

HEAT FLOW OF EXTRINSIC BIHARMONIC MAPS FROM A FOUR DIMENSIONAL MANIFOLD WITH BOUNDARY

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ABSTRACT. Let (M, g) be a four dimensional compact Riemannian manifold with boundary and (N, h) be a compact Riemannian manifold without boundary. We show the existence of a unique, global weak solution of the heat flow of extrinsic biharmonic maps from M to N under the Dirichlet boundary condition, which is regular with the exception of at most finitely many time slices. We also discuss the behavior of solution near the singular times. As an immediate application, we prove the existence of a smooth extrinsic biharmonic map from M to N under any Dirichlet boundary condition.

1. INTRODUCTION

Let (M, g) be a Riemannian manifold with or without boundary and (N, h) a Riemannian manifold without boundary and isometrically embedded in \mathbb{R}^L . For a nonnegative integer l and $1 \leq p < \infty$, the Sobolev space $W^{l,p}$ and Hölder space $C^{l+\alpha}$ ($0 < \alpha < 1$) are defined by:

$$W^{l,p}(M, N) := \left\{ u \in W^{l,p}(M, \mathbb{R}^L) \mid u(x) \in N \text{ for a.e. } x \in M \right\},$$

$$C^{l+\alpha}(M, N) := \left\{ u \in C^{l,\alpha}(M, \mathbb{R}^L) \mid u(x) \in N \forall x \in M \right\}.$$

On $W^{2,2}(M, N)$, there are two natural second order energy functionals defined by

$$F_2(u) = \int_M |\Delta u|^2 dv_g, \quad E_2(u) = \int_M |\tau(u)|^2 dv_g,$$

where Δ is the Laplace-Beltrami operator of (M, g) ,

$$\tau(u) = \Delta u + A(u)(\nabla u, \nabla u)$$

is the tension field of u , and $A(\cdot)(\cdot, \cdot)$ is the second fundamental form of (N, h) in \mathbb{R}^L .

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A map is called an extrinsic (or intrinsic, resp.) biharmonic map if u is a critical point of F_2 (or E_2 , resp.). The Euler-Lagrange equation for F_2 is (cf. [3, 11, 10, 24, 25, 26])

$$(1.1) \quad \begin{aligned} \Delta^2 u &= - \sum_{i=n+1}^L (\Delta \langle \nabla u, (d\nu_i \circ u) \nabla u \rangle + \nabla \cdot \langle \Delta u, (d\nu_i \circ u) \nabla u \rangle \\ &\quad + \langle \nabla \Delta u, (d\nu_i \circ u) \nabla u \rangle) \nu_i \circ u \\ &:= -f(u), \end{aligned}$$

where $\{\nu_i\}_{i=n+1}^L$ is a smooth local orthonormal frame field of the normal space of N . It is easy to see that

$$(1.2) \quad |f(u)| \leq C(|\nabla^3 u| |\nabla u| + |\nabla^2 u|^2 + |\nabla u|^4).$$

Regularity issues for extrinsic biharmonic maps in dimensions ≥ 4 were first studied by Chang etc. in [3] and for intrinsic biharmonic maps in dimension 4 by Ku [8] and alternative proofs by Wang [23] and Strzelecki [22] when the target manifold are the standard spheres \mathbb{S}^n . Wang extended the regularity result by [3] on biharmonic maps for general targets manifolds N in [24, 25], where he used a Coulomb gauge frame and Riesz potentials or Lorentz space estimates to prove that every weakly biharmonic maps from \mathbb{R}^4 to N is smooth and every stationary biharmonic map from $\mathbb{R}^m (m \geq 5)$ to N satisfies $\dim \mathcal{S} \leq m - 4$, i.e., the Hausdorff dimension of singular set is at most $m - 4$. Wang's partial regularity result was reproved by Lamm and Riviere [12] and Struwe [21] extending the lower order gauge theory technique developed in [16, 17]. See also Scheven [18] for partial regularity result for minimizing extrinsic biharmonic maps and Breiner and Lamm [2] for recent development and references therein.

The negative gradient flow for extrinsic biharmonic maps from a closed manifold (compact without boundary) was first studied by Lamm [10], where he proved the long time existence of global smooth solution when either the dimension of M is at most 3 or under a small initial energy condition in dimension 4. In general, a finite time singularity may develop in dimension 4 [4, 15]. Motivated by the heat flow of harmonic maps from surfaces by Struwe [20], it is natural to consider whether an extrinsic biharmonic map heat flow in dimension 4 has a global weak solution, which is regular outside at most finite many singularities. In this direction Gastel [5] and Wang [26] independently established a global weak solution for extrinsic biharmonic map heat flow in dimension 4, which is singular at most at finite time slices, but the problem of at most finite many singularities remains open (cf. Remark 1.2 of [26]).

In this paper we will study the extrinsic biharmonic map heat flow from a 4-dimensional compact manifold with boundary, i.e., we consider a solution $u \in C^{4+\alpha}(M \times (0, T), N)$ of

$$(1.3) \quad \partial_t u + \Delta^2 u = -f(u)$$

$$(1.4) \quad u(\cdot, 0) = u_0,$$

$$(1.5) \quad u|_{\partial M} = g,$$

$$(1.6) \quad \partial_\nu u|_{\partial M} = h,$$

where $u_0 \in W^{2,2}(M, N)$, $g \in C^{4+\alpha}(\partial M, N)$, and $h \in C^{3+\alpha}(\partial M, T_g N)$.

For $x_0 \in \overline{M}$, let $B_R^M(x_0)$ denote the closed geodesic ball in \overline{M} with center x_0 and radius $R > 0$, and set

$$E(u(t); B_R^M(x_0)) := \int_{B_R^M(x_0)} |\nabla^2 u(t)|^2 dx + \left(\int_{B_R^M(x_0)} |\nabla u(t)|^4 dx \right)^{\frac{1}{2}},$$

for $0 < R < \frac{1}{4} \text{inj}_M$, here inj_M denotes the injectivity radius of \overline{M} .

The main result of this work is:

Theorem 1. *For $\dim M = 4$, given any maps $u_0 \in W^{2,2}(M, N)$, $g \in C^{4+\alpha}(\partial M, N)$, and $h \in C^{3+\alpha}(\partial M, T_g N)$, there exists a unique global weak solution $u \in L^\infty(\mathbb{R}_+, W^{2,2}(M, N))$ of (1.3)-(1.6), with $u_t \in L^2(M \times \mathbb{R}_+)$, satisfying:*

(1) For any $0 < T < \infty$,

$$(1.7) \quad 2 \int_0^T \int_M |u_t|^2 dv_g dt + F_2(u(T)) \leq F_2(u_0),$$

and $F_2(u(\cdot, t))$ is monotonically non-increasing with respect to $t \geq 0$.

(2) There exist a positive integer K depending only on u_0, g, h, M, N , and $0 < T_1 < \dots < T_K \leq \infty$, which is characterized by the condition

$$(1.8) \quad \limsup_{t \uparrow T_k} \max_{x \in \overline{M}} E(u(t); B_R^M(x)) > \epsilon_1 \quad \text{for all } R > 0,$$

where $\epsilon_1 > 0$ is the constant given by Theorem 5 below, such that $u \in C_{loc}^{4+\alpha, 1+\frac{\alpha}{4}}(M \times (\mathbb{R}_+ \setminus \cup_{k=1}^K \{T_k\}), N)$.

(3) For each $k \in \{1, \dots, K\}$, there exist sequences $t_i^k \uparrow T_k$, $x_i^k \rightarrow x^k \in \overline{M}$, and $r_i^k \rightarrow 0$ such that

(i) if $x^k \in M$, there exists a non-constant biharmonic map $\omega^k \in C^\infty \cap W^{2,2}(\mathbb{R}^4, N)$ such that

$$(1.9) \quad u_i^k(x) = u(x_i^k + r_i^k x, t_i^k) \rightarrow \omega^k \text{ in } C_{loc}^4(\mathbb{R}^4).$$

(ii) if $x^k \in \partial M$ and if $\limsup_{i \rightarrow \infty} \frac{\text{dist}(x_i^k, \partial M)}{r_i^k} \rightarrow \infty$, then statement (i) holds. If there

exists $0 \leq a < +\infty$ such that $\limsup_{i \rightarrow \infty} \frac{\text{dist}(x_i^k, \partial M)}{r_i^k} = a$, then there exists a non-constant biharmonic map $\omega^k \in C^\infty \cap W^{2,2}(\mathbb{R}_a^4, N)$, with $\omega = \text{constant}$, $\partial_\nu \omega = 0$ on $\partial \mathbb{R}_a^4$, such that

$$(1.10) \quad u_i^k(x) = u(x_i^k + r_i^k x, t_i) \rightarrow \omega^k \text{ in } C_{loc}^4(\mathbb{R}_a^{4+}),$$

where $\mathbb{R}_a^4 := \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \mid x^4 \geq -a\}$ and $\mathbb{R}_a^{4+} := \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \mid x^4 > -a\}$.

As an application of the heat flow of biharmonic maps, we obtain the following existence result.

Theorem 2. *Let u be the global solution of (1.3)-(1.6) obtained by Theorem 1. Then there exists $t_i \uparrow \infty$ such that $u(\cdot, t_i)$ converges weakly in $W^{2,2}(M)$ to a biharmonic map $u_\infty \in C^{4+\alpha}(\overline{M}, N)$ with boundary data $u_\infty|_{\partial M} = g$ and $\partial_\nu u_\infty|_{\partial M} = h$.*

The paper is organized as follows. In section 2, we prove a small energy regularity result for biharmonic maps, a main tool in our a priori estimates, and as a corollary, we obtain a gap theorem for biharmonic map under the Dirichlet boundary condition. At the end of this section, we prove several interpolation inequalities which will be used frequently in the subsequent sections. In section 3, we give a priori estimates for the heat flow and the uniform local $W^{4,2}$ estimates in time under the assumption of small energy on a ball. In section 4, we prove the main theorems, Theorem 1 and Theorem 2.

Throughout this paper, the letter C denotes a positive constant that depends only on M, N, u_0, g , whose values may vary from lines to lines. If it depends on some other quantity, then we will point it out. For example, $C(R)$ is a positive constant depends on R .

Additional Notations. For $\Omega \subset \mathbb{R}^4$ and $0 \leq s < t \leq \infty$, denote $\Omega_s^t = \Omega \times [s, t]$, $M_s^t = M \times [s, t]$, and $M^T = M \times [0, T]$. Also denote the standard Sobolev and Hölder spaces by $W_p^{m,n}(M^T)$ and $C^{m+\alpha, n+\beta}(M^T)$.

We denote B_R (or $B_R(0)$) as the standard ball in \mathbb{R}^4 with radius R and center 0. Denote $x' = (x^1, x^2, x^3) \in \mathbb{R}^3$,

$$B_R^+ := \left\{ (x', x^4) \mid |x'|^2 + |x^4|^2 \leq R^2, x^4 \geq 0 \right\}, \quad \partial^0 B_R^+ := \left\{ (x', x^4) \mid |x'|^2 + |x^4|^2 \leq R^2, x^4 = 0 \right\}$$

and

$$V(M_s^t) := \left\{ u : M \times [s, t] \rightarrow N \mid \sup_{s \leq \sigma \leq t} (\|\nabla^2 u\|_{L^2(M)} + \|\nabla u\|_{L^4(M)}) + \int_{M_s^t} (|\partial_t u|^2 + |\nabla^4 u|^2) dv_g dt < \infty \right\}.$$

2. SOME BASIC THEOREMS AND INTERPOLATION INEQUALITIES

In this section we prove several basic theorems, including the small energy regularity theorem and the gap theorem. At the end of this section, we derive some interpolation inequalities which will be used later.

