1+\omega_o\tau(1+\tau)).$$

Now using (2.19), (2.24), (2.26), (2.25), one has

$$||q_i^x|^\alpha-|q_i|^\alpha||X|\le 6\alpha(N+\omega_o\tau(1+\tau))M|\bar{x}-x_o|(\frac{5M}{4})^{\alpha-1}M|\bar{x}-\bar{y}|^{-1}\le c_3|\bar{x}-\bar{y}|^{\frac{\gamma}{2}-1}M^{1+\alpha}$$

and then (2.10) holds with

$$\tau_2=1-\frac{\gamma}{2}<1-\tau+(2-\alpha)\epsilon \text{ and } c_2=\alpha Nc_{\gamma,r}^{\frac{1}{2}}(2)^{\alpha-1}6(N-1+\omega_o\tau(1+\tau)).$$

Note finally that

$$|h(\bar{x},q^x)|+|h(\bar{y},q^y)|\le 2c_h\left(\frac{5M}{4}\right)^{1+\alpha}$$

and then (2.12) holds with $\tau_4=0$ and $c_4=2^{2+\alpha}c_h$.

3. Existence of solutions.

Using Perron's method, see e.g. [8], the existence's Theorem 1.2 will be proved once the following Propositions are known:

Proposition 3.1. *Suppose that Ω is a bounded domain in $\mathbb{R}^N$ and that F satisfies (H1), (H2), (H3), (H4). Suppose that h is continuous and it satisfies (1.5). Let u be a USC sub-solution of*

$$F(x, \nabla u, D^2u) + h(x, \nabla u) - \beta(u) \geq f \text{ in } \Omega$$

and v be a LSC super-solution of

$$F(x, \nabla v, D^2v) + h(x, \nabla v) - \beta(v) \leq k \text{ in } \Omega$$

where β, f and k are continuous. Suppose that either β is increasing and $f \geq k$ in Ω, or β is nondecreasing and $f > k$ in Ω.

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proposition 3.2. *Suppose that the assumptions in Proposition 3.1 hold, and that f is continuous and bounded and β is increasing. Suppose that $\underline{u}$ is a USC sub-solution, and $\overline{u}$ is a LSC super-solution of the equation*

$$F(x, \nabla u, D^2u) + h(x, \nabla u) - \beta(u) = f, \text{ in } \Omega,$$

such that $\underline{u} = \overline{u} = \varphi$ on $\partial\Omega$. Then there exists u a viscosity solution of the equation with $\underline{u} \leq u \leq \overline{u}$ in Ω, and $u = \varphi$ on $\partial\Omega$.

The proofs of these two Propositions can be done by using the classical tools, see [8].

Remark 1. One can get the same existence's result when $\beta = 0$, by using a standard approximation procedure and the stability of viscosity solutions.

Nevertheless the proof of Theorem 1.2 requires the existence of a super-solution which is zero on the boundary when $\beta = 0$ which is the object of the next proposition:

Proposition 3.3. *Suppose that Ω is a bounded $\mathcal{C}^2$ domain, and that F and h satisfy the hypothesis in Proposition 3.1. Then for any f continuous and bounded, there exist a super-solution and a sub-solution of*

$$F(x, \nabla u, D^2u) + h(x, \nabla u) = f \text{ in } \Omega$$

which are zero on the boundary.

Proof of Proposition 3.3 : Let diam(Ω) denote the diameter of Ω and we recall that the distance to the boundary d satisfies everywhere that d is semi concave or equivalently there exists C_1 such that

$$D^2 d \leq C_1 \mathrm{I}.$$

In the following we will make the computations as if d be $\mathcal{C}^2$, it is not difficult to see that the required inequalities hold also in the viscosity sense.