Theorem 3. (ε_1 -regularity)

(i) If $u \in W^{4,p}(B_1)$, $p > 1$, is an approximated biharmonic map with bi-tension field $\tau_2(u) \in L^p(B_1)$, i.e.

$$\Delta^2 u = -f(u) + \tau_2(u),$$

where $f(u)$ is defined in (1.1). Then there exists a constant $\varepsilon_1 > 0$ such that if $E(u; B_1) \leq \varepsilon_1$, then

$$\|u - \bar{u}\|_{W^{4,p}(B_{1/2})} \leq C(p, N) \left(\|\nabla^2 u\|_{L^2(B_1)} + \|\nabla u\|_{L^4(B_1)} + \|\tau_2(u)\|_{L^p(B_1)} \right),$$

where $\bar{u} = \frac{1}{|B_1|} \int_{B_1} u dx$ is the mean value of u over the unit ball.

(ii) If $u \in W^{4,p}(B_1^+)$, $p > 1$, is an approximated biharmonic map with tension field $\tau_2(u) \in L^p(B_1^+)$ and the Dirichlet boundary value

$$u|_{\partial^0 B_1^+} = g \quad \text{and} \quad \frac{\partial u}{\partial \bar{n}}|_{\partial^0 B_1^+} = h,$$

where $g \in C^4(\partial^0 B_1^+)$, $h \in C^3(\partial^0 B_1^+)$ and \vec{n} is the outward unit normal vector of $\partial^0 B_1^+$. Then there exists a constant $\epsilon_1 > 0$ such that if $E(u; B_1^+) \leq \epsilon_1$, then

$$\begin{aligned} \|u - \bar{u}\|_{W^{4,p}(B_{1/2}^+)} &\leq C(p, N) \left(\|\nabla^2 u\|_{L^2(B_1^+)} \right. \\ &\quad \left. + \|\nabla u\|_{L^4(B_1^+)} + \|\tau_2(u)\|_{L^p(B_1^+)} + \|g\|_{W^{4,2}(\partial^0 B_1^+)} + \|h\|_{W^{3,2}(\partial^0 B_1^+)} \right) \end{aligned}$$

where $\bar{u} := \frac{1}{|\partial^0 B_{1/2}^+|} \int_{\partial^0 B_{1/2}^+} u$ is the mean value of u over the boundary $\partial^0 B_{1/2}^+$.

Proof. Here we use the idea of [14] to give the proof of boundary estimate stated in (ii), and leave the interior estimate in (i) for interested readers since it is similar to (ii) and easier to obtain.

For convenience, assume $\bar{u} = 0$. Since u satisfies the Euler-Lagrange equation:

$$\Delta^2 u = \nabla^3 u \# \nabla u + \nabla^2 u \# \nabla^2 u + \nabla^2 u \# \nabla u \# \nabla u + \nabla u \# \nabla u \# \nabla u \# \nabla u + \tau_2(u).$$

Here $\#$ denotes some ‘product’ for which we are only interested in the properties such as

$$|a \# b| \leq C|a||b|.$$

For $0 < \sigma < 1$ and $\sigma' = \frac{1+\sigma}{2}$, let $\varphi \in C_0^\infty(B_{\sigma'}^+)$ be a cut-off function, satisfying $\varphi \equiv 1$ in B_σ^+ and $|\nabla^j \varphi| \leq \frac{4^j}{(1-\sigma)^j}$ for $j = 1, 2, 3, 4$. Direct computations show that

$$\begin{aligned} \Delta^2(\varphi u) &= \Delta(\varphi \Delta u + 2\nabla u \nabla \varphi + u \Delta \varphi) \\ &= \varphi \Delta^2 u + 4\nabla \Delta u \nabla \varphi + 2\Delta u \Delta \varphi + 4\nabla^2 u \nabla^2 \varphi + 4\nabla u \nabla \Delta \varphi + u \Delta^2 \varphi \\ &= (\nabla^3 u \# \nabla u + \nabla^2 u \# \nabla^2 u + \nabla^2 u \# \nabla u \# \nabla u + \nabla u \# \nabla u \# \nabla u \# \nabla u + \tau_2(u))\varphi \\ &\quad + \nabla^3 u \# \nabla \varphi + \nabla^2 u \# \nabla^2 \varphi + \nabla u \# \nabla^3 \varphi + u \nabla^4 \varphi \\ &= (\nabla^3(\varphi u) \# \nabla u + \nabla^2(\varphi u) \# \nabla^2 u + \nabla^2 u \# \nabla u \# \nabla(\varphi u) + \nabla u \# \nabla u \# \nabla u \# \nabla(\varphi u)) \\ &\quad + \nabla^3 u \# \nabla \varphi + \nabla^2 u \# \nabla^2 \varphi + \nabla u \# \nabla^3 \varphi + u \nabla^4 \varphi + \nabla^2 u \# \nabla u \# \nabla \varphi + \nabla^2 \varphi \# \nabla u \# \nabla u \\ &\quad + \nabla u \# \nabla u \# \nabla u \# \nabla \varphi + \varphi \tau_2(u). \end{aligned}$$

Assume first that $1 < p < \frac{4}{3}$. Observe that

$$\varphi u = \varphi g, \quad \frac{\partial(\varphi u)}{\partial \vec{n}} = \varphi h + \frac{\partial \varphi}{\partial \vec{n}} g \quad \text{on } \partial^0 B_1^+.$$

By the standard L^p theory (cf. [7]), we have

$$\begin{aligned}
& \|\nabla^4(\varphi u)\|_{L^p(B_1^+)} \leq \\
& C \left(\|\nabla u\|_{L^4(B_1^+)} \|\nabla^3(\varphi u)\|_{L^{\frac{4p}{4-p}}(B_1^+)} + \|\nabla^2 u\|_{L^2(B_1^+)} \|\nabla^2(\varphi u)\|_{L^{\frac{4p}{4-2p}}(B_1^+)} \right. \\
& + \|\nabla^2 u\|_{L^2(B_1^+)} \|\nabla u\|_{L^4(B_1^+)} \|\nabla(\varphi u)\|_{L^{\frac{4p}{4-3p}}(B_1^+)} + \|\nabla u\|_{L^4(B_1^+)}^3 \|\nabla(\varphi u)\|_{L^{\frac{4p}{4-3p}}(B_1^+)} \\
& + \frac{\|\nabla^3 u\|_{L^p(B_{\sigma'}^+)}}{1-\sigma} + \frac{\|\nabla^2 u\|_{L^p(B_{\sigma'}^+)}}{(1-\sigma)^2} + \frac{\|\nabla u\|_{L^p(B_{\sigma'}^+)}}{(1-\sigma)^3} \\
& + \frac{\|u\|_{L^p(B_{\sigma'}^+)}}{(1-\sigma)^4} + \frac{\|\nabla^2 u \# \nabla u\|_{L^p(B_{\sigma'}^+)}}{1-\sigma} + \frac{\|\nabla u \# \nabla u\|_{L^p(B_{\sigma'}^+)}}{(1-\sigma)^2} \\
& + \frac{1}{1-\sigma} \|\nabla u \# \nabla u \# \nabla u\|_{L^p(B_{\sigma'}^+)} + \|\varphi \tau_2(u)\|_{L^p(B_1^+)} \\
& \left. + \|\varphi g\|_{W^{4,p}(\partial^0 B_1^+)} + \|\varphi h + \frac{\partial \varphi}{\partial \bar{n}} g\|_{W^{3,p}(\partial^0 B_1^+)} \right).
\end{aligned}$$

By the Sobolev embedding, if ϵ_1 is chosen to be sufficiently small, then we get

$$\begin{aligned}
& \|\nabla^4(\varphi u)\|_{L^p(B_1^+)} \leq \\
& C \left(\frac{1}{1-\sigma} \|\nabla^3 u\|_{L^p(B_{\sigma'}^+)} + \frac{1}{(1-\sigma)^2} \|\nabla^2 u\|_{L^p(B_{\sigma'}^+)} + \frac{1}{(1-\sigma)^3} \|\nabla u\|_{L^p(B_{\sigma'}^+)} \right. \\
& + \frac{1}{(1-\sigma)^4} \|u\|_{L^p(B_{\sigma'}^+)} + \frac{1}{1-\sigma} \|\nabla^2 u \# \nabla u\|_{L^p(B_{\sigma'}^+)} + \frac{1}{(1-\sigma)^2} \|\nabla u \# \nabla u\|_{L^p(B_{\sigma'}^+)} \\
& + \frac{1}{1-\sigma} \|\nabla u \# \nabla u \# \nabla u\|_{L^p(B_{\sigma'}^+)} + \|\varphi \tau_2(u)\|_{L^p(B_1^+)} \\
& \left. + \|\varphi g\|_{W^{4,p}(\partial^0 B_1^+)} + \|\varphi h\|_{W^{3,p}(\partial^0 B_1^+)} + \|\frac{\partial \varphi}{\partial \bar{n}} g\|_{W^{3,p}(\partial^0 B_1^+)} \right).
\end{aligned}$$

Setting

$$\Psi_j(p) = \sup_{0 \leq \sigma \leq 1} (1-\sigma)^j \|\nabla^j u\|_{L^p(B_{\sigma'}^+)},$$

and noticing that $1-\sigma = 2(1-\sigma')$, $1 < p < \frac{4}{3}$, we have

$$\begin{aligned}
& \Psi_4(p) \\
& \leq C \left(\sum_{j=0}^3 \Psi_j(p) + \|\nabla^2 u \# \nabla u\|_{L^p(B_1^+)} + \|\nabla u \# \nabla u\|_{L^p(B_1^+)} \right. \\
& + \|\nabla u \# \nabla u \# \nabla u\|_{L^p(B_1^+)} + \|\varphi \tau_2(u)\|_{L^p(B_1^+)} \\
& \left. + \sup_{0 \leq \sigma \leq 1} (1-\sigma)^4 \left[\|\varphi g\|_{W^{4,p}(\partial^0 B_1^+)} + \|\varphi h\|_{W^{3,p}(\partial^0 B_1^+)} + \|\frac{\partial \varphi}{\partial \bar{n}} g\|_{W^{3,p}(\partial^0 B_1^+)} \right] \right) \\
& \leq C \left(\sum_{j=1}^3 \Psi_j(p) + \|\nabla^2 u\|_{L^2(B_1^+)} + \|\nabla u\|_{L^4(B_1^+)} + \|\tau_2(u)\|_{L^p(B_1^+)} \right. \\
& \left. + \|g\|_{W^{4,p}(\partial^0 B_1^+)} + \|h\|_{W^{3,p}(\partial^0 B_1^+)} \right).
\end{aligned}$$

Using the interpolation inequality (see [14])

$$\Psi_j(p) \leq \epsilon^{4-j} \Psi_4(p) + C\epsilon^{-j} \Psi_0(p), \quad j = 1, 2, 3, \quad \epsilon > 0,$$

we get, by choosing sufficiently small $\epsilon > 0$,

$$(2.1) \quad \begin{aligned} \Psi_4(p) &\leq C \left(\Psi_0(p) + \|\nabla^2 u\|_{L^2(B_1^+)} + \|\nabla u\|_{L^4(B_1^+)} + \|\tau_2(u)\|_{L^p(B_1^+)} \right. \\ &\quad \left. + \|g\|_{W^{4,2}(\partial^0 B_1^+)} + \|h\|_{W^{3,2}(\partial^0 B_1^+)} \right) \\ &\leq C \left(\|\nabla^2 u\|_{L^2(B_1^+)} + \|\nabla u\|_{L^4(B_1^+)} + \|\tau_2(u)\|_{L^p(B_1^+)} \right. \\ &\quad \left. + \|g\|_{W^{4,2}(\partial^0 B_1^+)} + \|h\|_{W^{3,2}(\partial^0 B_1^+)} \right), \end{aligned}$$

where we have used the Poincaré inequality in the last step.