Recall that $\sum_{i=1}^N (\partial_i d)^2 = 1$, hence

$$\sum_{i=1}^N |\partial_i d|^\alpha \leq N, \text{ while } \sum_1^N |\partial_i d|^{\alpha+2} \geq N^{-\frac{\alpha}{\alpha+2}}.$$

For some M large that will be chosen later, we define

$$\psi(x) = M(1 - \frac{1}{(1+d(x))^k}).$$

Clearly

$$\nabla\psi = M\frac{k\nabla d}{(1+d)^{k+1}},\ D^2\psi = \frac{Mk}{(1+d)^{k+2}}((1+d)D^2 d - (k+1)\nabla d \otimes \nabla d)$$

and then, one has

$$\begin{aligned} F(x,\nabla\psi, D^2\psi) &\le \frac{(Mk)^{\alpha+1}}{(1+d)^{k+2+(k+1)\alpha}}[(1+d)\mathcal{M}_\alpha^+(\nabla d, D^2 d) \\ &\qquad -(k+1)\mathcal{M}_\alpha^-(\nabla d, \nabla d\otimes\nabla d)] \\ &\le \frac{(Mk)^{\alpha+1}}{(1+d)^{k+2+(k+1)\alpha}}[N(1+d)\Lambda C_1 \sum |\partial_i d|^\alpha - (k+1)\lambda \sum |\partial_i d|^{\alpha+2}] \\ &\le \frac{(Mk)^{\alpha+1}}{(1+d)^{k+2+(k+1)\alpha}}[N^2(1+\operatorname{diam}(\Omega))\Lambda C_1 - \lambda(k+1)N^{-\frac{\alpha}{\alpha+2}}] \end{aligned}$$

and

$$h(x,\nabla\psi) \quad \le \quad C_h\frac{(Mk)^{\alpha+1}}{(1+d)^{(k+1)(1+\alpha)}}.$$

In particular we can choose k such that

$$\frac{1}{2}\lambda(k+1)N^{-\frac{\alpha}{\alpha+2}} = (1+\operatorname{diam}(\Omega))(\Lambda C_1 N^2 + C_h).$$

Hence

$$F(x,\nabla\psi, D^2\psi) + h(x,\nabla\psi) \quad \le \quad -\frac{(k+1)\lambda N^{-\frac{\alpha}{\alpha+2}}(Mk)^{\alpha+1}}{4(1+d)^{k+2+(k+1)\alpha}}.$$

For k as above we can choose M large enough in order that

$$F(x,\nabla\psi, D^2\psi) + h(x,\nabla\psi) \le -\|f\|_\infty.$$

A similar computation leads to:

$$F(x,\nabla(-\psi), D^2(-\psi)) + h(x,\nabla(-\psi)) \ge \|f\|_\infty.$$

4. The strong Maximum Principle

Theorem 4.1. *Under the hypothesis of Theorem 1.1, suppose that u is a supersolution of the equation $F(x,\nabla u, D^2u) \le 0$ in a domain Ω and that $u \ge 0$. Then either $u > 0$ in Ω or $u \equiv 0$.*

Proof. Without loss of generality we suppose that $u > 0$ on $B(x_1, R)$, with $R = |x_1 - x_o|$ and $u(x_o) = 0$, and we can assume that the annulus $\frac{R}{2} \le |x - x_1| \le \frac{3R}{2}$ is included in Ω. Let w be defined as

$$w(x) = m(e^{-c|x-x_1|} - e^{-cR})$$

for some c and m to be chosen.

For simplicity of the calculation we will suppose that $x_1 = 0$ and we denote by $r := |x - x_1| = |x|$. We choose m so that on $r = \frac{R}{2}$, $w \le u$ in the same spirit of simplicity we replace m by 1.

One has

$$\nabla w = \frac{-cx}{r} e^{-cr}, \ D^2 w = e^{-cr} \left((\frac{c^2}{r^2} + \frac{c}{r^3})(x \otimes x) - \frac{c}{r} \mathrm{I} \right)$$

and then, using the usual notation $\Theta(\nabla w)$, $H := \Theta(\nabla w) D^2 w \Theta(\nabla w)$, i.e.

$$H e^{c(\alpha+1)r} = \left(\frac{c}{r}\right)^\alpha \left((\frac{c^2}{r^2} + \frac{c}{r^3}) \vec{i} \otimes \vec{i} - \frac{c}{r} \vec{j} \otimes \vec{j} \right)$$

where $\vec{i} = \sum |x_i|^{\frac{\alpha}{2}} x_i e_i$ and $\vec{j} = \sum |x_i|^{\frac{\alpha}{2}} e_i$.