If $p \geq \frac{4}{3}$, we start by applying (2.1) with $p = \frac{16}{13}$ so that

$$\begin{aligned} \|u\|_{W^{4,\frac{16}{13}}(B_{7/8}^+)} &\leq C \left(\|\nabla^2 u\|_{L^2(B_1^+)} + \|\nabla u\|_{L^4(B_1^+)} + \|\tau_2(u)\|_{L^p(B_1^+)} \right. \\ &\quad \left. + \|g\|_{W^{4,2}(\partial^0 B_1^+)} + \|h\|_{W^{3,2}(\partial^0 B_1^+)} \right). \end{aligned}$$

This, combined with the Sobolev embedding theorem, implies that

$$\begin{aligned} \|\nabla^3 u\|_{L^{\frac{16}{9}}(B_{7/8}^+)} + \|\nabla^2 u\|_{L^{\frac{16}{5}}(B_{7/8}^+)} + \|\nabla u\|_{L^{16}(B_{7/8}^+)} \\ \leq C \left(\|\nabla^2 u\|_{L^2(B_1^+)} + \|\nabla u\|_{L^4(B_1^+)} + \|\tau_2(u)\|_{L^p(B_1^+)} + \|g\|_{W^{4,2}(\partial^0 B_1^+)} + \|h\|_{W^{3,2}(\partial^0 B_1^+)} \right). \end{aligned}$$

With this estimate, we can bound the $L^{\min\{\frac{8}{5}, p\}}$ -norm of the right hand side of the Euler-Lagrange equation of u . The interior L^p -estimate together (2.1) show that u is bounded in $W^{4,\min\{\frac{8}{5}, p\}}(B_{3/4}^+)$. The lemma can be finally proved by applying the standard bootstrapping method. \square

As a direct corollary of the above theorem, we can get the following gap theorem.

Theorem 4 (Gap-phenomena). *Suppose either $u \in C^\infty(\mathbb{R}^4, N)$ is a biharmonic map or $u \in C^\infty(\mathbb{R}_+^4, N)$ is a biharmonic map with the Dirichlet boundary condition:*

$$u|_{\partial\mathbb{R}_+^4} = \text{constant} \quad \text{and} \quad \frac{\partial u}{\partial \bar{n}}|_{\partial\mathbb{R}_+^4} = 0.$$

Then there exists a universal constant $\epsilon_0 > 0$ such that if either

$$\int_{\mathbb{R}^4} |\Delta u|^2 dx \leq \epsilon_0^2 \quad \text{or} \quad \int_{\mathbb{R}_+^4} |\Delta u|^2 dx \leq \epsilon_0^2,$$

then u is a constant map.

Proof. For simplicity, we only prove the upper half space case, since the proof of $u \in C^\infty(\mathbb{R}^4, N)$ is similar. By Poincaré's inequality and integration by parts, we have that for any $R > 0$, it holds

$$\frac{1}{4R^2} \int_{B_{2R}^+} |\nabla u|^2 dx \leq C \int_{B_{2R}^+} |\nabla^2 u|^2 dx \leq C \int_{\mathbb{R}_+^4} |\nabla^2 u|^2 dx = C \int_{\mathbb{R}_+^4} |\Delta u|^2 dx \leq C\epsilon_0^2.$$

Hence, by the standard elliptic estimates and Sobolev's embedding, we have

$$\int_{B_R^+} |\nabla^2 u|^2 dx + \left(\int_{B_R^+} |\nabla u|^4 dx \right)^{\frac{1}{2}} \leq C \left[\int_{B_{2R}^+} |\Delta u|^2 dx + \frac{1}{R^2} \int_{B_{2R}^+} |\nabla u|^2 dx \right] \leq C \epsilon_0^2.$$

Choosing $\epsilon_0 \ll \epsilon_1$ and applying both Theorem 3 and the Sobolev embedding, we have that for any $R > 0$, there holds

$$R \|\nabla u\|_{L^\infty(B_R^+)} \leq C (\|\nabla^2 u\|_{L^2(B_{2R}^+)} + \|\nabla u\|_{L^4(B_{2R}^+)}) \leq C.$$

Sending R to infinity yields that u is a constant map. \square

In the following we will prove several interpolation type inequalities, which will be used through the remaining sections.

Lemma 1. *For any $u \in W^{4,2}(M, N)$, we have*

$$(2.2) \quad \int_{B_R^M} |\nabla^3 u|^2 dx \leq CR^2 \int_{B_R^M} |\nabla^4 u|^2 dx + \frac{C}{R^2} \int_{B_R^M} |\nabla^2 u|^2 dx,$$

$$(2.3) \quad \left(\int_{B_R^M} |\nabla^3 u|^4 dx \right)^{\frac{1}{2}} \leq C \left(\int_{B_R^M} |\nabla^4 u|^2 dx + \frac{C}{R^4} \int_{B_R^M} |\nabla^2 u|^2 dx \right),$$

$$(2.4) \quad \int_{B_R^M} |\nabla^2 u|^4 dx \leq C \int_{B_R^M} |\nabla^2 u|^2 dx \left(\int_{B_R^M} |\nabla^4 u|^2 dx + \frac{1}{R^4} \int_{B_R^M} |\nabla^2 u|^2 dx \right),$$

$$(2.5) \quad \int_{B_R^M} |\nabla u|^8 dx \leq C \int_{B_R^M} |\nabla u|^4 dx \left[\int_{B_R^M} |\nabla^2 u|^2 dx \right. \\ \left. \times \left(\int_{B_R^M} |\nabla^4 u|^2 dx + \frac{1}{R^4} \int_{B_R^M} |\nabla^2 u|^2 dx \right) + \frac{1}{R^4} \int_{B_R^M} |\nabla u|^4 dx \right].$$

Proof. (2.2) is a standard interpolation inequality (cf. [6], page 173). By the Sobolev embedding $W^{1,2} \hookrightarrow L^4$ on B_R^M we get

$$\left(\int_{B_R^M} |\nabla^3 u|^4 dx \right)^{\frac{1}{2}} \leq C \left(\int_{B_R^M} |\nabla^4 u|^2 dx + \frac{1}{R^2} \int_{B_R^M} |\nabla^3 u|^2 dx \right),$$

then (2.3) is a consequence of (2.2). By Sobolev embedding $W^{1,\frac{4}{3}} \hookrightarrow L^2$, we have

$$\int_{B_R^M} |\nabla^2 u|^4 dx \leq \frac{C}{R^2} \|\nabla^2 u\|_{L^{\frac{4}{3}}(B_R^M)}^2 + C \|\nabla^2 u \# \nabla^3 u\|_{L^{\frac{4}{3}}(B_R^M)}^2 \\ \leq \frac{C}{R^2} \|\nabla^2 u\|_{L^2(B_R^M)}^2 \|\nabla^2 u\|_{L^4(B_R^M)}^2 + C \|\nabla^2 u\|_{L^2(B_R^M)}^2 \|\nabla^3 u\|_{L^4(B_R^M)}^2 \\ \leq \frac{1}{2} \int_{B_R^M} |\nabla^2 u|^4 dx + \frac{C}{R^4} \|\nabla^2 u\|_{L^2(B_R^M)}^4 + C \|\nabla^2 u\|_{L^2(B_R^M)}^2 \|\nabla^3 u\|_{L^4(B_R^M)}^2 \\ \leq \frac{1}{2} \int_{B_R^M} |\nabla^2 u|^4 dx + C \int_{B_R^M} |\nabla^2 u|^2 dx \left[\left(\int_{B_R^M} |\nabla^3 u|^4 dx \right)^{\frac{1}{2}} + \frac{1}{R^4} \int_{B_R^M} |\nabla^2 u|^2 dx \right] \\ \leq \frac{1}{2} \int_{B_R^M} |\nabla^2 u|^4 dx + C \int_{B_R^M} |\nabla^2 u|^2 dx \left(\int_{B_R^M} |\nabla^4 u|^2 dx + \frac{1}{R^4} \int_{B_R^M} |\nabla^2 u|^2 dx \right),$$

which implies (2.4).

By Sobolev embedding $W^{1,2} \hookrightarrow L^4$, we have

$$\begin{aligned}
\int_{B_R^M} |\nabla u|^8 dx &\leq C \frac{1}{R^4} \left(\int_{B_R^M} |\nabla u|^4 dx \right)^2 + C \left(\int_{B_R^M} |\nabla u|^2 |\nabla^2 u|^2 dx \right)^2 \\
&\leq C \int_{B_R^M} |\nabla u|^4 dx \left(\int_{B_R^M} |\nabla^2 u|^4 dx + \frac{1}{R^4} \int_{B_R^M} |\nabla u|^4 dx \right) \\
&\leq C \int_{B_R^M} |\nabla u|^4 dx \left(\int_{B_R^M} |\nabla^2 u|^2 dx \left(\int_{B_R^M} |\nabla^4 u|^2 dx + \frac{1}{R^4} \int_{B_R^M} |\nabla^2 u|^2 dx \right) \right. \\
&\quad \left. + \frac{1}{R^4} \int_{B_R^M} |\nabla u|^4 dx \right),
\end{aligned}$$

which implies (2.5), here in the last inequality we used (2.4). \square

3. A PRIORI ESTIMATES

In this section we will show some properties of the flow and some a priori estimates, including the monotonicity of the energy $F_2(u)$ and small energy regularity theorem of parabolic case, which will be needed in the next section for the existence result.

From now on, we will use η as a smooth cut off function satisfying the following properties:

$$\begin{aligned}
(3.1) \quad &\eta \in C^\infty(M), \quad 0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B_R^M(x_0), \quad \eta \equiv 0 \text{ on } M \setminus B_{2R}^M(x_0), \\
&\|\nabla^j \eta\|_{L^\infty} \leq \frac{C}{R^j} \quad (j = 1, 2),
\end{aligned}$$

where $x_0 \in M$ and $0 < R < \frac{1}{4} \text{inj}_M$.

Lemma 2. *Let $u \in V(M^T)$ be a solution of (1.3)-(1.6). Then for all $t \in [0, T]$, we have*

$$(3.2) \quad F_2(u(t)) + 2 \int_{M^t} |\partial_t u|^2 dv_g dt = F_2(u_0),$$

$$(3.3) \quad \int_M |\nabla^2 u|^2 dv_g + \left(\int_M |\nabla u|^4 dv_g \right)^{\frac{1}{2}} \leq C(F_2(u_0) + \|g\|_{W^{2,2}(M,N)}^2).$$

Moreover, $F_2(u(t))$ is absolutely continuous in $[0, T]$ and monotonically non-increasing.

Proof. Multiplying the equation (1.3) by $\partial_t u$ and integrating by parts, we have

$$\begin{aligned}
0 &= \int_{M^t} |\partial_t u|^2 dv_g dt + \int_{M^t} \Delta^2 u \partial_t u dv_g dt \\
&= \int_{M^t} |\partial_t u|^2 dv_g dt + \int_{M^t} \Delta u \partial_t \Delta u dv_g dt + \int_0^t \int_{\partial M} \partial_\nu \Delta u \partial_t u - \int_0^t \int_{\partial M} \Delta u \partial_t \partial_\nu u \\
&= \int_{M^t} |\partial_t u|^2 dv_g dt + \int_{M^t} \partial_t \left(\frac{1}{2} |\Delta u|^2 \right) dv_g dt,
\end{aligned}$$

where we used $\partial_t u|_{\partial M} = \partial_t \partial_\nu u|_{\partial M} = 0$. Hence (3.2) follows immediately. Moreover, it is easy to see $F_2(u(t))$ is absolutely continuous in $[0, T]$ and monotonically non-increasing.

For (3.3), we first use the L^2 -estimate for the Laplace operator Δ to get

$$\int_M |\nabla^2 u|^2 dv_g \leq C(F_2(u(t)) + \|g\|_{W^{2,2}(M,N)}^2) \leq C(F_2(u_0) + \|g\|_{W^{2,2}(M,N)}^2).$$

Then, by Sobolev's inequality we have

$$\int_M |\nabla(u - g)|^2 dv_g \leq C \int_M |\nabla^2(u - g)|^2 dv_g,$$

and hence

$$\int_M |\nabla u|^2 dv_g \leq C \left(\int_M |\nabla^2 u|^2 dv_g + \|g\|_{W^{2,2}(M,N)}^2 \right).$$

Observe that (3.3) is a consequence of the following Sobolev inequality

$$\left(\int_M |\nabla u|^4 dv_g \right)^{\frac{1}{2}} \leq C \left(\int_M |\nabla^2 u|^2 dv_g + \int_M |\nabla u|^2 dv_g \right).$$

This completes the proof. \square

With the help of Theorem 3, we have

Lemma 3. *There exists $\epsilon_1 > 0$ such that if $u \in V(M^T)$ is a solution of (1.3)-(1.6) satisfying $E(u(t); B_{2R}^M(x_0)) \leq \epsilon_1$ for some $R > 0$, then we have*

$$(3.4) \quad \int_{B_{2R}^M(x_0)} |\nabla^4 u|^2 dx + \frac{1}{R^2} \int_{B_{2R}^M(x_0)} |\nabla^3 u|^2 dx \leq C \int_{B_{2R}^M(x_0)} |\partial_t u|^2 dx + \frac{C}{R^4}.$$

Proof. Since u satisfies (1.3), we have that

(i) if $B_{2R}^M(x_0) \cap \partial M = \emptyset$, then by taking $\tau_2(u) = \partial_t u$ in Theorem 3 (i) and applying a standard scaling argument, we have

$$\begin{aligned} \int_{B_{2R}^M(x_0)} |\nabla^4 u|^2 dx + \frac{1}{R^2} \int_{B_{2R}^M(x_0)} |\nabla^3 u|^2 dx &\leq C \int_{B_{2R}^M(x_0)} |\partial_t u|^2 dx + \frac{CE(u(t); B_{2R}^M(x_0))}{R^4} \\ &\leq C \int_{B_{2R}^M(x_0)} |\partial_t u|^2 dx + \frac{C}{R^4}, \end{aligned}$$

(ii) if $B_{2R}^M(x_0) \cap \partial M \neq \emptyset$, then Theorem 3 (ii) implies that

$$\begin{aligned} &\int_{B_{2R}^M(x_0)} |\nabla^4 u|^2 dx + \frac{1}{R^2} \int_{B_{2R}^M(x_0)} |\nabla^3 u|^2 dx \\ &\leq C \int_{B_{2R}^M(x_0)} |\partial_t u|^2 dx + C \frac{E(u(t); B_{2R}^M(x_0)) + \|g\|_{W^{4,2}(\partial^0 B_{2R}^M(x_0))}^2 + \|h\|_{W^{3,2}(\partial^0 B_{2R}^M(x_0))}^2}{R^4} \\ &\leq C \int_{B_{2R}^M(x_0)} |\partial_t u|^2 dx + \frac{C}{R^4}. \end{aligned}$$

Here $\partial^0 B_{2R}^M(x_0) = \partial B_{2R}^M(x_0) \cap \partial M$. Hence the conclusion of the lemma follows. \square

From Lemma 2 and Lemma 3, we can easily obtain the following corollary.

Corollary 1. *Let $u \in V(M^T)$ be a solution of (1.3)-(1.6). Assume that there exists $R > 0$ such that*

$$\sup_{0 \leq t < T} E(u(t); B_{2R}^M(x_0)) \leq \epsilon_1.$$

Then we have for all $t \in [0, T)$

$$(3.5) \quad \int_{(B_R^M(x_0))^t} |\nabla^3 u|^2 \leq C + \frac{ct}{R^2},$$

$$(3.6) \quad \int_{(B_R^M(x_0))^t} |\nabla^4 u|^2 \leq C + \frac{ct}{R^4}.$$

Proof. Integrating (3.4) from 0 to t and applying Lemma 2 yields (3.5) and (3.6). \square

In the next step we derive an L^2 -estimate for $\partial_t u$, which in turn yields an L^2 -estimate for $\nabla^4 u$, and then we can apply both L^p and Schauder estimates to achieve the desired C^l estimates.

Lemma 4. *Let $u \in V(M^T) \cap_{\sigma>0} C^4(M_\sigma^T; N)$ be a solution of (1.3)-(1.6). Assume that there exists $R > 0$ such that*

$$\sup_{0 \leq t < T} E(u(t); B_{4R}^M(x_0)) \leq \epsilon_1.$$

Then there exists $0 < \delta < \min\{T, CR^4\}$ such that for all $s, t \in (0, T)$ with $s < t$ and $|t - s| < \delta$, we have

$$(3.7) \quad \sup_{s \leq t' \leq t} \int_M \eta^4 |\partial_t u(\cdot, t')|^2 dx \leq C \int_M \eta^4 |\partial_t u(\cdot, s)|^2 dx + \frac{C}{R^4},$$

where η is a cut off function, with support in $B_{2R}(x_0)$, defined as in (3.1).

Proof. Differentiating equation (1.3) with respect to t , multiplying the resulting equation with $\eta^4 \partial_t u$, and integrating over M and applying integration by parts, we get

$$(3.8) \quad \begin{aligned} & \frac{1}{2} \int_{M_s^t} \eta^4 \partial_t |\partial_t u|^2 + \int_{M_s^t} \eta^4 |\Delta \partial_t u|^2 + 2 \int_{M_s^t} \nabla \eta^4 \nabla \partial_t u \Delta \partial_t u + \int_{M_s^t} \Delta \eta^4 \partial_t u \Delta \partial_t u \\ & \leq C \int_{M_s^t} \eta^4 (|\nabla \Delta u| |\nabla u| |\partial_t u|^2 + |\nabla^2 u|^2 |\partial_t u|^2 + |\nabla u|^4 |\partial_t u|^2) \\ & := I_1 + I_2 + I_3. \end{aligned}$$

Without loss of generality, we may assume that

$$\sup_{s \leq t' \leq t} \int_M \eta^4 |\partial_t u(\cdot, t')|^2 = \int_M \eta^4 |\partial_t u(\cdot, t)|^2.$$

Let's first estimate I_1 . With the help of Hölder's inequality and the Sobolev embedding $W^{1,2}(M) \hookrightarrow L^4(M)$, we get

$$\begin{aligned} I_1 & \leq C \int_s^t \left(\int_{B_{2R}^M(x_0)} |\nabla u|^4 \right)^{\frac{1}{4}} \left(\int_M \eta^8 |\partial_t u|^4 \right)^{\frac{1}{2}} \left(\int_{B_{2R}^M(x_0)} |\nabla \Delta u|^4 \right)^{\frac{1}{4}} \\ & \leq C \epsilon_1^{\frac{1}{2}} \int_s^t \left(\int_M \eta^4 |\partial_t u|^2 + |\nabla \eta|^2 \eta^2 |\partial_t u|^2 + \eta^4 |\nabla \partial_t u|^2 \right) \left[\int_{B_{2R}^M(x_0)} |\nabla^4 u|^2 + \frac{1}{R^2} |\nabla^3 u|^2 \right]^{\frac{1}{2}} \end{aligned}$$

Since $\partial_t u|_{\partial M} = 0$, by integration by part we get

$$\begin{aligned} \int_M \eta^4 |\nabla \partial_t u|^2 &= - \int_M \Delta \partial_t u \partial_t u \eta^4 + 4 \nabla \partial_t u \partial_t u (\eta^3 \nabla \eta) \\ &\leq \int_M |\Delta \partial_t u \partial_t u \eta^4| + \frac{1}{2} \int_M \eta^4 |\nabla \partial_t u|^2 + C \int_M \eta^2 |\nabla \eta|^2 |\partial_t u|^2. \end{aligned}$$

Thus we have

$$\int_M \eta^4 |\nabla \partial_t u|^2 \leq C \int_M |\Delta \partial_t u \partial_t u \eta^4| + C \int_M \eta^2 |\nabla \eta|^2 |\partial_t u|^2.$$

Putting it into the estimate of I_1 , we have

$$\begin{aligned} I_1 &\leq C \epsilon_1^{\frac{1}{2}} \int_s^t \left(\int_M |\Delta \partial_t u \partial_t u \eta^4| + \eta^4 |\partial_t u|^2 + |\nabla \eta|^2 \eta^2 |\partial_t u|^2 \right) \left(\int_{B_{2R}^M} |\nabla^4 u|^2 + \frac{1}{R^2} |\nabla^3 u|^2 \right)^{\frac{1}{2}} \\ &\leq C \epsilon_1^{\frac{1}{2}} \int_s^t \left(\left(\int_M \eta^4 |\Delta \partial_t u|^2 \right)^{\frac{1}{2}} \left(\int_M \eta^4 |\partial_t u|^2 \right)^{\frac{1}{2}} + \int_M \eta^4 |\partial_t u|^2 + \int_M |\nabla \eta|^2 \eta^2 |\partial_t u|^2 \right) \\ &\quad \times \left(\int_{B_{2R}^M} |\nabla^4 u|^2 + \frac{1}{R^2} |\nabla^3 u|^2 \right)^{\frac{1}{2}} \\ &\leq C \epsilon_1^{\frac{1}{2}} \int_s^t \left[\left(\int_M \eta^4 |\Delta \partial_t u|^2 \right)^{\frac{1}{2}} \left(\int_M \eta^4 |\partial_t u|^2 \right)^{\frac{1}{2}} + \int_M \eta^4 |\partial_t u|^2 \right. \\ &\quad \left. + \left(\int_M \eta^4 |\partial_t u|^2 \right)^{\frac{1}{2}} \left(\int_M |\nabla \eta|^4 |\partial_t u|^2 \right)^{\frac{1}{2}} \right] \left[\int_{B_{2R}^M} |\nabla^4 u|^2 + \frac{1}{R^2} \int_{B_{2R}^M} |\nabla^3 u|^2 \right]^{\frac{1}{2}} \\ &\leq C \epsilon_1^{\frac{1}{2}} \left(\sup_{s \leq t' \leq t} \int_M \eta^4 |\partial_t u(\cdot, t')|^2 \right)^{\frac{1}{2}} \left(\int_{M_s^t} \eta^4 |\Delta \partial_t u|^2 + \eta^4 |\partial_t u|^2 + \frac{1}{R^4} \int_{M_s^t} |\partial_t u|^2 \right)^{\frac{1}{2}} \\ &\quad \times \left(C + \frac{C\delta}{R^4} \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore we obtain

$$(3.9) \quad I_1 \leq C \epsilon_1^{\frac{1}{2}} \left(\int_M \eta^4 |\partial_t u(\cdot, t)|^2 + \int_{M_s^t} \eta^4 |\Delta \partial_t u|^2 + C \right).$$

Similarly,

$$I_2 = \int_{M_s^t} \eta^4 |\nabla^2 u|^2 |\partial_t u|^2 \leq \int_s^t \left(\int_M \eta^8 |\partial_t u|^4 \right)^{\frac{1}{2}} \left(\int_{B_{2R}^M(x_0)} |\nabla^2 u|^4 \right)^{\frac{1}{2}}.$$

By (2.4), we get

$$\begin{aligned} \left(\int_{B_{2R}^M(x_0)} |\nabla^2 u|^4 \right)^{\frac{1}{2}} &\leq C \left(\int_{B_{2R}^M(x_0)} |\nabla^2 u|^2 \right)^{\frac{1}{2}} \left(\int_{B_{2R}^M(x_0)} |\nabla^4 u|^2 + \frac{1}{R^4} \right)^{\frac{1}{2}} \\ &\leq C \epsilon_1^{\frac{1}{2}} \left(\int_{B_{2R}^M(x_0)} |\nabla^4 u|^2 + \frac{1}{R^4} \right)^{\frac{1}{2}}. \end{aligned}$$

Then, by the same argument as in the estimates of I_1 , we get

$$(3.10) \quad I_2 \leq C \epsilon_1^{\frac{1}{2}} \left(\int_M \eta^4 |\partial_t u(\cdot, t)|^2 + \int_{M_s^t} \eta^4 |\Delta \partial_t u|^2 + C \right).$$

For I_3 , we have

$$I_3 = \int_s^t \int_M \eta^4 |\partial_t u|^2 |\nabla u|^4 \leq \int_s^t \left(\int_M \eta^8 |\partial_t u|^4 \right)^{\frac{1}{2}} \left(\int_{B_{2R}^M(x_0)} |\nabla u|^8 \right)^{\frac{1}{2}}.$$