Since, by hypothesis (H1), $F(x, \nabla w, D^2 w) \geq e^{-c(\alpha+1)r} \mathcal{M}^-(H)$, where $\mathcal{M}^-(X) := \inf_{\lambda I \leq A \leq \Lambda I} (\mathrm{trAX})$ is the extremal Pucci operator, we need to evaluate the eigenvalues of H and in particular prove that

$$\mathcal{M}^-(H) > 0.$$

For that aim let us note that $(\vec{i}, \vec{j})^\perp$ is in the kernel of H. We introduce $a = \frac{c^2}{r^2} + \frac{c}{r^3}$ and $b = -\frac{c}{r}$. Then the non zero eigenvalues of $H c^{-\alpha} e^{cr(1+\alpha)}$ are given by

$$\mu^\pm = \frac{a|\vec{i}|^2 + b|\vec{j}|^2}{2} \pm \sqrt{\left(\frac{a|\vec{i}|^2 + b|\vec{j}|^2}{2}\right)^2 - ab(|\vec{i}|^2 |\vec{j}|^2 - (\vec{i} \cdot \vec{j})^2)}.$$

Note that there exist constants $c_i(N, \alpha)$ for $i = 1, \cdots 4$, such that

$$c_1(N,\alpha) \left(\frac{R}{2}\right)^{\alpha+2} \leq c_1(N,\alpha) r^{\alpha+2} \leq |\vec{i}|^2 \leq c_2(N,\alpha) r^{\alpha+2} \leq c_2(N,\alpha) \left(\frac{3R}{2}\right)^{\alpha+2}$$

and

$$c_3(N,\alpha) \left(\frac{R}{2}\right)^{\alpha} \leq c_3(N,\alpha) r^{\alpha} \leq |\vec{j}|^2 \leq c_4(N,\alpha) r^{\alpha} \leq c_4(N,\alpha) \left(\frac{3R}{2}\right)^{\alpha}.$$

Note that one can choose c large enough in order that for some constant $c_5(N, \alpha)$

$$\begin{aligned} a|\vec{i}|^2 + b|\vec{j}|^2 &\geq c_1(N,\alpha) \left(\frac{R}{2}\right)^{\alpha+2} \frac{c^2}{r^2} - c_4(N,\alpha) \left(\frac{3R}{2}\right)^\alpha \frac{c}{r} \\ &\geq c_5(N,\alpha) c^2. \end{aligned}$$

On the other hand one can assume c large enough in order that

$$\begin{aligned} 4|ab|(|\vec{i}|^2 |\vec{j}|^2 - (\vec{i} \cdot \vec{j})^2) &\leq 4 \frac{c^3}{r^2} c_2(N,\alpha) c_4(N,\alpha) \left(\frac{3R}{2}\right)^{2\alpha+2} \\ &\leq c_6(N,\alpha) c^3 \\ &< \left[\left(\frac{\lambda+\Lambda}{\Lambda-\lambda}\right)^2 - 1 \right] (c_5(N,\alpha) c^2))^2 \\ &\leq \left[\left(\frac{\lambda+\Lambda}{\Lambda-\lambda}\right)^2 - 1 \right] \left(a|\vec{i}|^2 + b|\vec{j}|^2 \right)^2. \end{aligned}$$

In particular this implies

$$\lambda\mu^+ + \Lambda\mu^- = (\frac{a|\vec{i}|^2 + b|\vec{j}|^2}{2}) \quad \times$$
$$\left((\lambda+\Lambda) + (\lambda-\Lambda)\sqrt{1 + 4\frac{|ab|(|\vec{i}|^2|\vec{j}|^2 - (\vec{i}\cdot\vec{j})^2)}{(a|\vec{i}|^2 + b|\vec{j}|^2)^2}}\right) > 0$$

i.e. $\mathcal{M}^-(H) > 0$. Using the comparison principle in the annulus $\{\frac{R}{2} \leq |x - x_1| \leq \frac{3R}{2}\}$ one obtains that $u \geq w$.