By (2.5), we get

$$\left(\int_{B_{B_{2R}^M(x_0)}} |\nabla u|^8 \right)^{\frac{1}{2}} \leq C \left(\int_{B_{2R}^M} |\nabla u|^4 \right)^{\frac{1}{2}} \left(\int_{B_{2R}^M} |\nabla^4 u|^2 + \frac{1}{R^4} \right)^{\frac{1}{2}}.$$

Then, by the same arguments as in the estimates of I_1, I_2 , we obtain

$$(3.11) \quad I_3 \leq C \epsilon_1^{\frac{1}{2}} \left(\int_M \eta^4 |\partial_t u(\cdot, t)|^2 + \int_{M_s^t} \eta^4 |\Delta \partial_t u|^2 + C \right).$$

Combining inequalities (3.8)-(3.11) yields

$$(3.12) \quad \begin{aligned} & \frac{1}{2} \int_{M_s^t} \eta^4 \partial_t |\partial_t u|^2 + \int_{M_s^t} \eta^4 |\Delta \partial_t u|^2 + 2 \int_{M_s^t} \nabla \eta^4 \nabla \partial_t u \Delta \partial_t u + \int_{M_s^t} \Delta \eta^4 \partial_t u \Delta \partial_t u \\ & \leq C \epsilon_1^{\frac{1}{2}} \left(\int_M \eta^4 |\partial_t u(\cdot, t)|^2 + \int_{M_s^t} \eta^4 |\Delta \partial_t u|^2 + C \right). \end{aligned}$$

By the Cauchy-Schwartz inequality, we have

$$2 \left| \int_{M_s^t} \nabla \eta^4 \nabla \partial_t u \Delta \partial_t u \right| \leq \frac{1}{4} \int_{M_s^t} \eta^4 |\Delta \partial_t u|^2 + 64 \int_{M_s^t} \eta^2 |\nabla \eta|^2 |\nabla \partial_t u|^2$$

and

$$(3.13) \quad \begin{aligned} \int_{M_s^t} \Delta \eta^4 \partial_t u \Delta \partial_t u &= \int_{M_s^t} (4\eta^3 \Delta \eta + 12\eta^2 |\nabla \eta|^2) \partial_t u \Delta \partial_t u \\ &\geq -\frac{1}{4} \int_{M_s^t} \eta^4 |\Delta \partial_t u|^2 - \frac{C}{R^4} \int_{M_s^t} |\partial_t u|^2 \\ &\geq -\frac{1}{4} \int_{M_s^t} \eta^4 |\Delta \partial_t u|^2 - \frac{C}{R^4}. \end{aligned}$$

Furthermore, by integration by parts and noting that $\partial_t u|_{\partial M} = 0$, we have

$$\begin{aligned} 64 \int_{M_s^t} \eta^2 |\nabla \eta|^2 |\nabla \partial_t u|^2 &= -64 \int_{M_s^t} \nabla(\eta^2 |\nabla \eta|^2) \nabla \partial_t u \partial_t u - 64 \int_{M_s^t} \eta^2 |\nabla \eta|^2 \Delta \partial_t u \partial_t u \\ &\leq \int_{M_s^t} \eta^2 |\nabla \eta|^2 |\nabla \partial_t u|^2 + \frac{1}{8} \int_{M_s^t} \eta^4 |\Delta \partial_t u|^2 + \frac{C}{R^4} \int_{M_s^t} |\partial_t u|^2. \end{aligned}$$

Therefore we get

$$(3.14) \quad 2 \int_{M_s^t} \nabla \eta^4 \nabla \partial_t u \Delta \partial_t u \geq -\frac{1}{2} \int_{M_s^t} \eta^4 |\Delta \partial_t u|^2 - \frac{C}{R^4}.$$

Combining inequalities (3.12), (3.14) and (3.13) and choosing ϵ_1 sufficiently small, we can finally achieve (3.7). \square

Now we can derive an L^2 estimate for $\nabla^4 u$ by the above lemma.

Lemma 5. *Let $u \in V(M^T) \cap_{\sigma>0} C^4(M_\sigma^T; N)$ be a solution of (1.3)-(1.6). Assume that there exists $R > 0$ such that*

$$\sup_{0 \leq t < T} E(u(t); B_{4R}^M(x_0)) \leq \epsilon_1.$$

Then there exists $0 < \delta < \min\{T, CR^4\}$ such that for all $t' \in [\frac{3\delta}{4}, T)$ we have

$$(3.15) \quad \int_{B_{\frac{R}{2}}^M(x_0)} |\nabla^4 u(\cdot, t')|^2 \leq C\left(\frac{1}{\delta} + \frac{1}{R^4}\right).$$

Proof. By Lemma 3, we have, for all $t' \in [\frac{3\delta}{4}, T)$

$$(3.16) \quad \int_{B_{\frac{R}{2}}^M(x_0)} |\nabla^4 u|^2(\cdot, t') dx \leq C \int_{B_R^M(x_0)} |\partial_t u|^2(\cdot, t') dx + \frac{C}{R^4}.$$

Let η be a cut off function as in Lemma 4. Without loss of generality, we assume that

$$\int_M \eta^4 |\partial_t u(\cdot, s)|^2 dx = \inf_{t' - \frac{\delta}{2} \leq s' \leq t' - \frac{\delta}{4}} \int_M \eta^4 |\partial_t u(\cdot, s')|^2.$$

Then Lemma 4 implies

$$\begin{aligned} \sup_{t' - \frac{\delta}{4} \leq t \leq t'} \int_M \eta^4 |\partial_t u(\cdot, t)|^2 &\leq C \int_M \eta^4 |\partial_t u(\cdot, s)|^2 + \frac{C}{R^4} \\ &= C \inf_{t' - \frac{\delta}{2} \leq s' \leq t' - \frac{\delta}{4}} \int_M \eta^4 |\partial_t u(\cdot, s')|^2 + \frac{C}{R^4} \\ &\leq \frac{C}{\delta} \int_{t' - \frac{\delta}{2}}^{t' - \frac{\delta}{4}} \int_M |\partial_t u|^2 + \frac{C}{R^4} \leq C\left(\frac{1}{\delta} + \frac{1}{R^4}\right). \end{aligned}$$

Therefore we have

$$(3.17) \quad \int_{B_R^M(x_0)} |\partial_t u|^2(\cdot, t') dx \leq C\left(\frac{1}{\delta} + \frac{1}{R^4}\right).$$

This completes the proof. \square

By Lemma 5, we have

Theorem 5. *There exists $\epsilon_1 > 0$, depending only on M, N, u_0, g , such that for $0 < T < \infty$, if u is a smooth solution of (1.3)-(1.6) satisfying*

$$(3.18) \quad \sup_{0 < t \leq T} \int_{B_{4R}^M(x_0)} |\nabla^2 u(\cdot, t)|^2 dx + \left(\int_{B_{4R}^M(x_0)} |\nabla u(\cdot, t)|^4 dx \right)^{\frac{1}{2}} \leq \epsilon_1,$$

for some $R < \frac{1}{2} \text{inj}_M$ and $x_0 \in M$, then we have

$$(3.19) \quad \max_{\frac{T}{2} \leq t \leq T} \|u\|_{C^k(B_{\frac{R}{4}}^M(x_0))} \leq C\left(k, R^{-1}, T, \|\nabla^2 u_0\|_{L^2(M)}, \|g\|_{C^k(\partial M)}, \|h\|_{C^{k-1}(\partial M)}\right).$$

Proof. It follows from Lemma 5 that u is uniformly bounded in $W^{4,2}(B_{\frac{R}{2}}^M(x_0))$ for $\frac{T}{2} \leq t \leq T$. It follows that $u_t + \Delta^2 u \in L^p(B_{\frac{R}{2}}^M(x_0) \times [\frac{T}{2}, T])$ for any $1 < p < \infty$. Therefore by the standard parabolic L^p -theory and Schauder estimate, we can get the desired estimate. \square

To prove our main theorem, we need to establish a lower bound estimate of the time interval for the existence of a smooth solution of (1.3)-(1.6). First we have

Lemma 6. *Let $u \in V(M^T)$ be a solution of (1.3)-(1.6). Assume that there exists $0 < R < 1$ such that*

$$\sup_{0 \leq t < T} E(u(t); B_{2R}^M(x_0)) \leq \epsilon_1.$$

Then we have for all $t \in [0, T)$

$$(3.20) \quad E(u(t); B_R^M(x_0)) \leq CE(u(0); B_{2R}^M(x_0)) + \frac{Ct}{R^4} + \frac{C\sqrt{t}}{R^2} + CR^2.$$

Proof. Multiplying (1.3) by $\eta^4 \partial_t u$ and integrating by parts, we get

$$\begin{aligned} & \int_{M^t} \eta^4 |\partial_t u|^2 dxdt + \frac{1}{2} \int_{M^t} \eta^4 \frac{\partial}{\partial t} |\Delta u|^2 dxdt \\ &= \int_{M^t} \Delta u \Delta \eta^4 \partial_t u dxdt + 2 \int_{M^t} \Delta u \nabla \eta^4 \nabla \partial_t u dxdt \\ &= - \int_{M^t} \Delta u \Delta \eta^4 \partial_t u dxdt - 2 \int_{M^t} \nabla \Delta u \nabla \eta^4 \partial_t u dxdt \\ &\leq \frac{1}{2} \int_{M^t} \eta^4 |\partial_t u|^2 dxdt + \frac{C}{R^4} \int_{B_{2R}^M(x_0)^t} |\Delta u|^2 dxdt + \frac{C}{R^2} \int_{B_{2R}^M(x_0)^t} |\nabla^3 u|^2 dxdt \\ &\leq \frac{1}{2} \int_{M^t} \eta^4 |\partial_t u|^2 dxdt + \frac{C}{R^4} \int_{B_{2R}^M(x_0)^t} |\Delta u|^2 dxdt + C\epsilon \int_{B_{2R}^M(x_0)^t} |\nabla^4 u|^2 dxdt \\ &\quad + \frac{C}{\epsilon R^4} \int_{B_{2R}^M(x_0)^t} |\nabla^2 u|^2 dxdt \\ &\leq \frac{1}{2} \int_{M^t} \eta^4 |\partial_t u|^2 dxdt + \frac{Ct}{R^4} + \frac{C\sqrt{t}}{R^2} \quad \text{by taking } \epsilon = \frac{\sqrt{t}}{R^2}. \end{aligned}$$

Then we have

$$(3.21) \quad \int_{M^t} \eta^4 |\partial_t u|^2 dxdt + \int_{M^t} \eta^4 \frac{\partial}{\partial t} |\Delta u|^2 dxdt \leq \frac{Ct}{R^4} + \frac{C\sqrt{t}}{R^2}.$$

Thus we obtain

$$(3.22) \quad \int_{B_R^M(x_0)} |\Delta u|^2(t) dx \leq \int_{B_{2R}^M(x_0)} |\Delta u|^2(0) dx + \frac{Ct}{R^4} + \frac{C\sqrt{t}}{R^2}.$$

Observe that

$$\partial_t \left(\frac{1}{2} |\nabla u|^2 \right) = \langle \nabla u, \nabla \partial_t u \rangle = \nabla \langle \nabla u, \partial_t u \rangle - \langle \Delta u, \partial_t u \rangle.$$

Multiplying this equality by η^4 , integrating it over M , and applying integration by parts and (1.3), we obtain

$$\begin{aligned}
\int_{M^t} \partial_t \left(\frac{1}{2} \eta^4 |\nabla u|^2 \right) dx dt &= \int_{M^t} \eta^4 \nabla \langle \nabla u, \partial_t u \rangle dx dt - \int_{M^t} \eta^4 \langle \Delta u, \partial_t u \rangle dx dt \\
&= - \int_{M^t} \nabla \eta^4 \langle \nabla u, \partial_t u \rangle dx dt - \int_{M^t} \eta^4 \langle \Delta u, \partial_t u \rangle dx dt \\
&\leq \frac{R^2}{4} \int_{M^t} \eta^4 |\partial_t u|^2 dx dt + \frac{C}{R^4} \int_{M^t} \eta^2 |\nabla u|^2 dx dt + \frac{C}{R^2} \int_{M^t} \eta^4 |\Delta u|^2 dx dt \\
&\leq \frac{R^2}{4} \int_{M^t} \eta^4 |\partial_t u|^2 dx dt + \frac{C}{R^2} \int_0^t \left(\int_{B_{2R}(x_0)} |\nabla u|^4 dx \right)^{\frac{1}{2}} dt + \frac{Ct}{R^2} \\
&\leq \frac{R^2}{4} \int_{M^t} \eta^4 |\partial_t u|^2 dx dt + \frac{Ct}{R^2} \\
&\leq \frac{R^2}{4} \int_{B_{2R}^M(x_0)} |\Delta u|^2(0) dx + C\sqrt{t} + \frac{Ct}{R^2}.
\end{aligned}$$

Thus,

$$(3.23) \quad \int_{B_R^M(x_0)} |\nabla u|^2(t) dx \leq \int_{B_{2R}^M(x_0)} |\nabla u|^2(0) dx + \frac{R^2}{4} \int_{B_{2R}^M(x_0)} |\Delta u|^2(0) dx + C\sqrt{t} + \frac{Ct}{R^2}.$$

Let $q \in C^{2+\alpha}(M, \mathbb{R}^N)$ be a harmonic function, satisfying

$$\begin{cases} \Delta q = 0 & \text{in } M, \\ q = g & \text{on } \partial M. \end{cases}$$

Then we have

$$\|q\|_{C^{2+\alpha}(M)} \leq C(M) \|g\|_{C^{2+\alpha}(M)},$$

and hence

$$\begin{aligned}
&\int_{B_R^M(x_0)} |\nabla^2(u - q)|^2(t) dx + \left(\int_{B_R^M(x_0)} |\nabla(u - q)|^4(t) dx \right)^{\frac{1}{2}} \\
&\leq C \int_{B_{\frac{3}{2}R}^M(x_0)} |\Delta(u - q)|^2(t) dx + \frac{C}{R^2} \int_{B_{\frac{3}{2}R}^M(x_0)} |\nabla(u - q)|^2(t) dx \\
&\leq C \int_{B_{2R}^M(x_0)} |\Delta u|^2(0) dx + \frac{C}{R^2} \int_{B_{2R}^M(x_0)} |\nabla u|^2(0) dx + \frac{Ct}{R^4} + \frac{C\sqrt{t}}{R^2} + \frac{C}{R^2} \int_{B_{2R}^M(x_0)} |\nabla q|^2 dx \\
&\leq C \int_{B_{2R}^M(x_0)} |\Delta u|^2(0) dx + C \left(\int_{B_{2R}^M(x_0)} |\nabla u|^4(0) dx \right)^{\frac{1}{2}} + \frac{Ct}{R^4} + \frac{C\sqrt{t}}{R^2} + CR^2.
\end{aligned}$$

This implies

$$\begin{aligned}
& \int_{B_R^M(x_0)} |\nabla^2 u|^2(t) dx + \left(\int_{B_R^M(x_0)} |\nabla u|^4(t) dx \right)^{\frac{1}{2}} \\
& \leq C \left(\int_{B_R^M(x_0)} |\nabla^2(u-q)|^2(t) dx + \int_{B_R^M(x_0)} |\nabla^2 q|^2(t) dx \right) \\
& \quad + C \left(\left(\int_{B_R^M(x_0)} |\nabla(u-q)|^4(t) dx \right)^{\frac{1}{2}} + \left(\int_{B_R^M(x_0)} |\nabla q|^4 dx \right)^{\frac{1}{2}} \right) \\
& \leq C \int_{B_{2R}^M(x_0)} |\Delta u|^2(0) dx + C \left(\int_{B_{2R}^M(x_0)} |\nabla u|^4(0) dx \right)^{\frac{1}{2}} + \frac{Ct}{R^4} + \frac{C\sqrt{t}}{R^2} + CR^2,
\end{aligned}$$

which implies (3.20). This completes the proof. \square

According to Lemma 6, we have

Lemma 7. *There exists $0 < \epsilon_2 \ll \epsilon_1 < \frac{inj_M}{2}$ such that if $u_0 \in C^\infty(M, N)$, $g \in C^\infty(\partial M, N)$, and $h \in C^\infty(\partial M, T_g N)$ satisfies*

$$(3.24) \quad \sup_{x \in M} \left(\int_{B_{2R}^M(x)} |\nabla^2 u_0|^2 dx + \left(\int_{B_{2R}^M(x)} |\nabla u_0|^4 dx \right)^{\frac{1}{2}} \right) \leq \epsilon_2^2,$$

for some $R \in (0, \epsilon_2)$. Then there exists $T_1 \geq O(R^4 \epsilon_1^2)$ and a unique solution $u \in C^\infty(M \times [0, T_1], N)$ to (1.3)-(1.6).

Proof. Let $T_1 > 0$ be the maximum time interval such that there exists a smooth solution $u \in C^\infty(M \times [0, T_1], N)$ of (1.3)-(1.6). Let $T'_1 > 0$ be the maximum time such that

$$(3.25) \quad \sup_{0 \leq t \leq T'_1} \sup_{x \in M} E(u(t); B_R^M(x)) \leq \epsilon_1.$$

By Theorem 5, we know $T_1 \geq T'_1$. By Lemma 6, we get

$$\begin{aligned}
\epsilon_1 &= E(u(T'_1); B_R^M(x)) \leq CE(u(0); B_{2R}^M(x)) + \frac{CT'_1}{R^4} + \frac{C\sqrt{T'_1}}{R^2} + CR^2 \\
&\leq \frac{\epsilon_1}{2} + \frac{CT'_1}{R^4} + \frac{C\sqrt{T'_1}}{R^2} \\
&\leq \frac{3\epsilon_1}{4} + \frac{CT'_1}{\epsilon_1 R^4}.
\end{aligned}$$

This implies $T_1 \geq T'_1 \geq O(R^4 \epsilon_1^2)$. \square

4. EXISTENCE RESULTS AND BEHAVIOR OF SOLUTIONS NEAR SINGULARITIES

In this section, we show the existence of the global weak solution of the extrinsic biharmonic map flow, which is regular with the exception of at most finitely many time slices. We also study the behavior of the solution near its singularities. Moreover, we get the existence of biharmonic maps with a fixed Dirichlet boundary data. Both Theorem 1 and Theorem 2 will be proved in this section.

Proof of Theorem 1. Step 1. From [19] and [1] we see that there exists a sequence of maps $\phi_l \in C^{4+\alpha}(M, N)$ such that $\phi_l = g, \partial_\nu \phi_l = h$ on ∂M , and

$$\phi_l \rightarrow u_0 \quad \text{strongly in } W^{2,2}(M, N).$$

Step 2. The short-time existence. Since $\phi_l \rightarrow u_0$ in $W^{2,2}(M, N)$, there exists a $R \in (0, \frac{inj_M}{2})$ such that

$$\sup_l \sup_{x \in M} \left(\int_{B_{2R}(x)} |\nabla^2 \phi_l|^2 dx + \left(\int_{B_{2R}(x)} |\nabla \phi_l|^4 dx \right)^{\frac{1}{2}} \right) \leq \epsilon_2^2,$$

where ϵ_2 is given in Lemma 7. By the short-time existence theory in [9], there exist $T_l > 0$ and $u_l \in C^{4+\alpha, 1+\frac{\alpha}{4}}(M \times [0, T_l], N)$ which solves (1.3) with the boundary-initial data (g, h) . Then Lemma 7 implies that $T_l \geq O(R^4 \epsilon_1^2)$ and Theorem 5 implies that we have uniformly $C_{loc}^{4+\alpha, 1+\frac{\alpha}{4}}$ estimates of u_l in $M \times (0, O(R^4 \epsilon_1^2)]$. Hence we may assume that u_l converges to u weakly in $W^{2,2}(M, N)$, strongly in $W^{1,2}(M, N)$ and in $C^{4+\alpha, 1+\frac{\alpha}{4}}(M \times [\rho, O(R^4 \epsilon_1^2)], N)$ for any $\rho > 0$. It is clear that $u \in C_{loc}^{4+\alpha, 1+\frac{\alpha}{4}}(M \times (0, O(R^4 \epsilon_1^2)), N)$ is a classical solution of (1.3). The short-time existence theory guarantees the existence of a solution to (1.3) using $u(O(R^4 \epsilon_1^2))$ as the new initial data so that the solution can be continued to a larger time interval. Assume that T_1 is the maximum time interval such that $u \in C_{loc}^{4+\alpha, 1+\frac{\alpha}{4}}(M \times (0, T_1), N)$ solves (1.3)-(1.6). Repeating this argument, the solution can be continued until the first time of energy concentration exceeds ϵ_1 , that is, the condition

$$(4.1) \quad \lim_{r \downarrow 0} \limsup_{t \uparrow T_1} \sup_{x \in M} E(u(t), B_r(x)) > \epsilon_1$$

reaches. Set

$$(4.2) \quad S(T_1) := \left\{ (x, T_1) \mid x \in M, \lim_{r \rightarrow 0} \limsup_{t \uparrow T_1} E(u(t), B_r(x)) > \epsilon_1 \right\},$$

which is called as the singularity set of u at time T_1 . It is an open question if $S(T_1)$ is a finite set.

Step 3. Behavior of the solution u near its first singular time T_1 . By the standard blowup argument, there exist sequences $t_i^1 \nearrow T_1$, $x_i^1 \rightarrow x_0 \in \bar{M}$, and $r_i^1 \rightarrow 0$ such that

$$(4.3) \quad E(u(t_i^1), B_{r_i^1}^M(x_i^1)) = \sup_{\substack{(x,t) \in \bar{M} \times [T_1 - \delta^2, t_i^1] \\ r \leq r_i^1}} E(u(t), B_r^M(x)) = \frac{\epsilon_1}{C_0},$$

where C_0 is a positive constant to be determined later. Assume that $B_{2r_i^1}(x_0)$ is covered by m balls of radius r_i^1 contained in \bar{M} and let $C_0 = m$, then we see that

$$\sup_{T_1 - \delta^2 \leq t < T_1} E(u(t); B_{2r_i^1}^M(x_0)) \leq \epsilon_1.$$

By Lemma 6, for any $T_1 - \delta^2 \leq s \leq t_i^1 < T_1$, we have

$$E(u(t_i^1); B_{r_i^1}^M(x_0)) \leq CE(u(s); B_{2r_i^1}^M(x_0)) + C \frac{t_i^1 - s}{(r_i^1)^4} + C \frac{\sqrt{t_i^1 - s}}{(r_i^1)^2} + C(r_i^1)^2.$$

Set $T = \frac{\epsilon_1^2}{16C^2C_0^2}$. Then we have

$$(4.4) \quad E(u(s); B_{2r_i^1}^M(x_0)) \geq \frac{\epsilon_1}{2CC_0}$$

for any $s \in [t_i^1 - T(r_i^1)^4, t_i^1]$, when i is sufficiently large.