Observe that w touches u by below on x_o, and then, since w is $\mathcal{C}^2$ around x_o, by the definition of viscosity solution

$$F(x_o, \nabla w(x_o), D^2 w(x_o)) \leq 0.$$

This contradicts the above computation. □

Remark 2. As it is well known, the above proof can be used to see that on a point of the boundary where the interior sphere condition is satisfied, the Hopf principle holds.

Appendix A. Proof of Lemma 2.2

The proof of Lemma 2.2 is based on the following Lemma by Ishii

Lemma A.1. [14] *Let A be a symmetric matrix on $\mathbb{R}^{2N}$. Suppose that $U \in USC(\mathbb{R}^N)$ and $V \in USC(\mathbb{R}^N)$ satisfy $U(0) = V(0)$ and, for all $(x,y) \in (\mathbb{R}^N)^2$,*

$$U(x) + V(y) \leq \frac{1}{2}({}^t x, {}^t y) A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then, for all $\iota > 0$, there exist $X_\iota^U \in S$, $X_\iota^V \in S$ such that

$$(0, X_\iota^U) \in \bar{J}^{2,+}U(0), \ \ (0, X_\iota^V) \in \bar{J}^{2,+}V(0)$$

and

$$-(\frac{1}{\iota} + |A|)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_\iota^U & 0 \\ 0 & X_\iota^V \end{pmatrix} \leq (A + \iota A^2).$$

We can now start the proof of Lemma 2.2. The second order Taylor's expansion for Φ around $(\bar{x}, \bar{y})$, gives that for all $\epsilon > 0$ there exists $r > 0$ such that, for $|x - \bar{x}|^2 + |\bar{y} - y|^2 \leq r^2$,

$$\begin{aligned} u(x) - u(\bar{x}) \quad &- \quad \langle (MD_1\Phi(\bar{x},\bar{y}) + 2M)(\bar{x} - x_o), x - \bar{x}\rangle + \\ +v(\bar{y}) - v(y) \quad &- \quad \langle (MD_2\Phi(\bar{x},\bar{y}) + 2M)(\bar{y} - x_o), y - \bar{y}\rangle \\ &\leq \quad \frac{1}{2}\left({}^t(x-\bar{x}), {}^t(y-\bar{y})\right)\left(MD^2\Phi(\bar{x},\bar{y}) + \epsilon \mathrm{I}\right)\begin{pmatrix} x - \bar{x} \\ y - \bar{y}\end{pmatrix} \\ &\quad\ +M(|x-\bar{x}|^2 + |y - \bar{y}|^2). \end{aligned}$$

We now introduce the functions U and V defined, in the closed ball $|x - \bar{x}|^2 + |y - \bar{y}|^2 \leq r^2$, by

$$U(x) = u(x + \bar{x}) - \langle MD_1\Phi(\bar{x},\bar{y}) + 2M(\bar{x} - x_o), x\rangle - u(\bar{x}) - M|x|^2$$

and

$$V(y) = -v(y + \bar{y}) - \langle MD_2\Phi(\bar{x},\bar{y}) + 2M(\bar{y} - x_o), y\rangle + v(\bar{y}) - M|y|^2$$

which we extend by some convenient negative constants in the complementary of that ball (see [14] for details). Observe first that

$$(0, X^U) \in \overline{J}^{2,+} U(0), \ (0, X^V) \in \overline{J}^{2,-} V(0)$$

is equivalent to

$$(MD_1\Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), X^U + 2M\mathrm{I}) \in \overline{J}^{2,+} u(\bar{x})$$

and

$$(-MD_2\Phi(\bar{x}, \bar{y}) - 2M(\bar{y} - x_o), -X^V - 2M\mathrm{I}) \in \overline{J}^{2,-} v(\bar{y}).$$