Case 1. $\limsup_{i \rightarrow \infty} \frac{\text{dist}(x_i^1, \partial M)}{r_i^1} \rightarrow \infty$. By passing to a subsequence, we may assume

$$\lim_{i \rightarrow \infty} \frac{\text{dist}(x_i^1, \partial M)}{r_i^1} \rightarrow \infty.$$

Assume $t_i^1 - \frac{\delta^2}{4} > T_1 - \delta^2$ and define

$$B_i := \{x \in \mathbb{R}^4 | x_i^1 + r_i^1 x \in B_\delta^M(x_0)\},$$

and

$$v_i(x, t) := u(x_i^1 + r_i^1 x, t_i^1 + (r_i^1)^4 t), \quad \forall x \in B_i, \quad -\frac{\delta^2}{4(r_i^1)^4} \leq t \leq 0.$$

It is easy to see that $B_i \rightarrow \mathbb{R}^4$ as $i \rightarrow \infty$. Then v_i satisfies

$$(4.5) \quad \partial_t v_i + \Delta^2 v_i = -f(v_i),$$

along with the boundary condition

$$(4.6) \quad \begin{cases} v_i(x, t) = g(x_i^1 + r_i^1 x), & \text{if } x_i^1 + r_i^1 x \in \partial M; \\ \partial_\nu v_i(x, t) = r_i^1 h(x_i^1 + r_i^1 x), & \text{if } x_i^1 + r_i^1 x \in \partial M. \end{cases}$$

By Lemma 2, we have

$$(4.7) \quad \int_{-T}^0 \int_{B_i} |\partial_t v_i|^2 dx dt \leq \int_{t_i^1 - (r_i^1)^4 T}^{t_i^1} \int_M |\partial_t u|^2 dv_g dt \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

and

$$(4.8) \quad \sup_{\frac{\delta^2}{4(r_i^1)^4} \leq t \leq 0} E(v_i, B_i) \leq \sup_{T_1 - \delta^2 \leq t \leq T_1} E(u) \leq C.$$

By (4.3), we can see that

$$\sup_{-T \leq t \leq 0} \sup_{x \in B_i} E(v_i, B_1(x) \cap B_i) \leq \sup_{\substack{(x, t) \in \overline{M} \times [T_1 - \delta^2, t_i^1] \\ r \leq r_i^1}} E(u(t), B_r(x)) = \frac{\epsilon_1}{2C_0}.$$

Hence, for any $x \in \mathbb{R}^4$, when i is sufficiently large, we have

$$(4.9) \quad \sup_{-T \leq t \leq 0} E(v_i, B_1(x)) \leq \frac{\epsilon_1}{2C_0}.$$

Combining (4.9) with Theorem 5, we have

$$(4.10) \quad \sup_{-\frac{T}{2} \leq t \leq 0} \|v_i(\cdot, t)\|_{C^{4+\alpha}(B_{1/2}(x))} \leq C,$$

which yields

$$(4.11) \quad \sup_{-\frac{T}{2} \leq t \leq 0} \|v_i(\cdot, t)\|_{C^{4+\alpha}(B_R)} \leq C(R), \quad \forall R > 0.$$

From (4.7) and (4.11), we can find $\sigma_i \in [-\frac{T}{2}, 0]$ such that as $i \rightarrow \infty$, there holds

$$(4.12) \quad \int_{B_i} |\partial_t v_i|^2(x, \sigma_i) dx \rightarrow 0$$

and

$$(4.13) \quad \|v_i(\cdot, \sigma_i)\|_{C_{loc}^{4+\alpha}(\mathbb{R}^4)} \leq C.$$

Therefore, there exists a subsequence of $v_i(\cdot, \sigma_i)$ and a limit map $v \in C^4(\mathbb{R}^4, N)$ such that

$$(4.14) \quad v_i(\cdot, \sigma_i) \rightarrow v \quad \text{in } C^4(B_R), \quad \forall R > 0.$$

Setting $t = \sigma_i$ in the equation (4.5) and letting $i \rightarrow \infty$, it is easy to see that v is a biharmonic map with

$$\frac{\epsilon_1}{2CC_0} \leq E(v; \mathbb{R}^4) \leq C,$$

where the above inequality follows from (4.4) and (4.8). Taking $t_i^1 + (r_i^1)^4 \sigma_i$ as the new time sequence, then we get that

$$u_i(x) = v_i(x, \sigma_i) = u(x_i^1 + r_i^1 x, t_i^1 + (r_i^1)^4 \sigma_i)$$

is the desired sequence in the theorem.

Case 2. $\limsup_{i \rightarrow \infty} \frac{\text{dist}(x_i^1, \partial M)}{r_i^1} < \infty$. After taking a subsequence, we may assume that $\frac{\text{dist}(x_i^1, \partial M)}{r_i^1} \rightarrow a$ as $i \rightarrow \infty$. Then

$$B_i \rightarrow \mathbb{R}_a^4 := \{(x', x^4) | x^4 \geq -a\},$$

where $x' := (x^1, x^2, x^3) \in \mathbb{R}^3$. Noting that for any $x \in \{x^4 = -a\}$, $x_i^1 + r_i^1 x \rightarrow x_0$. Moreover,

$$(4.15) \quad \begin{cases} v_i(x, t) = g(x_i^1 + r_i^1 x), & \text{if } x_i^1 + r_i^1 x \in \partial M; \\ \partial_\nu v_i(x, t) = r_i^1 h(x_i^1 + r_i^1 x), & \text{if } x_i^1 + r_i^1 x \in \partial M. \end{cases}$$

By Theorem 5 and (4.3), for any $B_R(0) \subset \mathbb{R}^4$, $R > 0$, we have

$$(4.16) \quad \sup_{-\frac{T}{2} \leq t \leq 0} \|v_i(\cdot, t)\|_{C^{4+\alpha}(B_R(0) \cap B_i)} \leq C.$$

Using a similar argument as in **Case 1**, we can obtain $v \in C^4(\mathbb{R}_a^4, N)$ satisfying

$$(4.17) \quad \frac{\epsilon_1}{2CC_0} \leq E(v; \mathbb{R}_a^4) \leq C,$$

and a sequence $\sigma_i \in [-\frac{T}{2}, 0]$ such that as $i \rightarrow \infty$, there hold

$$(4.18) \quad \|v_i(\cdot, \sigma_i) - v\|_{C^4(B_i \cap B_R(0))} \rightarrow 0,$$

for any $R > 0$. Moreover, v is a biharmonic map with the boundary condition

$$(4.19) \quad \begin{cases} v(x) = g(x_0), & \text{on } \partial \mathbb{R}_a^4; \\ \partial_\nu v(x) = 0, & \text{on } \partial \mathbb{R}_a^4. \end{cases}$$

Step 4. Global existence of weak solutions. Let $(x_0, T_1) \in S(T_1)$ be as in Step 3. Then we claim

$$(4.20) \quad \lim_{r \rightarrow 0} \limsup_{t \uparrow T_1} \int_{B_r(x_0)} |\Delta u(\cdot, t)|^2 dx \geq \epsilon_0^2.$$

In fact, by (4.14), (4.18) and Theorem 4, we have

$$\begin{aligned} \epsilon_0^2 &\leq \lim_{R \rightarrow \infty} \int_{B_R(0)} |\Delta v|^2 dx = \lim_{R \rightarrow \infty} \lim_{i \rightarrow \infty} \int_{B_R(0)} |\Delta v_i|^2(\cdot, \sigma_i) dx \\ &= \lim_{R \rightarrow \infty} \lim_{i \rightarrow \infty} \int_{B_{r_i R}(x_i)} |\Delta u|^2(\cdot, t_i + r_i^4 \sigma_i) dx \\ &\leq \lim_{r \rightarrow 0} \limsup_{t \uparrow T_1} \int_{B_r(x_0)} |\Delta u(\cdot, t)|^2 dx. \end{aligned}$$

Next, we claim that there is a unique weak limit $u(\cdot, T_1) \in W^{2,2}(M, N)$ such that

$$\lim_{t \uparrow T_1} u(\cdot, t) = u(\cdot, T_1) \text{ weakly in } W^{2,2}(M, N).$$

In fact, by Lemma 2, for any sequence $t_i \rightarrow T_1$, there exists a subsequence (also denoted by t_i) such that $u(\cdot, t_i) \rightarrow u(\cdot, T_1)$ weakly in $W^{2,2}(M)$ as $i \rightarrow \infty$. So, we just need to show the weak limit $u(\cdot, T_1)$ is independent of the choice of the time sequences. Let $s_i \rightarrow T_1$ be another time sequence and the corresponding weak limit $\hat{u}(\cdot, T_1)$. Note that

$$\begin{aligned} &\int_M |u(\cdot, T_1) - \hat{u}(\cdot, T_1)|^2 dx \\ (4.21) \quad &= \int_M \langle u(\cdot, T_1) - \hat{u}(\cdot, T_1), u(\cdot, T_1) - u(\cdot, t_i) \rangle dx + \int_M \langle u(\cdot, T_1) - \hat{u}(\cdot, T_1), u(\cdot, t_i) \\ &\quad - u(\cdot, s_i) \rangle dx + \int_M \langle u(\cdot, T_1) - \hat{u}(\cdot, T_1), u(\cdot, s_i) - \hat{u}(\cdot, T_1) \rangle dx. \end{aligned}$$

Since

$$\int_M |u(\cdot, t_i) - u(\cdot, s_i)|^2 dx = \int_M \left| \int_{s_i}^{t_i} \frac{\partial u}{\partial t} dt \right|^2 dx \leq |s_i - t_i| \int_{M_{s_i}^{t_i}} \left| \frac{\partial u}{\partial t} \right|^2 dx dt,$$

$\int_{M_{T_1}^{T_1}} \left| \frac{\partial u}{\partial t} \right|^2 dx dt \leq C$ (see Lemma 2), $u(\cdot, t_i) \rightarrow u(\cdot, T_1)$, $u(\cdot, s_i) \rightarrow \hat{u}(\cdot, T_1)$ weakly in $W^{2,2}(M)$ by sending $i \rightarrow \infty$ in (4.21), we obtain

$$\int_M |u(\cdot, T_1) - \hat{u}(\cdot, T_1)|^2 dx = 0.$$

Thus $u(\cdot, T_1) = \hat{u}(\cdot, T_1)$. It is easy to see that

$$\int_M |\Delta u(\cdot, T_1)|^2 dx \leq \int_M |\Delta u_0|^2 dx - \epsilon_0^2.$$

Now we use $u(\cdot, T_1)$ as the initial condition and (g, h) as the boundary condition to extend the above solution beyond T_1 to obtain a weak solution $u : M \times (0, T_2) \rightarrow N$ for some $T_2 > T_1$ by piecing together the solutions at T_1 . Then we see that $u \in C_{loc}^{4+\alpha, 1+\frac{\alpha}{4}}(M \times ((0, T_2) \setminus \{T_1\}), N)$. Iterating this process, we obtain a global solution defined on $M \times [0, \infty)$. Let $\{T_k\}_{k=1}^K$ be all the possible singular times. Then we have

$$\begin{aligned} \int_M |\Delta u(\cdot, T_K)|^2 dx &\leq \liminf_{t_i \uparrow T_K} \int_M |\Delta u(\cdot, t_i)|^2 dx - \epsilon_0^2 \\ &\leq \int_M |\Delta u_0|^2 dx - K \epsilon_0^2, \end{aligned}$$

which implies

$$K \leq \frac{\int_M |\Delta u_0|^2 dx}{\epsilon_0^2}.$$

Hence there are at most finitely many singular time slices.

Step 5. Uniqueness. The only thing left to be proven is uniqueness and we only need to prove uniqueness of the short time solution constructed above and the full uniqueness follows by iteration. Let $u, v : [0, t_0) \rightarrow N$ be two constructed smooth (for $t > 0$) solutions and set $w := u - v$. Then

$$\begin{aligned} \partial_t w + \Delta^2 w &= f(u) - f(v) \\ &= \nabla^3 u \# \nabla u + \nabla^2 u \# \nabla^2 u + \nabla^2 u \# \nabla u \# \nabla u + \nabla u \# \nabla u \# \nabla u \# \nabla u \\ &\quad - (\nabla^3 v \# \nabla v + \nabla^2 v \# \nabla^2 v + \nabla^2 v \# \nabla v \# \nabla v + \nabla v \# \nabla v \# \nabla v \# \nabla v). \end{aligned}$$

Multiply this equation with w and integrate over $(0, s) \times M$. By partial integration ($w = \partial_\nu w = 0$ on ∂M for any $t \in (0, s)$), we can get rid of derivatives of order > 2 (cf. [5]). Simplifying terms by using Young's inequality we get

$$\begin{aligned} &\frac{1}{2} \int_M |w(s)|^2 + \int_{M^s} |\Delta w|^2 \\ (4.22) \quad &\leq C \sum_{k=1}^2 \int_{M^s} |w|^2 (|\nabla^k u| + |\nabla^k v|)^{\frac{4}{k}} + C \sum_{k=1}^2 \int_{M^s} |\nabla w|^2 (|\nabla^k u| + |\nabla^k v|)^{\frac{2}{k}} \\ &\quad + C \sum_{l=0}^1 \sum_{k=1}^{2-l} \int_{M^s} |\nabla^2 w| |\nabla^l w| (|\nabla^k u| + |\nabla^k v|)^{\frac{2-l}{k}} \\ &:= I_4 + I_5 + I_6. \end{aligned}$$

To make it more clear how the above inequality is obtained, let us give the details of the estimates of the highest order term of $(f(u) - f(v))w$ (we denote it by $\varphi(u) \cdot \nabla^3 u \cdot \nabla u$) as follows.

$$\begin{aligned} &\int_{M^s} (\varphi(u) \cdot \nabla^3 u \cdot \nabla u - \varphi(v) \cdot \nabla^3 v \cdot \nabla v) w \\ &= \int_{M^s} [(\varphi(u) - \varphi(v)) \cdot \nabla^3 u \cdot \nabla u + \varphi(v) \cdot \nabla^3 w \cdot \nabla u + \varphi(v) \cdot \nabla^3 v \cdot \nabla w] w \\ &= \int_{M^s} \nabla^2 u \# \nabla w \# \nabla u \# w + \nabla^2 u \# \nabla^2 u \nabla \# w^2 + \nabla^2 w \# \nabla u \# \nabla v \# w + \nabla^2 w \# \nabla^2 u \# w \\ &\quad + \nabla^2 w \# \nabla u \# \nabla w + \nabla^2 v \# \nabla v \# \nabla w \# w + \nabla^2 v \# \nabla^2 w \# w + \nabla^2 v \# \nabla w \# \nabla w \\ &\leq I_4 + I_5 + I_6, \end{aligned}$$

where

$$\varphi(u) \cdot \nabla^3 u \cdot \nabla u := \varphi_{AB}^{ijkl}(u) \cdot \nabla_{ijk}^3 u^A \cdot \nabla_l u^B$$

and the last inequality follows from Young's inequality and following property

$$|\nabla^k \varphi| \leq C(N) \quad \text{and} \quad |\varphi(u) - \varphi(v)| \leq C(N)(u - v).$$

In the following, let's estimate the right hand side of (4.22). By Hölder's inequality, the Sobolev inequality, the Poincaré inequality and partial integration, we get

$$\begin{aligned}
& \int_{M^s} |w|^2 (|\nabla u| + |\nabla v|)^4 \\
& \leq \left(\int_{M^s} |w|^4 \right)^{\frac{1}{2}} \left(\int_{M^s} |\nabla u|^8 + |\nabla v|^8 \right)^{\frac{1}{2}} \\
& \leq C \left(\int_{M^s} |\nabla u|^8 + |\nabla v|^8 \right)^{\frac{1}{2}} \left(\int_0^s \left(\int_M |\nabla w|^2 + |w|^2 \right)^2 \right)^{\frac{1}{2}} \\
(4.23) \quad & \leq C \left(\int_{M^s} |\nabla u|^8 + |\nabla v|^8 \right)^{\frac{1}{2}} \left(\int_0^s \left(\int_M |\nabla w|^2 \right)^2 \right)^{\frac{1}{2}} \\
& \leq C \left(\int_{M^s} |\nabla u|^8 + |\nabla v|^8 \right)^{\frac{1}{2}} \left(\int_0^s \int_M |w|^2 \int_M |\nabla^2 w|^2 \right)^{\frac{1}{2}} \\
& \leq C \left(\int_{M^s} |\nabla u|^8 + |\nabla v|^8 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 \right)^{\frac{1}{2}} \left(\int_{M^s} |\nabla^2 w|^2 \right)^{\frac{1}{2}} \\
& \leq C \left(\int_{M^s} |\nabla u|^8 + |\nabla v|^8 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right),
\end{aligned}$$

and similarly we have

$$\begin{aligned}
(4.24) \quad & \int_{M^s} |w|^2 (|\nabla^2 u| + |\nabla^2 v|)^2 \\
& \leq C \left(\int_{M^s} |\nabla^2 u|^4 + |\nabla^2 v|^4 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right).
\end{aligned}$$

Now let's estimate I_5 term by term,

$$\begin{aligned}
& \int_{M^s} |\nabla w|^2 |\nabla u|^2 = - \int_{M^s} w \Delta w |\nabla u|^2 - \int_{M^s} w \nabla w \nabla u \cdot \nabla^2 u \\
& \leq \epsilon \int_{M^s} |\nabla^2 w|^2 + C(\epsilon) \int_{M^s} w^2 |\nabla u|^4 + \frac{1}{2} \int_{M^s} |\nabla w|^2 |\nabla u|^2 + \int_{M^s} w^2 |\nabla^2 u|^2.
\end{aligned}$$

Therefore we get

$$\int_{M^s} |\nabla w|^2 |\nabla u|^2 \leq \epsilon \int_{M^s} |\nabla^2 w|^2 + C(\epsilon) \int_{M^s} w^2 |\nabla u|^4 + \int_{M^s} w^2 |\nabla^2 u|^2.$$

Thus by (4.23) and (4.24) we have

$$\begin{aligned}
& \int_{M^s} |\nabla w|^2 (|\nabla u|^2 + |\nabla v|^2) \\
& \leq \epsilon \int_{M^s} |\nabla^2 w|^2 + C(\epsilon) \left(\left(\int_{M^s} |\nabla u|^8 + |\nabla v|^8 \right)^{\frac{1}{2}} + \left(\int_{M^s} |\nabla^2 u|^4 + |\nabla^2 v|^4 \right)^{\frac{1}{2}} \right) \\
& \quad \times \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right).
\end{aligned}$$

In addition,

$$\begin{aligned}
\int_{M^s} |\nabla w|^2 |\nabla^2 u| &= - \int_{M^s} w \Delta w |\nabla^2 u| - \int_{M^s} w \nabla w \cdot \nabla |\nabla^2 u| \\
&\leq \epsilon \int_{M^s} |\nabla^2 w|^2 + C(\epsilon) \int_{M^s} w^2 |\nabla^2 u|^2 - \frac{1}{2} \int_{M^s} \nabla w^2 \cdot \nabla |\nabla^2 u| \\
&= \epsilon \int_{M^s} |\nabla^2 w|^2 + C(\epsilon) \int_{M^s} w^2 |\nabla^2 u|^2 + \frac{1}{2} \int_{M^s} w^2 \Delta |\nabla^2 u| \\
&\leq \epsilon \int_{M^s} |\nabla^2 w|^2 + C(\epsilon) \int_{M^s} w^2 |\nabla^2 u|^2 + C \left(\int_{M^s} w^4 \right)^{\frac{1}{2}} \left(\int_{M^s} |\nabla^4 u|^2 \right)^{\frac{1}{2}} \\
&\leq \epsilon \int_{M^s} |\nabla^2 w|^2 + C(\epsilon) \left(\int_{M^s} |\nabla^2 u|^4 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right) \\
&\quad + C \left(\int_{M^s} |\nabla^4 u|^2 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right).
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\int_{M^s} |\nabla w|^2 (|\nabla^2 u| + |\nabla^2 v|) \\
&\leq \epsilon \int_{M^s} |\nabla^2 w|^2 + C(\epsilon) \left(\int_{M^s} |\nabla^2 u|^4 + |\nabla^2 v|^4 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right) \\
&\quad + C \left(\int_{M^s} |\nabla^4 u|^2 + |\nabla^4 v|^2 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right).
\end{aligned}$$

In summation of the above estimates we have

$$\begin{aligned}
(4.25) \quad I_5 &\leq \epsilon \int_{M^s} |\nabla^2 w|^2 + C(\epsilon) \left(\int_{M^s} |\nabla u|^8 + |\nabla v|^8 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right) \\
&\quad + C(\epsilon) \left(\int_{M^s} |\nabla^2 u|^4 + |\nabla^2 v|^4 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right) \\
&\quad + C \left(\int_{M^s} |\nabla^4 u|^2 + |\nabla^4 v|^2 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right).
\end{aligned}$$

We are left to estimate the last summation I_6 in (4.22). Note that by Young's inequality

$$(4.26) \quad I_6 \leq \epsilon \int_{M_0^s} |\nabla^2 w|^2 + C(\epsilon) (I_4 + I_5).$$

Therefore from inequalities (4.22)-(4.26) we obtain

$$\begin{aligned}
&\int_M |w(s)|^2 + \int_{M^s} |\Delta w(s)|^2 \\
&\leq \epsilon \int_{M^s} |\nabla^2 w|^2 + C(\epsilon) \left(\int_{M^s} |\nabla u|^8 + |\nabla v|^8 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right) \\
&\quad + C(\epsilon) \left(\int_{M^s} |\nabla^2 u|^4 + |\nabla^2 v|^4 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right) \\
&\quad + C(\epsilon) \left(\int_{M^s} |\nabla^4 u|^2 + |\nabla^4 v|^2 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right).
\end{aligned}$$

By the standard elliptic estimate, noting that $w = 0$ on ∂M , we have $\int_M |\nabla^2 w|^2 \leq C \int_M |\Delta w|^2$. Choosing $\epsilon = \frac{1}{2C}$, we obtain

$$\begin{aligned} & \int_M |w(s)|^2 + \int_{M^s} |\nabla^2 w(s)|^2 \\ & \leq C \left(\int_{M^s} |\nabla u|^8 + |\nabla v|^8 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right) \\ & \quad + C \left(\int_{M^s} |\nabla^2 u|^4 + |\nabla^2 v|^4 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right) \\ & \quad + C \left(\int_{M^s} |\nabla^4 u|^2 + |\nabla^4 v|^2 \right)^{\frac{1}{2}} \left(\sup_{t \in (0, s)} \int_M |w(t)|^2 + \int_{M^s} |\nabla^2 w|^2 \right). \end{aligned}$$

Hence for solutions $u, v \in V(M^T)$ by the interpolation inequalities of Lemma 1 we see that we can choose s small enough such that $C(\int_{M^s} |\nabla u|^8 + |\nabla v|^8)^{\frac{1}{2}}$, $C(\int_{M^s} |\nabla^2 u|^4 + |\nabla^2 v|^4)^{\frac{1}{4}}$ and $C(\int_{M^s} |\nabla^4 u|^2 + |\nabla^4 v|^2)^{\frac{1}{2}}$ are all smaller than $\frac{1}{4}$. Without loss of generality, we assume that

$$\sup_{t \in [0, s]} \int_M |w(t)|^2 = \int_M |w(s)|^2.$$

Hence we obtain that $\sup_{t \in [0, s]} \int_M |w(t)|^2 = 0$, i.e., $u \equiv v$ on $[0, s]$. \square

Proof of Theorem 2. By Theorem 1, we see that there exists a time sequence $\{t_i\}$, $t_i \rightarrow +\infty$ as $i \rightarrow +\infty$, such that $\frac{\partial u(t_i, \cdot)}{\partial t_i} \rightarrow 0$ in $L^2(M, N)$ and $u(\cdot, t_i)$ converges weakly in $W^{2,2}(M, N)$ to a map $u_\infty \in W^{2,2}(M, N)$ with Dirichlet boundary data $u = g$ and $\partial_\nu u = h$ on ∂M , where $g \in C^{4+\alpha}(\partial M, N)$ and $h \in C^{3+\alpha}(\partial M, T_g N)$. Denote $u(t_i) = u_i$ and $g_i = -\frac{\partial u_i}{\partial t_i}$, then $\Delta^2 u_i + f(u_i) = g_i$. Note that $g_i \rightarrow 0$ in $L^2(M, N)$ and hence in $(W^{2,2}(M, N))^*$, by the weak compactness theorem of Zheng [27], we see that $u_\infty \in W^{2,2}(M, N)$ is a biharmonic map. Then we see that $u \in C^{4+\alpha, 1+\frac{\alpha}{4}}(\bar{M}, N)$ by the interior regularity theorem of Wang [24] and boundary regularity theorem of Lamm and Wang [13]. \square

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