We can apply Lemma A.1, which gives that, for any $\iota > 0$, there exists (X_ι, Y_ι) such that

$$(MD_1\Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), X_\iota) \in \bar{J}^{2,+} u(\bar{x})$$

and

$$(-MD_2\Phi(\bar{x}, \bar{y}) - 2M(\bar{y} - x_o), -Y_\iota) \in \bar{J}^{2,-} v(\bar{y})$$

Choosing ϵ such that $2\epsilon\iota|MD^2\Phi(\bar{x}, \bar{y})| + \epsilon + \iota(\epsilon)^2 < 1$, one gets

$$\begin{aligned} -(\frac{1}{\iota} + |MD^2\Phi| + 1) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &\leq \begin{pmatrix} X_\iota - 2M\mathrm{I} & 0 \\ 0 & Y_\iota - 2M\mathrm{I} \end{pmatrix} \\ &\leq (MD^2\Phi + \iota(MD^2\Phi)^2) + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{aligned}$$

This ends the proof of Lemma 2.2.

Finally, as promised in the introduction, we here check that $\mathcal{M}_\alpha^+(q, X)$ satisfies (H4). First, recalling the properties of the Pucci's operators we get

$$\begin{aligned} \mathcal{M}_\alpha^+(p, X) &\leq \mathcal{M}_\alpha^+(p, X) + \mathcal{M}^+(\Theta_\alpha(p)X\Theta_\alpha(p) - \Theta_\alpha(q)X\Theta_\alpha(q)) \\ &\leq \mathcal{M}_\alpha^+(p, X) + (\Lambda + \lambda)|(\Theta_\alpha(p)X\Theta_\alpha(p) - \Theta_\alpha(q)X\Theta_\alpha(q))| \\ &= \mathcal{M}_\alpha^+(p, X) + \frac{\Lambda + \lambda}{2}(|(\Theta_\alpha(p) - \Theta_\alpha(q))X(\Theta_\alpha(p) + \Theta_\alpha(q)) \\ &+ ((\Theta_\alpha(p) + \Theta_\alpha(q))X\Theta_\alpha(p) - \Theta_\alpha(q)|) \end{aligned}$$

Then one has using for X symmetric $||X|| \leq |X| \leq \sqrt{N}||X||$, and observing that for any matrices A B, $||AB|| = ||BA||$

$$\begin{aligned} |(\Theta_\alpha(p) - \Theta_\alpha(q))X(\Theta_\alpha(p) + \Theta_\alpha(q)) &+ ((\Theta_\alpha(p) + \Theta_\alpha(q))X\Theta_\alpha(p) - \Theta_\alpha(q)| \\ &\leq \sqrt{N}||(\Theta_\alpha(p) - \Theta_\alpha(q))X(\Theta_\alpha(p) + \Theta_\alpha(q)) \\ &+ ((\Theta_\alpha(p) + \Theta_\alpha(q))X\Theta_\alpha(p) - \Theta_\alpha(q)|| \\ &\leq 2\sqrt{N}||X(\Theta_\alpha(p) - \Theta_\alpha(q))(\Theta_\alpha(p) + \Theta_\alpha(q))|| \\ &\leq 2\sqrt{N}||X|| \ ||(\Theta_\alpha(p) - \Theta_\alpha(q))(\Theta_\alpha(p) + \Theta_\alpha(q))|| \\ &\leq 2\sqrt{N}|X| \ ||(\Theta_\alpha(p))^2 - (\Theta_\alpha(q))^2|| \\ &\leq 2\sqrt{N}|X| \sum_i ||p_i|^\alpha - |q_i|^\alpha| \end{aligned}$$

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Dipartimento di Matematica "G. Castelnuovo", Sapienza Universitá di Roma, Piazzale Aldo Moro 5, 00185 Roma, and, Université de Cergy-Pontoise, 2 av. Adolphe Chauvin, 95302, France

E-mail address: isabeau@mat.uniroma1.it, demengel@u-cergy.fr